# Knot Theory:
## The Yang-Baxter Equation, Quantum Groups and Computation of the Homfly Polynomial.

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**Abstract**

This report gives an overview of knot theory and some of its applications. We look at how computing technology can aid us in calculating knot invariants, and construct an algorithm to calculate the Homfly polynomial. We also look at a number of relations to physics and algebra, through the use of abstract tensor diagrams. In particular we see a connection to the quantum group $SL(2)_q$, and the Yang-Baxter equation.

# Contents


**1 Introduction** **3**
- 1.1 What is a knot? . . . 3
- 1.2 Why study knots? . . . 4
- 1.3 Contents . . . 5

**2 Knot theory** **7**
- 2.1 Knot isotopy . . . 7
- 2.2 Knot diagrams . . . 8
- 2.3 Knot invariants . . . 10
- 2.4 Knot polynomials . . . 12
- 2.5 Summary . . . 17

**3 Computing in knot theory** **18**
- 3.1 Dowker notation . . . 18
- 3.2 Constructing the algorithm . . . 20
  - 3.2.1 Skein relation . . . 20
  - 3.2.2 Loops . . . 21
  - 3.2.3 Trivial diagrams . . . 21
  - 3.2.4 Link diagrams . . . 23
  - 3.2.5 Does the algorithm finish? . . . 25
- 3.3 Algorithm for calculating the Homfly polynomial . . . 26
- 3.4 An example calculation . . . 27
- 3.5 Summary . . . 32

**4 Knots, physics and algebra** **33**
- 4.1 Diagrammatic tensors . . . 33
- 4.2 Knots as tensor diagrams . . . 35
- 4.3 Topological invariance - the Yang-Baxter equation . . . 36
- 4.4 The quantum group $SL(2)_q$ . . . 39
- 4.5 Summary . . . 41

**5 What is a quantum group?** **43**
- 5.1 Hopf algebra . . . 43
- 5.2 $SL(2)_q$ as a Hopf algebra . . . 44

5.3 Quasitriangular Hopf algebras . . . . . 47
5.4 The Yang-Baxter equation . . . . . 48
5.5 Summary . . . . . 49

**6 Conclusion** **50**

**Bibliography** **52**

**A Proof of Theorem 2** **54**

**B Calculating the values $B$ and $C$ for the bracket polynomial** **56**

**C Homfly polynomial of the figure eight knot** **58**

**D Verifying multiplication $\mu$ in $SL(2)_q$ is well defined** **66**

# Chapter 1

# Introduction

Knots have been around since prehistoric times, and remain a vital part of everyday life today. They are used by sailors, climbers, fishermen and surgeons, as well as for such mundane tasks as tying a shoelace [1]. But amongst these utilitarian uses, knots also appear in an aesthetic way throughout history: they are found in manuscripts throughout the Middle Ages, in Eastern architecture, and the Celts are renowned for their use of knots when decorating objects such as megaliths and burial stones [2]. It is even thought that the ancient Incas may have used knotted strings, called khipu, as a form of writing [3]. But how do we turn our intuitive idea of such a common object into a mathemtical study? And what can we learn, that we have not already learned from thousands of years of practical experience?

## 1.1 What is a knot?

The first question we must ask ourselves is, what exactly is a knot? And how can we describe knots mathematically? Before embarking on any technical definitions, we would like to spend a little time thinking about what we would intuitively call a knot.

It should be safe to assume that we have all at some point come across a knot, whether it is while tying a shoelace or untangling a piece of string. Nearly all of the knots we will have experienced have one thing in common: they all started life from at least one long piece of untangled string, (or other similar material). Although this may seem obvious, it gives us a good starting point for our mathematical definition. Perhaps the simplest way to model a long, thin material like this is to look at a curve in $\mathbb{R}^3$, while not allowing any self-intersections (string cannot pass through itself). We are not concerned with the relative thickness of the material here - we want a knot made from string to be the same as a knot made from rope - so looking at a one dimensional curve is fine.

Our next problem comes from the fact that the materials we make knots

out of are not rigid. In 'the real world', when saying two knots are the same, we would allow for a certain amount of bending or reshaping of the material, and we even allow for their relative sizes to be different. The obvious mathematical technique to take into account reshaping and resizing is to look at our curve from a topological viewpoint. We will look at this in more detail in section 2.1.

The problem we have with allowing bending and reshaping of the material making up our knot, is that we introduce the ability to untie the knot. If we are to begin a study of knots, our definition must somehow stop this unknotting from happening, while maintaining the flexibility of our string. Perhaps the easiest way to do this would be to fix the ends of our string, stopping them from moving - this way we would never have any fear of the ends slipping through the knot. We can achieve the same effect however, by simply joining the two ends together, forming a loop.

We are now ready to attempt a definition of a knot.

**Definition** : A knot is a closed curve in $\mathbb{R}^3$ with no self-intersections.

We would like to emphasise that with this definition, any knot that we study from now on may still be formed from a piece of string, by tying it first and then simply sticking the two free ends together.

We may also study *links*, which are simply collections of knots (which may in turn be intertwined). Each individual closed curve making up a link is called a *component* of the link. Therefore a knot is nothing more than a one-component link [4]. Informally, we often refer to a link as a knot, this will rarely cause any problems, but we will attempt to be clear where this is not the case.

So, we are able to describe these everyday objects in a mathematical context. But why go to the effort?

## 1.2 Why study knots?

Knot theory is one of the most active areas of research in mathematics today, and its techniques can be found in such wide ranging areas as fluid dynamics, solar physics, DNA research, and quantum computation [5]. However, it was only as late as the twentieth century that mathematicians really began to seriously study knots [2].

Early scientific interest started in the late 1800s, when Lord Kelvin (William Thomson) suggested that atoms may be made from knotted vortices in the fabric of the ether - a substance that at the time was believed to permeate all of space [6]. He believed that the different elements may then be determined by the different possible knots.

Kelvin's theory proved to be wrong - the Michelson-Morley experminent in 1887 suggesting that there was in fact no ether. But the theory had

started the first real mathematical problem involving knots: how can you classify the different types of knots? Peter Guthrie Tait published the first paper to address this issue in 1877, and by 1900 he and C. N. Little had almost completed enumerating all knots with up to ten crossings [7].

It was the solid introduction of topology to mathematics at the turn of the century that really allowed the beginnings of knot theory as we know it; work done by M. Dehn and J. Alexander introduced algebraic methods into the theory, and the first book about knots, *Knotentheorie* was published by K. Reidemeister in 1932. By 1970, knot theory had become a well-developed area of topology [7].

The discovery of the Jones polynomial by Vaughan Jones in 1984 not only showed a connection between knot theory and different areas of mathematics (operator algebras, braid theory, quantum groups), but also to physics (statistical models) [2], [8]. The theory, borne from the desire to descibe the chemical elements, has not lost its scientific ties. In the 1980s, biochemists discovered knotting in DNA molecules, and since, synthetic chemists think that it may be possible to create knotted molecules, the properties of which are determined by the types of knots [6]. We have even found fish (called *Myxine*, or slime eels) that deliberately tie themselves up into knots to help escape from predators [2].

Perhaps knot theory is interesting due to the wide reaching relationships that it appears to create, or perhaps it is simply the familiarity of its most fundamental ideas. Whatever its appeal, there is great hope that it may become accepted as a fundamental theory of mathematics, science and nature, and that it may spark new levels of understanding.

"When we finally understand [the] deepest nature [of knots], profound physical applications will blossom. And it will be beautiful" [5].

## 1.3 Contents

This report aims to give an overview of some of the key ideas in knot theory, as well as to introduce some topics that are of current interest in research. Due to the wide ranging areas that knot theory relates to, it is not possible to cover them all. We will however see relations to a few of these different areas, in particular with regards to computing, physics and algebra.

Chapter 2 is an introduction to the mathematical theory of knots. We look at the problem of comparing different knots, introduce knot diagrams, and see a number of useful knot invariants.

Chapter 3 looks at how modern advances in computing technology has helped with the problem of distinguishing different knots, how we may utilise this technology for our mathematical theory, and some of the restrictions we must face. We will also construct an algorithm that may be used by a computer to calculate the Homfly polynomial.

In chapter 4 we will see how knot diagrams can be used to represent algebraic operations, in particular looking at how knot diagrams can relate to matrix multiplication. In this way we will find some suprising relationships with physics, through quantum mechanics and the Yang-Baxter equation. We will also look at how the topology of knot theory relates to algebraic structures, such as Hopf algebras and quantum groups.

Chapter 5 will then look at Hopf algebras in more detail. We will look at quasitriangular Hopf algebras, return to the Yang-Baxter equation, and see why $SL(2)_q$ is a quantum group.

The final chapter will give an overview of the report, and will discuss some of the many areas that we could look at following on from this report, as well as some areas that are currently being explored in knot theory today.

Some of the longer proofs, examples and verifications that we come across can be found in the appendices.

# Chapter 2

# Knot theory

This chapter looks at some of the fundamental building blocks of knot theory. Some of the material will be used later in this report, while some is included simply to give an idea of different techniques used in knot theory. The ideas used in this chapter can be found in most introductory books or courses on knot theory, such as [4], [6], [7], [9] and [10].

## 2.1 Knot isotopy

Now that we have an understanding of what a knot is, we begin with what is perhaps the most fundamental problem of knot theory: when is one knot the same as another knot (often known as the *comparison problem*)?

We begin by defining an equivalence relation between knots, using *ambient isotopy*. An ambient isotopy of a space $X \subset \mathbb{R}^3$ is an isotopy of $\mathbb{R}^3$ that carries $X$ with it. We are required to use ambient isotopy (as opposed to homotopy, or isotopy for example), as we do not want the curve that forms our knot to be able to pass through itself, or be able to 'shrink' away.

**Definition** : Two knots $K_1$ and $K_2$ are equivalent (ambient isotopic) if there is an isotopy $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $h(K_1, 0) = K_1$ and $h(K_1, 1) = K_2$. [9]

We then say that two knots are equivalent if they can be deformed into each other using this ambient isotopy. From now on, when we say a 'knot' we are generally referring to a whole equivalence class, so if we say two knots are different, we mean they are in different equivalence classes. In the same way, if we want to know if one knot is *the same* as another knot, we are in fact asking whether they are in the same equivalence class.

The comparison problem now becomes a case of finding ambient isotopies (or showing there are none) between knots. As is the case in most of topology however, trying to define specific maps is very difficult; it would be virtually impossible to define a specific ambient isotopy between most knots.

To aid us in solving this problem, we try to simplify how we can represent knots.

## 2.2 Knot diagrams

A very natural and useful way to look at knots is by using knot diagrams. As we have seen, we define a knot as a closed curve in space, therefore we may look at the projection of this curve onto a plane; the knot is thus represented by a planar curve. Where one part of the projection passes over another we include *crossings*, making a break in the strand that corresponds to the lower part of the curve in space. A few examples of knot diagams are shown in figure 2.1.

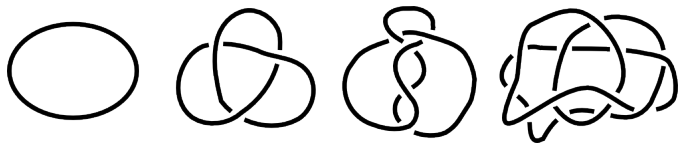


Figure 2.1: The standard diagram of the unknot (left), and three different diagrams of the trefoil knot.

In some regards the comparison problem now becomes very easy. A knot corresponds to a regular presentation (knot diagram), so the problem is simplified to just comparing the relavent knot diagrams. Unfortunately any knot can have many different diagrams. In the 1920s however, Kurt Reidemeister proved a theorem, which theoretically solves everything:

**Theorem 1 (Theorem of Reidemeister)** *If two knots (or links) are equivalent, their diagrams are related by a sequence of Reidemeister moves (see figure 2.2).*

*Where the Reidemeister moves (R1, R2 and R3) are three pairs of possible changes to a knot diagram, assuming with each move that you only change the diagram locally as shown, leaving the rest of the diagram alone.*

*Proof: See [11] page 238.*

Now, to show that two knots are equivalent, all we have to do is find a sequence of Reidemeister moves that turn the diagram of the first knot into the diagram of the second. An example of a sequence of such moves is shown in figure 2.3.

Unfortunately, there are a few problems. Suppose we have two different diagrams of a knot. How do we go about finding the sequence of Reidemeister moves that will turn one diagram into the other? Where do we start? Perhaps if we start by trying any move that does not increase the number of crossings? But if there are many different moves we could choose, how

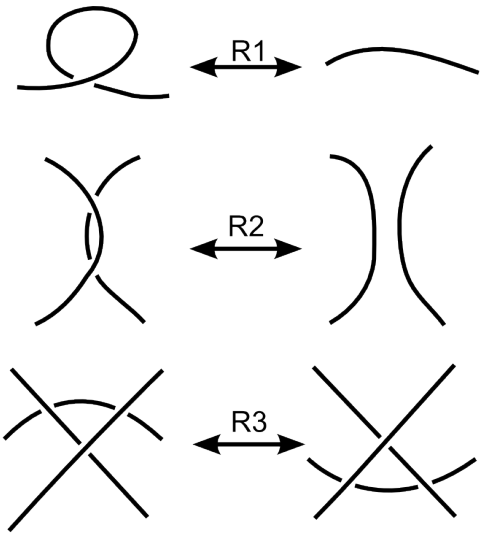



Figure 2.2: The Reidemeister moves.

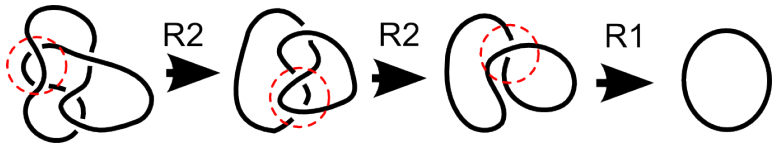



Figure 2.3: Example showing the Reidemeister moves that change the first diagram into the standard diagram of the unknot. The red circles indicate where each move takes place.

do we know which one to do first? Even if this doesn't matter, will we simplify a knot as much as possible if we only apply these kinds of moves? The answer to this last question is unfortunately, no. There are cases where we must first increase the number of crossings before we can simplify the diagram further. Two examples of such a diagram are shown in figure 2.4, the diagrams shown are both in fact the unknot, but to find a sequence of Reidemeister moves from this diagram to the standard unknot diagram, as seen in figure 2.1, would require us to first perform a Reidemeister move that would increase the number of crossings.

Therefore, although in principle we can find a sequence of Reidemeister moves between two equivalent diagrams, there is no obvious way of knowing which moves we should actually perform. Even harder is trying to distinguish two different knots using just Reidemeister moves; we would need to show that there is no sequence of moves between the two diagrams, but as there are so many possible changes to a diagram how would we even know if we had exhausted all possibilities, and is it even possible?

Clearly we need a quicker way to distinguish knots.

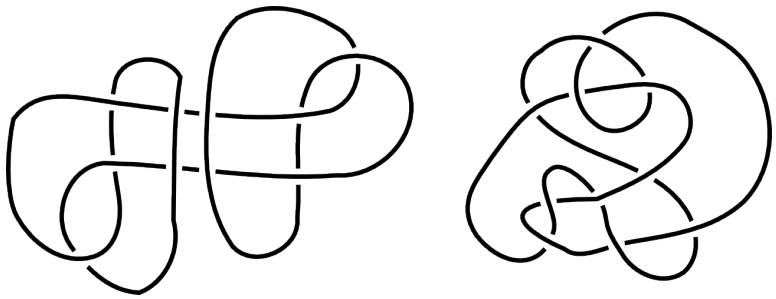


Figure 2.4: Two diagrams of the unknot, where a sequence of Reidemeister moves relating these to the standard unknot diagram requires us to first increase the number of crossings.

## 2.3 Knot invariants

Luckily there is a much easier way to show two knots are distinct, through the use of knot invariants.

**Definition** : A knot invariant is any function $i$ of knots which depends only on their equivalence classes.

Thus, if $K$ and $K'$ are two equivalent knots, $K \cong K'$, then $i(K) = i(K')$. Therefore, if $i(K) \neq i(K')$ then $K \ncong K'$. [4]

However, a complete invariant of knots has yet to be found, that is for all known invariants the reverse does not hold: if $i(K) = i(K')$ then $K$ need not be equivalent to $K'$.

When (or perhaps if) a complete invariant is ever found, we will have a way of distinguishing all knots. For now, we will look at a number of examples of (incomplete) knot invariants, starting with 3-colourability, which gives us our first easy way of distinguishing many knots from the unknot.

**Definition (3-colourability)** : A knot diagram is called 3-colourable if each arc can be assigned one of three colours, satisfying the following rules:

- at least two of the colours are used,
- at any crossing, either all three colours appear, or only one appears.

(Where an arc is an unbroken section of the knot diagram).

Example: the standard unknot diagram is not 3-colourable. But the standard diagram for the trefoil knot is 3-colourable, an example of such a colouring is shown in figure 2.5.
We now need to show this is a knot invariant.

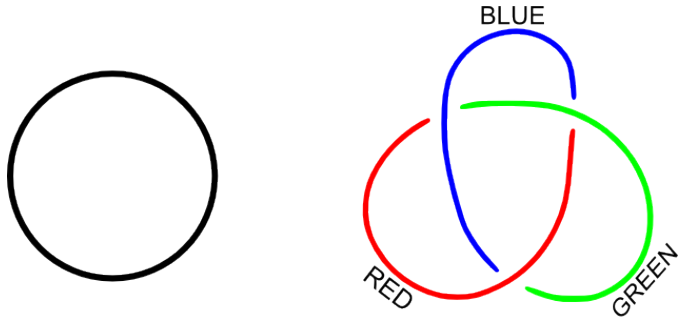



Figure 2.5: The unknot (not 3-colourable) and a diagram of a 3-colouring of the standard trefoil knot.

**Theorem 2 (Invariance of 3-colourability)** *If a diagram of a knot $K$ is 3-colourable, then every diagram of $K$ is 3-colourable. Hence, we may say the knot $K$ itself is 3-colourable. (From the example above, this proves that the trefoil knot is indeed not the unknot).*

*Proof: See appendix A.*

There is a natural way to generalize this idea of colouring: instead of using three colours for our 3-colourability invariant, we may instead label arcs with integers, 0, 1 and 2 - this way the relationship between arcs at a crossing (previously: all being the same colour, or all different) becomes an algebraic condition on integers (mod 3). The natural question then is whether we can look at similar relationships, but working in a different modulo.

But we have looked long enough at these kinds of invariants, and instead refer the reader to [7] for further information.

We now look at two invariants (the crossing number, and the unknotting number), which both share the idea of looking at the minimal value of a property of a knot. As we will see, both are quite obviously invariant by their definition, but both are very much more difficult to compute than other invariants, such as 3-colourability.

**Definition (Crossing number)** : The crossing number $c(K)$ of a knot (or link) $K$ is the minimum number of crossings in any diagram $D$ of $K$. So if $c(D)$ is the number of crossings in a diagram $D$, [4]:

$$c(K) = min\{c(D) : D \text{ is a diagram of } K\ \}.$$

**Definition (Unknotting number)** : The unknotting number $u(K)$ of a knot $K$ is the minimum, over all diagrams $D$ of $K$, of the minimal number of crossing changes required to turn $D$ into a diagram of the unknot, $u(D)$, [4]:

$$u(K) = min\{u(D) : D \text{ is a diagram of } K\}.$$

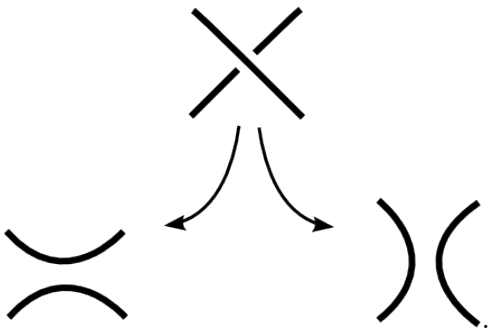
These are both clearly invariants, as for a given knot all possibilities of isotopies are already taken into account in the definitions. This is also exactly why they are so difficult to compute. Suppose we have a diagram of a knot, how do we know if this particular diagram has minimal crossing number? Much like the problems surrounding the Reidemeister moves, technically we would have to check all possible diagrams.

Perhaps the most interesting invariants however, are polynomial valued knot invariants, known as knot polynomials. We will look at these in the next section.

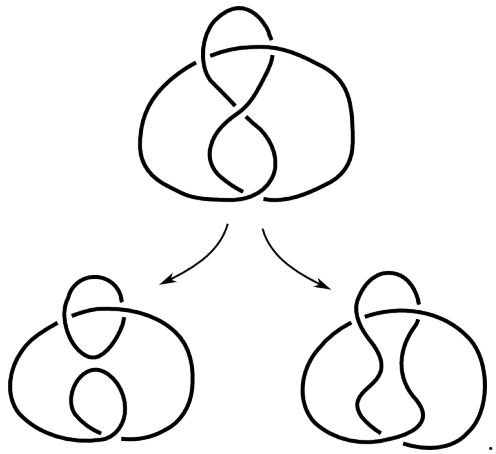
## 2.4 Knot polynomials

Knot polynomials are particularly useful, as not only are they (relatively) simple to compute, but they also manage to distinguish large numbers of different knots. There are a number of different knot polynomials, the first of which was found by J. Alexander in 1928 [7]. We will begin by looking at the more recent Kauffman bracket polynomial, which was not discovered until 1987 [2].

What makes the bracket polynomial particularly interesting, is the way that it relates to state sums - an idea commonly used in physics. Its definition stems from the idea of spliting a knot diagram into a number of different states, by splicing crossings as shown:



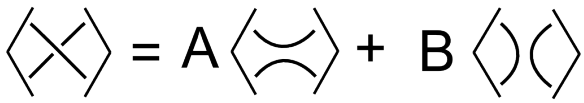
In this way, a knot diagram can be decomposed into a number of states. The bracket polynomial is then defined as a particular summation over these states. An example is shown here:



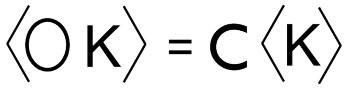
In this example we have decomposed a diagram of the figure eight knot into two states. This method can then be repeated to establish a complete family of state diagrams of trivial knots.

**Definition (The bracket polynomial)** : The Kauffman bracket polynomial $\langle K \rangle$ of a knot (or link) $K$, is a Laurent polynomial defined by the rules:

- it satisfies the skein relation (shown below), which is a relation between the bracket polynomial of diagrams that are different only inside a small neighbourhood as shown,

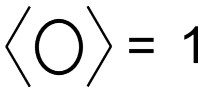



- it satisfies:




  (where $\langle 0\ K \rangle$ is the disjoint union of a knot diagram $K$ and the crossingless diagram of the unknot 0),

- and the normalisation:




For this to be a knot invariant it is sufficient to ensure that the bracket polynomial remains invariant under the three Reidemeister moves. By imposing the fact that we want it to be invariant under Reidemeister move R2, we can determine that $B = A^{-1}$, and $C = -A^2 - A^{-2}$, (the derivation of this is shown in appendix B).

With these values for $B$ and $C$ the third Reidemeister move is also satisfied. However, although this is indeed invariant under R2 and R3 moves, R1 moves give different vaules for the bracket polynomial, see figure 2.6. Invariance under the second and third Reidemeister moves, but not the first, is known as regular isotopy, and has its own interpretation as an invariant of the topological embeddings of knotted, linked and twisted bands in three-dimensional space [8], (see also [13] for a more in depth discussion of this idea). In fact, it is usually possible to normalise a regular isotopy invariant to obtain an ambient isotopy invariant [8].

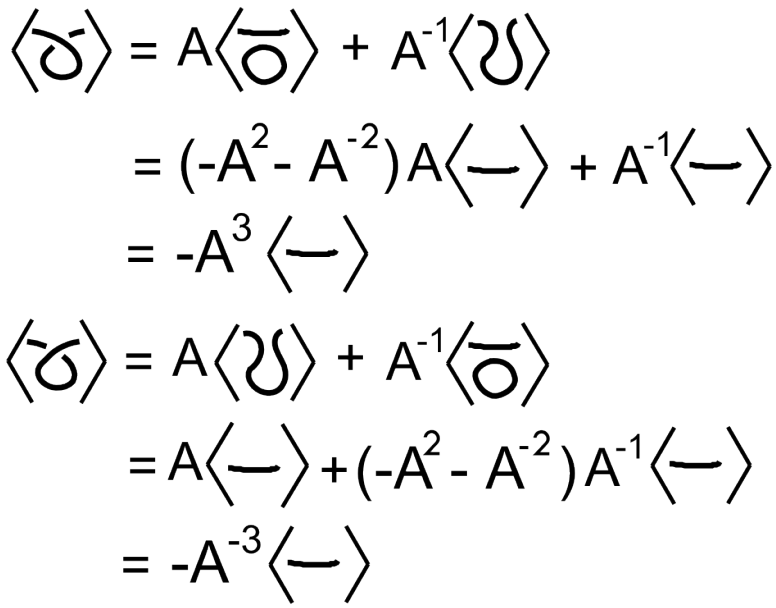



Figure 2.6: The bracket polynomial's 'failure' under R1 moves.

The way we may normalise the bracket polynomial to be an ambient isotopy invariant of knot diagrams, requires the notion of an oriented knot and two definitions - the crossing sign, and the writhe.

**Definition** : A knot (or link) is said to be oriented if it has been given a direction along its curve. Equivalentely each arc in its knot diagram is assigned a direction so that at each crossing the orientations appear as one of the two possibilities as seen in figure 2.7. The crossing sign (or crossing orientation) is the label (+1) or (-1) to the crossing.

**Definition** : The writhe of a diagram $D$, denoted $w(D)$, is the sum of the crossing signs (+1's and -1's) of all of the crossings in the given diagram.

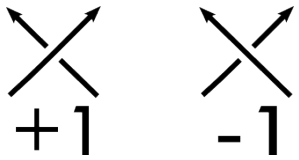



Figure 2.7: The crossing signs of an oriented knot diagram.

It is easy to see that the writhe is invariant under R2 and R3 moves, and each R1 move changes the writhe by $\pm 1$.

We now have all of the ingredients to alter the bracket polynomial to make it an invariant for oriented knots and links. This new polynomial is often called the $X$ polynomial:

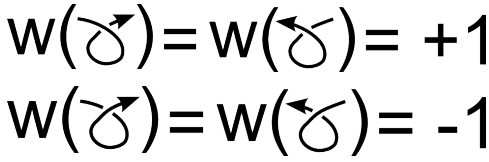
**Definition** : The $X$ polynomial of an oriented knot (or link) K is defined to be

$$X(K) = (-A^3)^{-w(K)}\langle K\rangle.$$

**Theorem 3 (Invariance of the $X$ polynomial)** *The $X$ polynomial is an invariant of oriented knot diagrams.*

*Proof: It is again sufficient to check that the $X$ polynomial remains invariant under the three Reidemeister moves. We already know that the bracket and the writhe of a knot are invariant under R2 and R3 moves, and so $X(K)$ is also invariant under R2 and R3 moves. We thus only need to check R1 moves.*

*Note:*




*if we denote a knot that includes a loop of one of the first two types as $K^+$ and a knot that includes a loop of one of the second two types as $K^-$, then:*

$$\begin{aligned} X(K^+) &= (-A^3)^{-w(K^+)}\langle K^+\rangle \\ &= (-A^3)^{-(w(K^+)-1)})(-A^{-3})\langle K^+\rangle \\ &= (-A^3)^{-(w(K^+)-1)})(-A^{-3})(-A^3)\langle -\rangle \\ &= (-A^3)^{-(w(K^+)-1)}\langle -\rangle \\ &= X(-). \\ \textit{Similarly, } X(K^-) &= X(-). \end{aligned}$$

*Where $\langle - \rangle$ denotes the bracket polynomial of the knot $K^+$ after performing an R1 move at the given loop, and $X(-)$ denotes the $X$ polynomial after performing an R1 move.*
*Thus the $X$ polynomial is invariant under R1 moves.*

□

The $X$ polynomial is equivalent to another invariant knot polynomial, known as the Jones polynomial. Now that we have constructed the $X$ polynomial, the Jones polynomial is easy to define by a simple change of variable:

**Definition (Jones polynomial)** : The Jones polynomial of a knot $K$, denoted $V(K)$ is a polynomial in the variable $t$, obtained from the $X$ polynomial via the transformation $A \to t^{-1/4}$.
That is:

$$V(K)(t) = X(K)(t^{-1/4}).$$

When viewed in this way, the Jones polynomial looks like nothing more than a simple variation on the $X$ polynomial (and hence the bracket polynomial). However, the importance of the Jones polynomial is really seen when we look at where its original definition came from.

The Jones polynomial was discovered by Vaughan Jones in 1984 [9], and is widely accepted to have reinvigorated the study of knot theory. Jones' original construction however, came from studying operator algebras, braid theory and statistical mechanics. The fact that it can so easily be defined as an invariant of knots, as we have done here, suggests a significant connection between all of these (previously, very different) branches of mathematics and physics.

After the discovery that the Jones polynomial could be defined using skein relations (as in the definition of the bracket polynomial), many people tried to come up with a general case. The result of this was the Homfly polynomial, named after six researches who all published their results in 1985 [2]. This is a general form of a linear skein relation for oriented knots. There are are a number of forms of the Homfly polynomial, but they only differ by a change of variables.

**Definition (Homfly polynomial)** : The Homfly polynomial of an oriented knot $K$, denoted $P(K)$ is a polynomial in two variables $v$ and $z$, defined by the following rules:

- the skein relation: (where the diagram below is a relation between the Homfly polynomial of diagrams that are different only in small neighbourhoods as shown)

- $P(0) = 1$,
  where (0) denotes the diagram of the unknot with no crossings.
- $P(K_1 \sqcup K_2) = (v^{-1} - v)z^{-1}P(K_1)P(K_2)$,
  where $(K_1 \sqcup K_2)$ denotes the disjoint union of diagrams $K_1$ and $K_2$.

The proof that this is a knot invariant is similar to that of the $X$ polynomial.

An example calculation can be seen in figure 2.8.

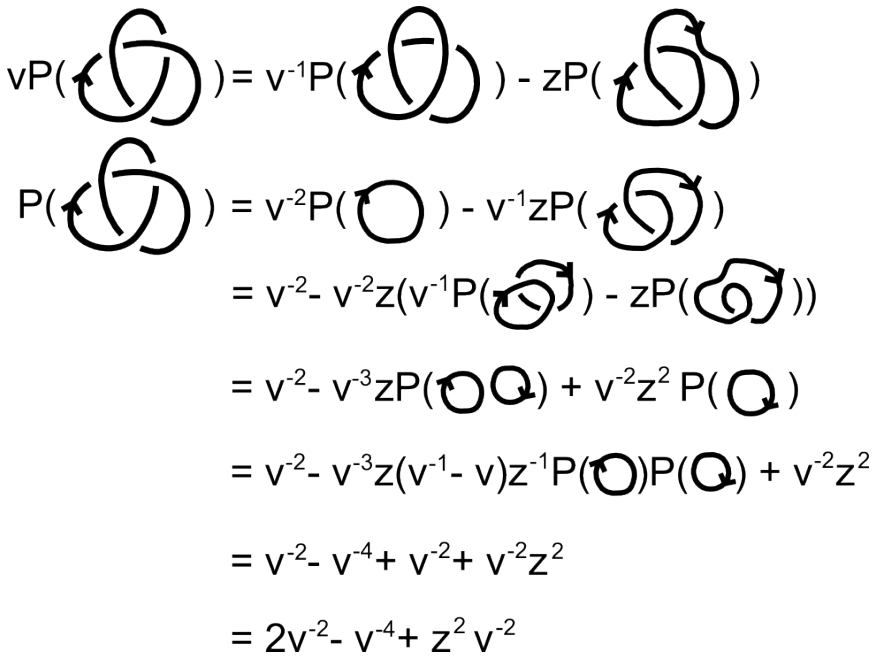



Figure 2.8: Example showing how to calculate the Homfly polynomial of the trefoil knot.

It is the Homfly polynomial that we will look at in the next section, when we discuss using a computer to calculate knot polynomials.

## 2.5 Summary

This section has given a brief overview of knot theory. We have seen some of the problems we must face when distinguishing even simple knots, and these problems are only exacerbated when dealing with larger knots. The next section looks at how we may use computers to help resolve some of these problems.

# Chapter 3

# Computing in knot theory

Now that we have a reasonable collection of knot invariants, we may feel that we are in a strong position to distinguish many different knots. However, although this is true, the actual task of calculating many of these invariants by hand, in particular knot polynomials, is a labourious task. This section explores how computers can be used to speed up the process. We begin by looking at the techniques used in [12]. Using these techniques, and some of our own, we then construct our own algorithm that could be implemented in a computer program to calculate the Homfly polynomial of a knot. Our aim in this chapter is to find a way that a computer may aid someone to calculate the Homfly polynomial of a knot diagram, which requires little work by the user.

## 3.1 Dowker notation

We first have to understand the limitations of computers. Although their abilities to calculate very specific tasks very quickly are far beyond that of a human being, they do not have the ability to think for themselves; it is up to the human programmer to provide complete instructions detailing exactly what the computer must do. We also require that the information input is in a format that the computer can do simple calculations, and answer true/false questions.

Furthermore, a computer is not able to 'see' a knot, (unless perhaps it was using some specialized graphics program, but these are in general very intensive on a computer's memory), instead they require lists of numbers and characters to be input in order to do any calculations on them. We therefore need a new notation for our knots that can be read by a computer.

Although there are a few ways this could be achieved, a popular choice of notation is the Dowker notation. It also has the direct benefit of being very easy to compute given a knot diagram:

**Construction** : Starting with a diagram of a knot, choose any point on the

knot to be the 'origin'. From the origin, choose a direction along the knot, and move around the diagram in that direction, the construction will be completed when you have made one full circuit of the diagram and reached the origin again.

As you move along the diagram, assign to each of the $x$ crossings a number, $1, \ldots, n$, (where $n = 2x$) in order each time you encounter it (thus assigning two different numbers to each crossing, one as you reach the crossing through an overcrossing arc, and one as an undercrossing arc).

Once this is done for all crossings (that is, you have reached the origin again), form ordered pairs $(U_x, O_x)$ for each crossing $x$, where $U_x$ is the number assigned to the undercrossing, and $O_x$ is the number assigned to the overcrossing.

The set of ordered pairs is the Dowker notation for the particular knot presentation.

Example: see figure 3.1.

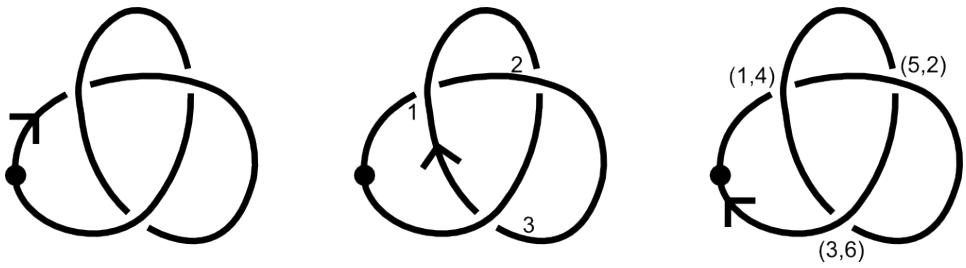



Figure 3.1: Example showing the calculation of a Dowker notation for the trefoil knot, $T$. We begin by choosing an origin (represented by the black dot), and a direction (shown by the arrow). As we move around the knot we assign numbers, in order, to the crossings (see centre diagram). When we arrive at each crossing for the second time, we form an ordered pair of the numbers assigned for the undercrossing arc and the overcrossing arc, where the first of the pair represents the undercrossing and the second the overcrossing (see diagram on right). The Dowker notation for this choice of origin and direction of the trefoil knot is therefore $T = \{(1,4),(5,2),(3,6)\}$.

A given knot may have many different Dowker notations, however once an origin and direction have been fixed, the knot has a unique Dowker notation.

It is worth noting here that it is also possible to draw a knot diagram given a Dowker notation, and so we do not lose any information by encoding a knot in this way. It is not true however that any Dowker notation represents a drawable knot diagram. For our purposes we need not worry about this, as we will be assuming that we begin with a knot diagram and construct the Dowker notation from it. However, this fact could be used

within a program that implements our algorithm, in order to check that any Dowker notation input into the program does in fact represent a drawable knot - if it does not then the user must have made an error in calculating the Dowker notation of their knot diagram. Here, we will simply focus on calculating the Homfly polynomial.

If we are now given a knot diagram we are able to calculate the Dowker notation, and therefore we have a way of storing the given knot onto a computer. The next section will look at how the Dowker notation can be implemented to calculate the Homfly polynomial using a computer algorithm.

## 3.2 Constructing the algorithm

For this section the reader may find it useful to refer to the completed algorithm in section 3.3, and the example of how to calculate the Homfly polynomial of a knot using diagrams in figure 2.8.

As we will see, following through all of the steps in the algorithm is a long winded process, and certainly not the quickest way for a person to calculate the Homfly polynomial. What we must remember is that this algorithm is intended to be implemented and run on a computer, and so would take no more than a few seconds to complete.

Our algorithm assumes that the user calculates the Dowker notation themself, and then inputs this into the algorithm (or ultimately, the computer program). We then also ask that the user calculate the crossing orientations (also called the crossing signs, see section 2.4) of each of the crossings. We will now look at how we may calculate the Homfly polynomial using just this information.

### 3.2.1 Skein relation

Throughout the calculation, we will have to apply the skein relation for the Homfly polynomial, as stated in section 2.4. This poses a few problems, as the skein relation may require us to change crossings, or remove them altogether.

Changing a crossing (from a positive crossing to a negative crossing, or vice versa) is achieved by simply swapping the numbers corresponding to the over and the under crossing components. That is, $(U_x, O_x)$ becomes $(O_x, U_x)$, and the crossing orientation is opposite.

The next problem we need to address is how we may alter the Dowker notation to denote removed crossings. A simple way to do this is by replacing the ordered pair $(U_x, O_x)$ at the corresponding crossing, with the unordered pair $\{U_x, O_x\}$. However, we need to think a little about how we may interpret this, in order to draw the resulting knot or link.

If we have a Dowker notation with removed crossings, it is possible to find the components of the knot or link. We start with some number in the Dowker notation, and continue through the proceeding numbers, until we reach a removed crossing - we then 'switch' to the other number in the removed pair, and continue from there. If we arrive at the highest number in the Dowker notation before returning back to the original number, we must return back to 1, and continue from there. This method continues until we return to the original number that we started with. The sequence of numbers that we have now found represents one component of the link. If we have exhausted all numbers in the notation, then the Dowker notation has only one component, and so must simply represent a knot, (if not then the notation must represent a link). If we have not come across all numbers yet, we then pick the lowest number not found in the first component, and continue in the same way to calculate the sequence of numbers that correspond to this next component of the link. This is done until all numbers have been exhausted, and thus, all components found. It is easiest to see how this works in an example, see figure 3.2.

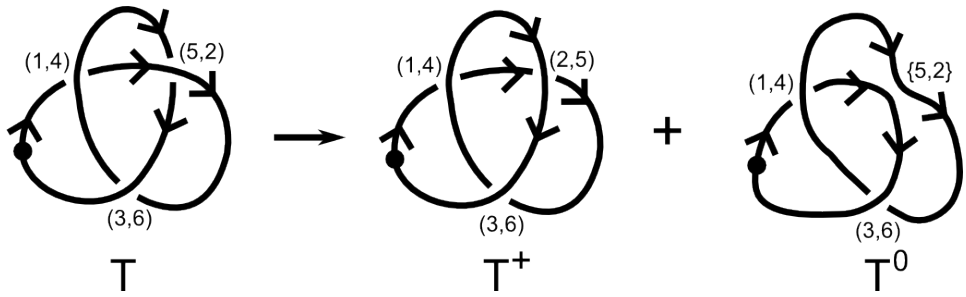
### 3.2.2 Loops

Using this notation for removed crossings, we can simplify our diagrams and speed up calculation of our algorithm, by checking for loops where we may apply the first Reidemeister move. It is possible to find situations where this is possible, from the Dowker notation, as any crossing of the form $(n, n+1)$ or $(n+1, n)$ must cause a loop, see figure 3.3. In fact, any crossing that is made up from consecutive numbers in the same component must cause a loop. By looking for these in the algorithm, we can simply remove crossings such as this, as explained in section 3.2.1; this is equivalent to performing the first Reidemeister move. As the Homfly polynomial is invariant under the first Reidemeister move, this will still give the correct value.

### 3.2.3 Trivial diagrams

We can simplify our calculation further, by checking for diagrams of the unknot - as we know that the unknot $U$ has Homfly polynomial $P(U) = 1$. So our first check is to see if we have a trivial diagram of the unknot. This is a simple matter of calculating the number of crossings, which may be done by calculating the number of ordered pairs in the Dowker notation. If there are no ordered pairs, then the Dowker notation must represent the trivial diagram of the unknot, and we are done.

We also know that any knot (but not link) with less than three crossings must be the unknot - this is easy to check. So, if our diagram only has one component (that is, it is a knot), then if the number of crossings is less than three, then the diagram represents the unknot, and so $P(K) = 1$.




$$T = \{(1,4),(5,2),(3,6)\}$$
$$T^+ = \{(1,4),(2,5),(3,6)\}$$
$$T^0 = \{(1,4),\{5,2\},(3,6)\}$$

Figure 3.2: Example showing how the skein relation is applied to the Dowker notation of the trefoil knot. The left hand diagram shows a Dowker notation for the trefoil knot $T$. Supposing we applied the skein relation for the Homfly polynomial on the crossing denoted $(5,2)$, we would get: $vP(T) = v^{-1}P(T^+) - zP(T^0)$, where $T^+$ is the same diagram but with the crossing $(5,2)$ changed to $(2,5)$, and $T^0$ is the diagram with the crossing removed, $\{5,2\}$, as shown. Notice that the crossing orientation of $(5,2)$ in $T$ was $-1$ (negative), and in $T^+$ the crossing orientation of $(2,5)$ is $+1$ (positive). The components of $T^0$ are calculated as such: starting with 1, the next number is 2, which is part of a removed crossing, so we 'switch' to the other number in the removed pair, 5. The next number is then 6, which returns back to our original number 1, and so we have found a component of the link. The number 3 has not been used yet, and so must be part of a different component. This second component is found as we move on to 4, then to 5, which is part of a removed crossing, so we switch to 2, which returns us to 3. All numbers are used, so we conclude that there are 2 components in this diagram, $\{1,6\}$ and $\{3,4\}$.

To check for other diagrams of the unknot, it is useful to use the following lemma.

**Lemma** : A descending or ascending diagram always represents a trivial knot [9].

Where a diagram is called ascending if it possible to choose a starting point and a direction around the knot, so that each crossing is first encountered on the under-crossing strand. Similarly, a descending diagram is where each crossing is first encountered on the over-crossing strand.

If our diagram has only one component (so that it is a knot), then it is easy to check for an ascending or descending diagram; in this case we have chosen to only look for ascending diagrams, which is explained further in section 3.2.5. To see if a knot diagram is ascending using only the Dowker

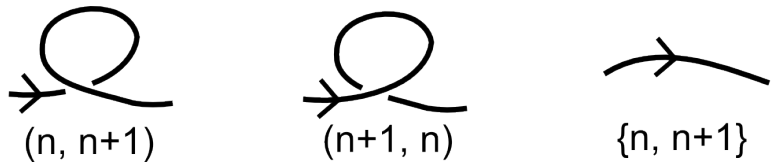



Figure 3.3: Diagrams showing how crossings of the form $(n, n+1)$, and $(n+1, n)$ form a loop, and can then be removed.

notation, we only need to look at each crossing (each ordered pair) in turn. If the undercrossing number is less than the overcrossing number for all crossings ($U_x < O_x$ for all crossings $x$), then our diagram is ascending. This is because - using the origin and direction as decided in the construction of the Dowker notation as our starting point - $U_x < O_x$ signifies that we encounter the crossing $x$ in the undercrossing component first. If this is true for all crossings, then this satisfies the definition of an ascending diagram, and so our diagram is trivial and has Homfly polynomial $P(K) = 1$.

We now have two methods for checking if we have a trivial diagram of a knot, but we still need a way to use these methods if we are looking at a link diagram.

### 3.2.4 Link diagrams

The first thing we need to be able to check, is whether components in a link diagram are disjoint, or 'linked together'. We can find out if a component is linked, by looking at where it crosses other components in the diagram. Assuming we have calculated which numbers in the Dowker notation correspond to which components (see section 3.2.1 for how this is done), then it is easy to know which crossings we should be looking at - simply discard any pairs that have both numbers in a single component, or both numbers in components other than the one we are looking at. The remaining crossings correspond to where this component crosses other components.

Now that we have chosen the relavent crossings, we can check if the component is disjoint from the rest of the diagram, by seeing if it sits 'above' or 'below' the rest of the diagram. This is acheived by looking at the numbers within these crossings, that are elements of the component we are interested in. If all of these numbers are undercrossing parts, then the component must sit 'below' the rest of the diagram; if the numbers are all overcrossing parts then the component sits 'above'. In either of these situations, the component we are looking at is disjoint from the rest of the diagram. If these numbers are a mixture of undercrossing and overcrossing parts, then we cannot deduce if it is disjoint.

If we establish that the components are disjoint, we apply the third

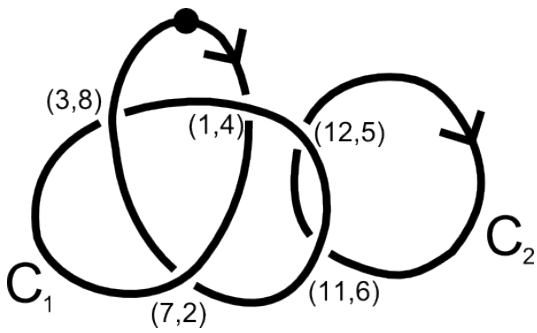
defining rule for the Homfly polynomial:

$$P(K_1 \sqcup K_2) = (v^{-1} - v)z^{-1}P(K_1)P(K_2),$$

(if the diagram can be separated into disjoint components $K_1, K_2$). If we cannot deduce if the components are disjoint, then we apply the skein relation again.

Our next problem however, is how to change the Dowker notation to distinguish these separate components $K_1, K_2$.

Supposing we find a disjoint component $C_i$ of a link diagram $L$, we need a way to distinguish this component, and the link diagram without this component (which we denote $L - C_i$). To do this we allow $C_i$ to be represented by the pairs of numbers in the Dowker notation of $L$ that are made up *only* from elements of $C_i$. The diagram $L - C_i$ is then made up from the Dowker notation of $L$ without *any* pairs involving $C_i$. See the example in figure 3.4.




$$L = \{(1,4),(7,2),(3,8),(12,5),(11,6),\{9,14\},\{10,13\}\}$$

$$\begin{aligned}
\text{Components: } C_1 &= 1,2,3,4,5,6,7,8 \\
C_2 &= 11,12. \\
\text{Separated Dowker notations: } C_1 &= \{(1,4),(7,2),(3,8)\} \\
C_2 &= \emptyset \\
L - C_1 &= \{\{9,14\},\{10,13\}\} \\
L - C_2 &= \{(1,4),(7,2),(3,8),\{9,14\},\{10,13\}\}.
\end{aligned}$$

Figure 3.4: Example showing how to separate the Dowker notations of disjoint components of a link $L$. The Dowker notation for the link $L$ is shown above. From this we may calculate the components $C_1, C_2$ as usual. The separated Dowker notations are then calculated by choosing the correct pairs of numbers from the Dowker notation of the link, as described in section 3.2.4, the results are shown above.

The only complication that we must face is that, by removing some crossings, we may remove some numbers in a sequence (for example, in figure 3.4 the remaining trefoil knot, component $C_1$, is described by numbers $\{1, 2, 3, 4, 7, 8\}$ - we are missing 5 and 6). This is overcome by simply ignoring any gaps in a sequence; if a number is not represented then simply move on to the next number until you arrive at a number that is represented, (starting again at 1 if you have reached the highest number in the notation), until you reach your original number again. There is no problem with doing this, as when separating components in this way, we only ever remove crossings between disjoint components and so the sequence of numbers should continue in the desired way.

### 3.2.5 Does the algorithm finish?

We now have all of the required techniques to make up the algorithm, however we need to ensure that the algorithm eventually stops and does not get stuck in a loop.

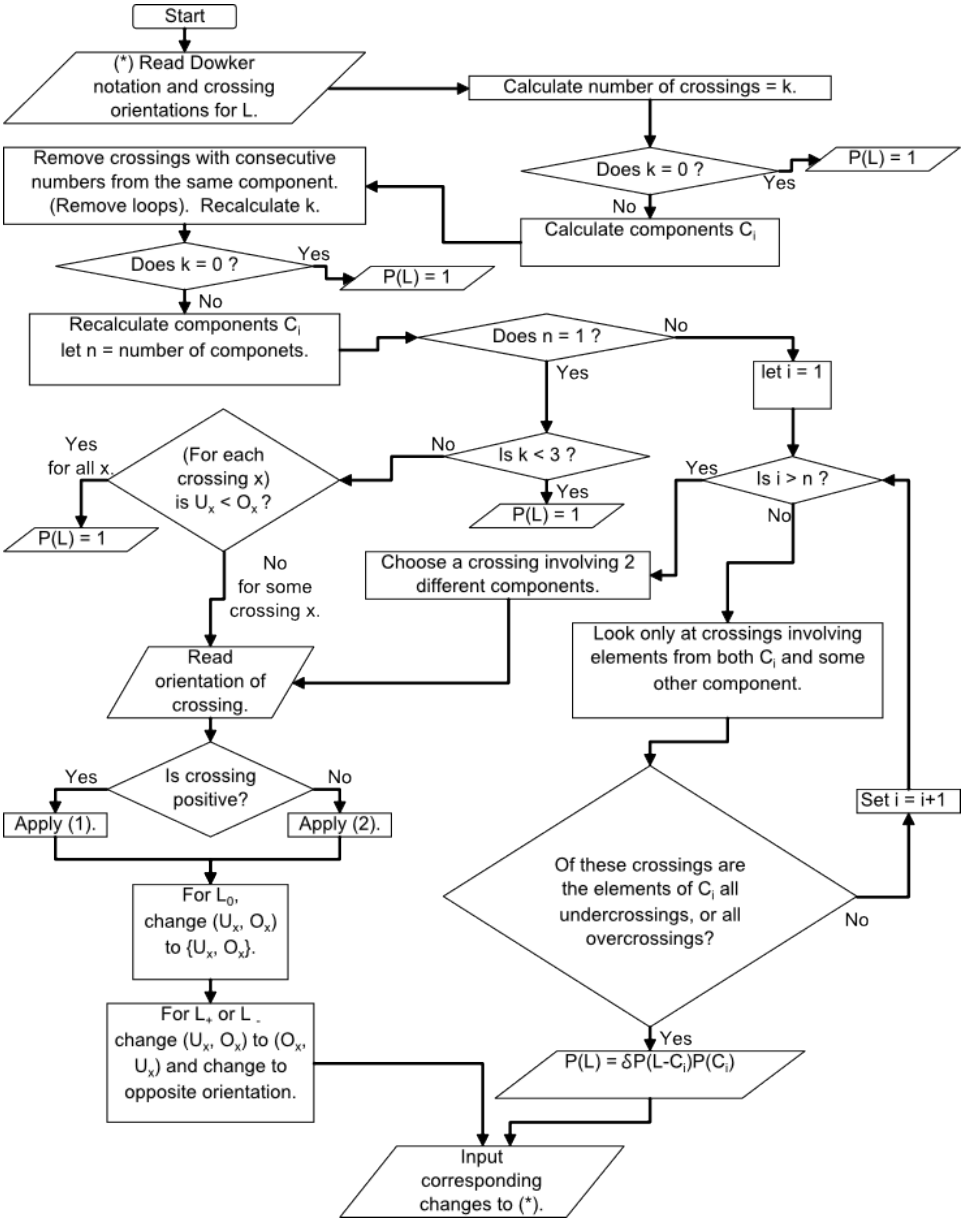
Notice that when calculating the Homfly polynomial, and applying the skein relation, we must remove crossings and swap crossing orientations. Although removing crossings will always eventually lead to trivial components, swapping the crossing orientations does not guarantee that we simplify the diagram.

To ensure that we always apply the skein relations in a way that will simplify our diagram, we aim to change the crossings so that they will lead to an ascending or descending diagram. In this way the diagram will always be simplified, ultimately resulting in trivial components. We have chosen here to aim for ascending diagrams, which may be acheived by only ever applying the skein relation to a crossing $x$, that has $U_x > O_x$ (or to crossings between separate components). In this way, we will always end up with a diagram where all crossings satisfy $U_x < O_x$, and is therefore ascending. We could equally have chosen to apply the skein relations in a way to ensure that we had descending diagrams.

By constructing the algorithm in this way, although it may require that we run through the algorithm multiple times (for each new knot or link that the skein relation produces), we will always be simplifying our original diagram, and hence the algorithm will eventually finish.

We will now see the completed algorithm in the form of a flow chart, and run through an example calculation.

## 3.3 Algorithm for calculating the Homfly polynomial




Where:

$$(1)\ P(L_+) = v^2 P(L_-) + zvP(L_0)$$
$$(2)\ P(L_-) = v^{-2} P(L_+) - zv^{-1} P(L_0)$$
$$\delta = (v^{-1} - v)z^{-1}.$$

Explanations of the steps are found in section 3.2.

## 3.4 An example calculation

In this section we see how the algorithm would work for the trefoil knot. Another example calculation can be found in appendix C, where we use the algorithm to calculate the Homfly polynomial of the figure eight knot. We have not attempted to lay these calculations out exactly how a computer program would calculate them, as this would not only depend greatly on the programming language used, but would almost certainly be harder to follow. Instead we lay them out within a table: in the left column, which step in the algorithm we have reached; in the centre column, how this step may be evaulated; in the right hand column, the result of this step. In this way it should be relatively straight forward to follow the calculation through the flowchart in section 3.3.

We will use the Dowker notation for the trefoil knot that we calculated in figure 3.1: $T = \{(1,4),(5,2),(3,6)\}$. We may also calculate that this has respective crossing orientations $\{-1,-1,-1\}$.

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $T = \{(1,4),(5,2),(3,6)\}$, $\{-1,-1,-1\}$ |
| Calculate number of crossings $= k$. | 3 crossings | $k = 3$ |
| Does $k = 0$? | $k = 3$ | No. |
| Calculate components. | $1 \to 2 \to 3 \to 4 \to 5 \to 6$. | Components: $\{1,2,3,4,5,6\}$. |
| Remove crossings with consecutive numbers from the same component. | No loops. | $k = 3$ |
| Does $k = 0$? | $k = 3$ | No. |
| Recalculate components. let $n =$ number of components. | $1 \to 2 \to 3 \to 4 \to 5 \to 6$. | $n = 1$, components: $\{1,2,3,4,5,6\}$. |
| Does $n = 1$? | $n = 1$. | Yes. |
| Is $k < 3$? | $k = 3$ | No. |
| For each crossing $x$ is $U_x < O_x$? | Crossing 1: $1 < 4$ Yes. Crossing 2: $5 > 2$ No. | No for $x = 2$. |
| Read orientation of crossing. | Crossing 2 is negative. | $-1$ |
| Is crossing positive? | Negative. | No. |
| Apply (2). | | $P(T) = v^{-2}P(L_+) - zv^{-1}P(L_0)$ |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| For $L_0$ change $(U_x, O_x)$ to $\{U_x, O_x\}$ | $(5,2) \to \{5,2\}$. | $L_0 = \{(1,4),\{5,2\},(3,6)\}$ Orientation: $\{-1,-1\}$. |
| For $L_+$, change $(U_x, O_x)$ to $(O_x, U_x)$ and change orientation. | $(5,2) \to (2,5)$ $-1 \to +1$ | $L_+ = \{(1,4),(2,5),(3,6)\}$ Orientation: $\{-1,+1,-1\}$. |

We now have that the Homfly polynomial of the trefoil knot is:

$$P(T) = v^{-2}P(L_+) - zv^{-1}P(L_0),$$

and we have the Dowker notation of $L_+$ and $L_0$. The program must now run through the algorithm again for both $L_+$ and $L_0$, the order this is done does not matter - the order here was chosen simply to make the calculation easier to follow. We will first evaluate $L_+$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{(1,4),(2,5),(3,6)\}$, $\{-1,+1,-1\}$ |
| Calculate number of crossings $= k$ | 3 crossings | $k = 3$. |
| Does $k = 0$ ? | $k = 3$ | No. |
| Calculate components. | $1 \to 2 \to 3 \to$ $4 \to 5 \to 6$. | Components: $\{1,2,3,4,5,6\}$. |
| Remove loops. | No loops. | $k = 3$ |
| Does $k = 0$? | $k = 3$ | No. |
| Recalculate components let $n =$ number of components. | $1 \to 2 \to 3 \to$ $4 \to 5 \to 6$ | $n = 1$, components: $\{1,2,3,4,5,6\}$. |
| Does $n = 1$? | $n = 1$. | Yes. |
| Is $k < 3$? | $k = 3$ | No. |
| For each crossing $x$ is $U_x < O_x$? | Crossing 1: $1 < 4$ Yes. Crossing 2: $2 < 5$ Yes. Crossing 3: $3 < 6$ Yes. | Yes for all $x$. |
| $P(L) = 1$ | | $P(L) = 1$. |

The Homfly polynomial of the trefoil knot is now:

$$P(T) = v^{-2}(1) - zv^{-1}P(L_0).$$

We now look at the algorithm computing the Homfly polynomial of $(L_0)$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation | | $L = \{(1,4),\{5,2\},(3,6)\}$, $\{-1,-1\}$. |
| Calculate number of crossings $= k$ | 2 crossings | $k = 2$. |
| Does $k = 0$? | $k = 2$ | No. |
| Calculate components. | $1 \to (5) \to 6$, $3 \to 4 \to (2)$. | Components: $C_1 = \{1,6\}$, $C_2 = \{3,4\}$. |
| Remove loops. | No loops. | $k = 2$ |
| Does $k = 0$? | $k = 2$ | No. |
| Recalculate components, let $n =$ number of components. | $1 \to (5) \to 6$, $3 \to 4 \to (2)$. | $n = 2$, components: $C_1 = \{1,6\}$, $C_2 = \{3,4\}$. |
| Does $n = 1$? | $n = 2$. | No. |
| Let $i = 1$. | | $i = 1$. |
| Is $i > n$? | $i = 1$, $n = 2$. | No. |
| Look only at crossings involving elements from both $C_1$ and some other component. | $1 \in C_1$ and $4 \in C_2$, so look at $(1,4)$, $3 \in C_2$ and $6 \in C_1$, so look at $(3,6)$. | Look at crossings: $\{(1,4),(3,6)\}$. |
| Of these, are all elements of $C_1$ under, or all over? | 1 is under, 6 is over. | No. |
| Set $i = i + 1$. | Set $i = 1 + 1$ | $i = 2$ |
| Is $i > n$? | $i = 2$, $n = 2$ | No. |
| Look only at crossings involving elements from both $C_2$ and some other component. | $1 \in C_1$ and $4 \in C_2$, so look at $(1,4)$, $3 \in C_2$ and $6 \in C_1$, so look at $(3,6)$. | Look at crossings: $\{(1,4),(3,6)\}$. |
| Of these, are all elements of $C_2$ under, or all over? | 3 is under, 4 is over. | No. |
| Set $i = i + 1$. | Set $i = 2 + 1$ | $i = 3$ |
| Is $i > n$? | $i = 3$, $n = 2$. | Yes. |
| Choose a crossing involving two different components. | First crossing: $(1,4)$ involves $C_1$ and $C_2$. | $(1,4)$. |
| Read orientation. | (1,4) is negative. | $-1$ |
| Is crossing positive? | Negative. | No. |
| Apply (2). | | $P(L) = v^{-2}P(L_+) - zv^{-1}P(L_0)$ |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| For $L_0$, change $(U_x, O_x)$ to $\{U_x, O_x\}$ | $(1,4) \to \{1,4\}$ | $L_0 = \{\{1,4\},\{5,2\},(3,6)\}$, Orientation: $\{-1\}$. |
| For $L_+$, change $(U_x, O_x)$ to $(O_x, U_x)$ and change orientation. | $(1,4) \to (4,1)$ $-1 \to +1$ | $L_+ = \{(4,1),\{5,2\}(3,6)\}$ $\{+1,-1\}$ |

The Homfly polynomial of the trefoil knot is now:

$$P(T) = v^{-2} - zv^{-1}(v^{-2}P(L_+) - zv^{-1}P(L_0)),$$

where we have the Dowker notation for these new $L_+$ and $L_0$ given in the table above, and must now calculate the Homfly polynomial of these using the algorithm. We start with $L_0$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{\{1,4\},\{5,2\},(3,6)\}$. $\{-1\}$ |
| Calculate number of crossings $= k$ | 1 crossing. | $k = 1$. |
| Does $k = 0$? | $k = 1$ | No. |
| Calculate components, | $3 \to 6$. | Components: $C_1 = \{3,6\}$ |
| Remove loops. Recalculate $k$. | $3 \to 6$ Remove (3,6) | $L = \{\{1,4\},\{5,2\},\{3,6\}\}$ $k = 0$. |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$. |

We now have that the Homfly polynomial of the trefoil knot is:

$$P(T) = v^{-2} - zv^{-1}(v^{-2}P(L_+) - zv^{-1}(1)).$$

We now calculate the Homfly polynomial of this $L_+$ :

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{(4,1),\{5,2\},(3,6)\}$. { +1, -1 } |
| Calculate number of crossings $= k$ | 2 crossings. | $k = 2$. |
| Does $k = 0$? | $k = 2$ | No. |
| Calculate components. | $1 \to (5) \to 6$, $3 \to 4 \to (2)$. | Components: $C_1 = \{1,6\}$ $C_2 = \{3,4\}$ |
| Remove loops. | No loops. | $k = 2$ |
| Does $k = 0$? | $k = 2$ | No. |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Recalculate components, let $n =$ number of components. | $1 \to (5) \to 6$, $3 \to 4 \to (2)$. | $n = 2$, components: $C_1 = \{1, 6\}$ $C_2 = \{3, 4\}$ |
| Does $n = 1$? | $n = 2$ | No. |
| Let $i = 1$ | | $i = 1$ |
| Is $i > n$? | $i = 1$, $n = 2$ | No. |
| Look only at crossings involving elements from both $C_1$ and some other component. | $1 \in C_1$, and $4 \in C_2$, so look at $(4, 1)$. $3 \in C_2$ and $6 \in C_1$, so look at $(3, 6)$. | Look at crossings: $\{(4, 1), (3, 6)\}$. |
| Of these, are all elements of $C_1$ under, or all over? | 1 is over, 6 is over. | Yes. |
| $P(L) = \delta P(L - C_1)P(C_1)$ | $L - C_1 = \{\{5, 2\}\}$, $C_1 = \emptyset$ | $P(L) = \delta P(L - C_1)P(C_1)$ |

The Homfly polynomial of the trefoil knot is now:

$$P(T) = v^{-2} - zv^{-1}(v^{-2}(\delta P(L - C_1)P(C_1)) - zv^{-1}).$$

We now need to calculate the Homfly polynomial of these $(L - C_1)$ and $C_1$. We start with $C_1$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \emptyset$ |
| Calculate number of crossings $= k$. | No crossings. | $k = 0$ |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

The Homfly polynomial of the trefoil knot is now:

$$P(T) = v^{-2} - zv^{-1}(v^{-2}(\delta P(L - C_1)(1)) - zv^{-1}).$$

We now calculate the Homfly polynomial of $L - C_1$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{\{5, 2\}\}$ |
| Calculate number of crossings $= k$ | No crossings. | $k = 0$ |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

So the Homfly polynomial of the trefoil knot is:

$$\begin{aligned} P(T) &= v^{-2} - zv^{-1}(v^{-2}(\delta(1)) - zv^{-1}) \\ &= v^{-2} - zv^{-3}(v^{-1} - v)z^{-1} + z^2v^{-2} \\ &= v^{-2} - v^{-4} + v^{-2} + z^2v^{-2} \\ &= 2v^{-2} - v^{-4} + z^2v^{-2}. \end{aligned}$$

This is indeed the correct answer for the Homfly polynomial (see figure 2.8).

If this algorithm were to be implemented by a computer program it would only require that the user calculates the Dowker notation of their given knot diagram, and the crossing orientations. This is a much quicker, and easier task than calculating the Homfly polynomial by hand.

## 3.5 Summary

We have now seen some techniques that allow us to use computers to aid us in storing information and doing calculations on knots. We have also used these techniques to create an algorithm that may be used to construct a computer program that calulates the Homfly polynomial of a knot, given the Dowker notation and crossing orientations. This chapter has shown some of the difficulties that we come across when trying to transfer mathematical ideas into a computer program, but also some of the benefits that we ultimately gain. Using the techniques discussed here, the algorithm could easily be modified to calculate any other invariant that is constructed from similar skein relations.

The next chapter will move on to look in more detail at how knot theory relates to a number of different areas.

# Chapter 4

# Knots, physics and algebra

We now have some knot theory under our belts and an appreciation for the difficulties surrounding the subject, but we may still ask: why do we bother? In this chapter we will show a few relationships between knot theory and various other fields of mathematics and physics. We have already seen one such example; the Jones polynomial, giving us a link to operator algebras and statistical models. Here, we will look at a relationship to quantum mechanics, the Yang-Baxter equation, and to algebraic structures, in particular the quantum group $SL(2)_q$. The content of this chapter is primarily based on [8], [11], [13], and [14].

## 4.1 Diagrammatic tensors

We begin by looking at representing matrix algebra through the use of diagrams.

A matrix $M = (M^i_j)$ with indices $i$ and $j$ can be represented as a box with an upper strand corresponding to the upper index $i$, and a lower strand corresponding to the lower index $j$ (see figure 4.1).

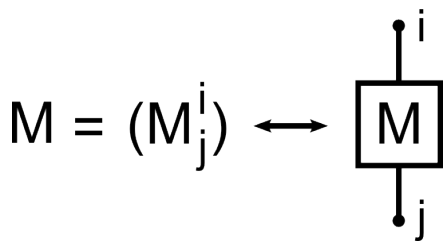



Figure 4.1: A diagrammatic representation of a matrix $M$.

We can now very neatly interpret matrix multiplication using this diagramatic language. The product of two matrices $M$ and $N$ may be written: $(MN)^i_j = \sum_k M^i_k N^k_j = M^i_k N^k_j$ (using Einstein summation convention). We

then represent this in our diagrammatic language by allowing any line that connects two index strands to denote a summation over an index value for the line. Multiplication of two matrices is then represented by joining up the corresponding index strands with a line (see figure 4.2).

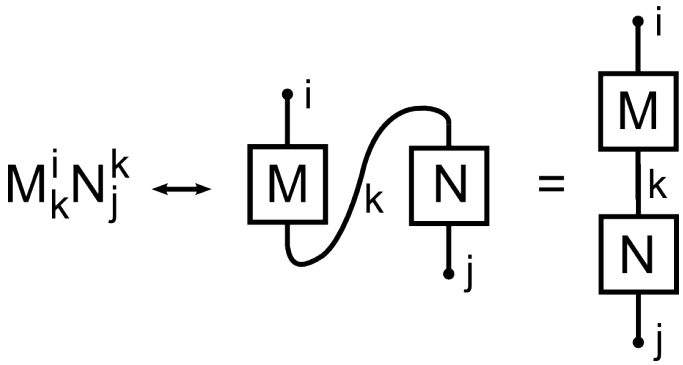



Figure 4.2: A diagram of matrix multiplication.

In the same way, any strand connecting two indices may denote a Kronecker delta. An example of this can be seen in figure 4.3, where a strand with upper index $k$ and lower index $l$ represents

$$\delta^k_l = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

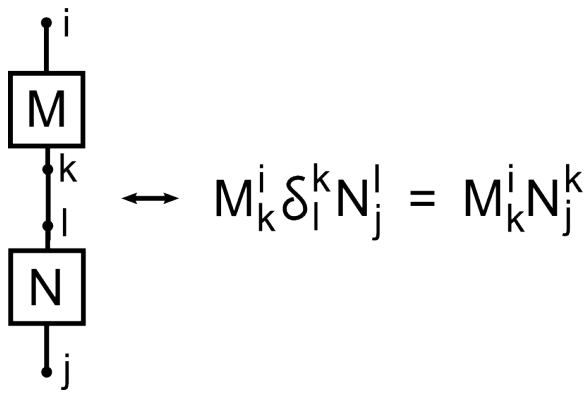



Figure 4.3: Diagram demonstrating the use of the Kronecker delta.

More generally, we may denote a tensor like object with multiple upper and lower strands, ordered from left to right (see figure 4.4).

We now use this idea of diagrammatic tensors to represent a knot diagram as a contracted tensor.

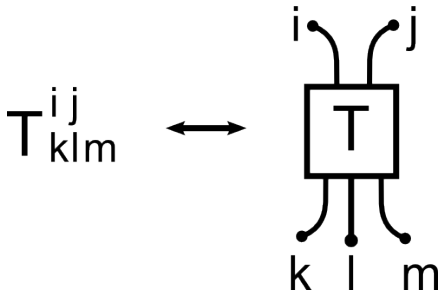



Figure 4.4: A diagrammatic representation of a tensor $T^{ij}_{klm}$.

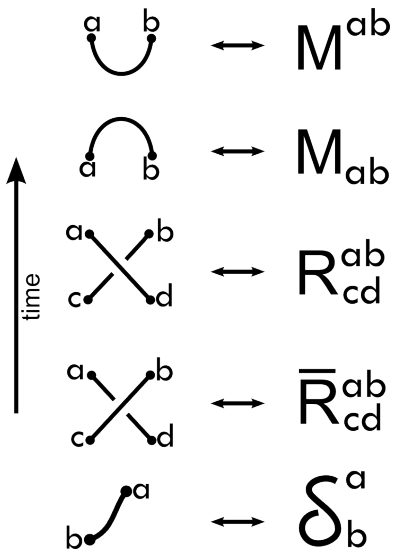


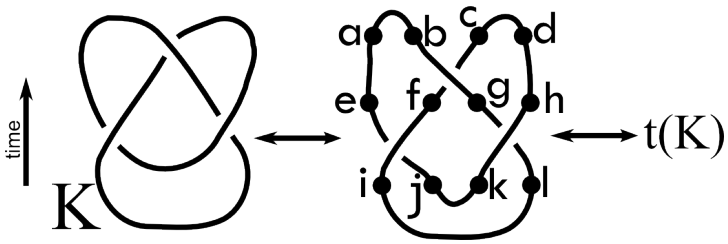

Figure 4.5: Assignment of matrices to knot diagrams.

## 4.2 Knots as tensor diagrams

There are a number of different ways we could interpret a knot as a tensor diagram. Here we will look at one particular case that leads naturally to the Yang-Baxter equation and to the notion of a quantum group, both of which we will discuss in more detail later. This construction can in fact be motivated by the idea of an amplitude in quantum mechanics, we will first describe the construction before discussing this motivation.

We begin by arranging our knot diagrams with respect to a given direction (referred to as the 'time' direction). It is then possible to represent any knot diagram so that it is decomposed into maxima, minima, two possible types of crossing, and curves with no critical points in relation to the time direction. To each of these possibilities we assign a matrix, as shown in figure 4.5.

Once we have assigned matrices to our knot diagram, we once again




$$t(K) = M_{ab} M_{cd} \delta^a_e \delta^d_h R^{bc}_{fg} \bar{R}^{ef}_{ij} \bar{R}^{gh}_{kl} M^{jk} M^{il}.$$

Figure 4.6: Example of tensor contraction for a diagram of the trefoil knot.

allow connected strands to represent summation over an index value for the line. As a knot is a closed loop, the diagram will have no free strands, and therefore no free indices, and so any knot diagram $K$ is mapped to a specific contracted tensor, which we will denote $t(K)$. An example is shown in figure 4.6.

In some ways, the idea of assigning a specific direction to a knot diagram seems perfectly natural if we consider the construction of tensor diagrams in the previous section; we need to not only distinguish upper and lower indices, but also maintain an order (left to right) of these indices. However, it is when considering this direction as 'time' that we may understand a relation to quantum mechanics.

If we consider a knot diagram sitting in a spacetime plane (with time represented vertically and space horizontally), we may interpret each section of the knot diagram as a quantum mechanical event. Minima may represent the creation of two 'particles'; maxima the annihilation of two particles; and crossings show some form of interaction. This way a knot diagram may represent a vacuum to vacuum process. In quantum mechanics, "the probability amplitude for the concatenation of processes is obtained by summing the products of the amplitude of the intermediate configurations in the process over all possible internal configurations [13]". So, $t(K)$ may represent a vacuum to vacuum expectation for the process shown by the knot diagram.

This relation to quantum mechanics is an interesting area of study in itself, discussed in more detail in [8] and [13]. Our next step however is to look at how the topology of knot diagrams may relate to the algebra of the corresponding matrices.

## 4.3 Topological invariance - the Yang-Baxter equation

It is now interesting to consider how demanding invariance under topological moves may affect the matrices that we assign to the knot diagrams. If we wish to look at regular isotopies of knot diagrams (that is the equivalence

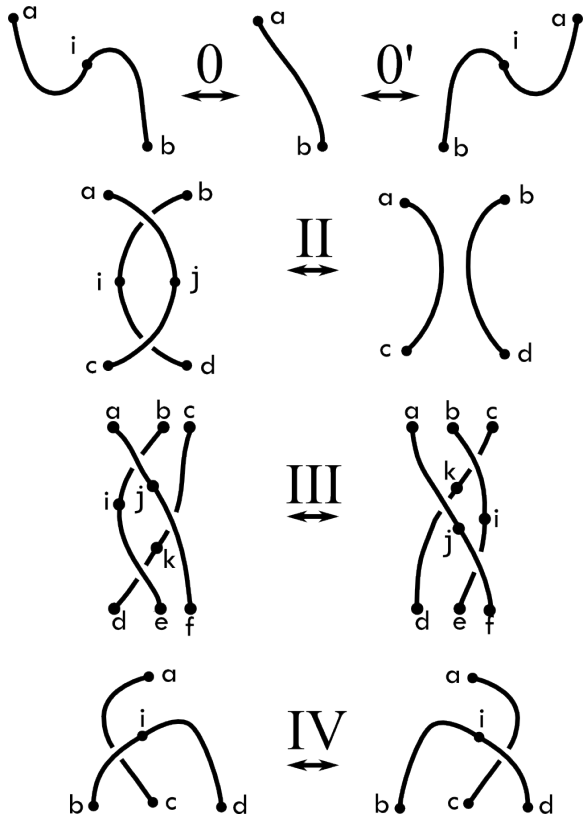



Figure 4.7: The augmented Reidemeister moves for regular isotopy of knot diagrams arranged with respect to 'time' (running vertically).

relations generated by the second and third Reidemeister moves), we must use an 'augmented list of Reidemeister moves' to account for the fact that the diagrams are arranged with respect to the time direction. A list of the augmented Reidemesiter moves (as used in [11]) is shown in figure 4.7. We then demand that the tensor contraction $t(K)$ is invariant under these regular isotopies, and obtain a number of constraints for the given matrices $M$, $R$ and $\bar{R}$.

The corresponding algebraic equations to these topological moves are thus:

$$\begin{aligned}
\textbf{0 and 0':}\ & M^{ai}M_{ib} = \delta^a_b = M_{bi}M^{ia}, \\
\textbf{II:}\ & R^{ab}_{ij}\bar{R}^{ij}_{cd} = \delta^a_c\delta^b_d, \\
\textbf{III:}\ & R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} = R^{bc}_{ki}R^{ak}_{dj}R^{ji}_{ef}, \\
\textbf{IV:}\ & \bar{R}^{ai}_{bc}M_{id} = R^{ia}_{cd}M_{bi}.
\end{aligned}$$

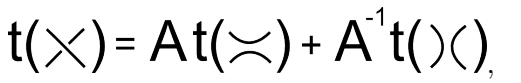
We therefore establish that these may be satisfied if $R$ and $\bar{R}$ are inverse

matrices (from II), and if $M_{ab}$ and $M^{ab}$ are inverse matrices (from 0 and 0'). Perhaps the most interesting of these equations however, is given by the third Reidemeister move (III):

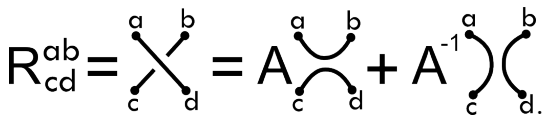
$$R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ki} R^{ak}_{dj} R^{ji}_{ef}.$$

This is in fact the **Yang-Baxter equation**, one of the principal laws governing the evolution of statistical models [2], it was first established in regard to problems of exactly solved models [11]. The Yang-Baxter equation now has many applications to theoretical physics and mathematical physics, in particular to integrable systems and representation theory [15], [16] - we will return to the Yang-Baxter equation in section 5.4. For now, we hope to appreciate how remarkable it is to find such a non-trivial equation manifest in what is a comparatively simple topological move.

We may establish further constraints on the $R$ and $M$ matrices if we wish our tensor $t(K)$ to satisfy the equations for the bracket polynomial (as defined in section 2.4). We therefore want $t(K)$ to satisfy:



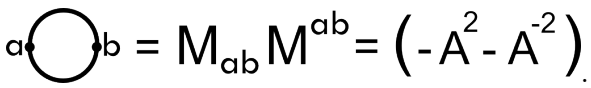

which may be achieved by taking the $R$-matrix to be:




An equivalent diagram may also be found for $\bar{R}$. Assigning matrices as before, we establish the algebraic equations:

$$R^{ab}_{cd} = AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d$$
$$\bar{R}^{ab}_{cd} = A^{-1}M^{ab}M_{cd} + A\delta^a_c\delta^b_d.$$

To formulate the correct loop value for the bracket polynomial we also require:




To create a model for the bracket polynomial, we therefore define the $R$-matrix and its inverse $\bar{R}$ as above, and then we only need a pair of inverse matrices $M^{ab}$ and $M_{ab}$ that must satisfy the loop equation. This way we also find a solution to the Yang-Baxter equation due to the invariance of the bracket polynomial under the third Reidemeister move.

In the next section we will look further at these matrices $M^{ab}$ and $M_{ab}$.

## 4.4 The quantum group $SL(2)_q$

We now look at one particular set of solutions to these constraints.

If we assume that $M^{ab} = M_{ab}$, then we restrict ourselves to looking for a single matrix $M$ that satisfies $M^{ai}M_{ib} = M^2 = \delta^a_b = I$ (by condition 0 of the 'augmented Reidemeister moves' shown in figure 4.7), which must also give the correct loop value: $M_{ab}M^{ab} = \sum_{a,b}(M_{ab})^2 = -A^2 - A^{-2}$.

One way we may satisfy these conditions is by taking the matrix $M$ to be:

$$M = \begin{bmatrix} 0 & \sqrt{-1}A \\ -\sqrt{-1}A^{-1} & 0 \end{bmatrix}.$$

Indeed, we have

$$M^{ai}M_{ib} = I$$

$$\text{and } \sum_{a,b}(M_{ab})^2 = (\sqrt{-1}A)^2 + (-\sqrt{-1}A^{-1})^2 = -A^2 - A^{-2}.$$

From this it is then possible to write down the $R$-matrix, using the equation found earlier:

$$R^{ab}_{cd} = AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d,$$

that is,

$$R = AM \otimes M + A^{-1}I \otimes I.$$

So we may write,

$$R = \begin{bmatrix} 0 + A^{-1} & 0 + 0 & 0 + 0 & 0 + 0 \\ 0 + 0 & A(\sqrt{-1}A)(\sqrt{-1}A) + A^{-1} & A(\sqrt{-1}A)(-\sqrt{-1}A^{-1}) + 0 & 0 + 0 \\ 0 + 0 & A(-\sqrt{-1}A^{-1})(\sqrt{-1}A) + 0 & A(-\sqrt{-1}A^{-1})(-\sqrt{-1}A^{-1}) + A^{-1} & 0 + 0 \\ 0 + 0 & 0 + 0 & 0 + 0 & 0 + A^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & -A^3 + A^{-1} & A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & A^{-1} \end{bmatrix}.$$

This choice of $M$ is of particular interest, as it allows for the construction of an algebraic structure known as the quantum group $SL(2)_q$. To see why this is, we begin by rewriting $M$ as:

$$M = \sqrt{-1}\tilde{\epsilon}, \text{ where } \tilde{\epsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}.$$

We first concentrate on looking at the special case of the bracket with $A = 1$. So we have:

$$M = \sqrt{-1}\epsilon, \text{ where } \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We also look at the lemma:

**Lemma** : Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of commuting, associative scalars. Then,

$$P\epsilon P^T = det(P)\epsilon$$

where $P^T$ denotes the transpose of $P$. [13]

**Proof** : The proof is a simple matter of evaluating the left hand side of the the equation:

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b & d \\ -a & -c \end{bmatrix} \\
&= \begin{bmatrix} ab - ba & ad - bc \\ cb - da & cd - dc \end{bmatrix} \\
&= \begin{bmatrix} 0 & ad - bc \\ -ad + bc & 0 \end{bmatrix} \\
&= (ad - bc) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
&= det(P)\epsilon.
\end{aligned}$$

□

The algebraic structure that leaves this epsilon invariant is defined to be the set of $2 \times 2$ matrices $P$ satisfying:

$$P\epsilon P^T = \epsilon.$$

Now by the lemma, this structure is the set of $(2\times2)$ matrices with $det(P) = 1$. This set of matrices is known as $SL(2)$, the special linear group of degree two.

The calculus formed from the bracket equation for this specific case of the $M$ matrix corresponds directly to what is known as *binor calculus*, which was originally established by Roger Penrose in order to investigate the foundations of spin, angular momentum and the structure of space-time [14].

What is of particular interest about $SL(2)$ from an algebraic point of view however, is the fact that it forms a well defined group under matrix multiplication.

We now think of $\tilde{\epsilon}$ as being a form of deformation of $\epsilon$ (that is, $\epsilon$ is simply a special case of $\tilde{\epsilon}$ where we take $A = 1$). If we obtain the structure of a group by looking at the set of matrices that leave $\epsilon$ invariant, we may ask what structure leaves the deformed epsilon, $\tilde{\epsilon}$, invariant? We are then asking, what sorts of matrices $P$ will satisfy

$$P\tilde{\epsilon}P^T = \tilde{\epsilon} \text{ and } P^T\tilde{\epsilon}P = \tilde{\epsilon} \text{ ?}$$

Therefore, suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a$, $b$, $c$, $d$ are elements of an associative but not necessarily commutative ring. Assuming that $A$ commutes with the entries $a$, $b$, $c$, $d$, then we obtain

$$\begin{aligned} P\tilde{\epsilon}P^T &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ &= \begin{pmatrix} -A^{-1}b & Aa \\ -A^{-1}d & Ac \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ &= \begin{pmatrix} -A^{-1}ba + Aab & -A^{-1}bc + Aad \\ -A^{-1}da + Acb & -A^{-1}dc + Acd \end{pmatrix}. \end{aligned}$$

In the same way, $P^T\tilde{\epsilon}P = \begin{pmatrix} -A^{-1}ca + Aac & -A^{-1}cb + Aad \\ -A^{-1}da + Abc & -A^{-1}db + Abd \end{pmatrix}$.

Equating these to $\tilde{\epsilon}$ and making the change of variable $A = \sqrt{q}$ , then the above equations are equivalent to the set of equations:

$$\begin{cases} ba = qab \\ db = qbd \\ dc = qcd \\ ca = qac \\ bc = cb \\ ad - da = (q^{-1} - q)bc \\ ad - q^{-1}bc = 1. \end{cases}$$

It is now clear from this set of equations that to obtain a non-trivial result we must indeed take the elements $a$, $b$, $c$, $d$ to be from a non-commutative ring. However, it is of interest to note that even in this more general case, when $q = 1$ these equations demand that the elements of $P$ must commute amongst themselves, and that $ad - bc = 1$. In other words, the particular value $q = 1$ (equivalently, $A = 1$) means that the set of equations once again defines $SL(2)$.

This set of equations therefore defines some kind of generalization of $SL(2)$. It in fact defines the algebraic structure known as $SL(2)_q$, which is a particular quasitriangular Hopf algebra (also known as a **quantum group** [8]).

## 4.5 Summary

In this chapter we have seen how we may represent matrix multiplication in a diagrammatic way. Using this diagrammatic language we have then looked at knot diagrams, and seen how the topological constraints of these diagrams directly correspond to algebraic constraints. In this context the

Yang-Baxter equation may be viewed as equivalent to the topological move known as the third Reidemeister move.

Looking at a specific case of the matrices that we relate to our diagrams, we have seen how the topology of knot diagrams and the construction of the bracket polynomial, may lead us to a set of equations that define a special type of Hopf algebra, known as a quantum group. In order to fully describe this set of equations as a quantum group we must define a few other things, namely: comultiplication, the co-unit and the antipode. To do this we must first look at the definition and construction of a Hopf Algebra in a little more detail, which we will do in the next chapter.

# Chapter 5

# What is a quantum group?

At the end of the previous chapter we used the topology of knot diagrams to construct a particular type of Hopf algebra. In this chapter we will first define what is meant by a Hopf algebra, and verify that our construction does indeed satisfy this definition. We will then look at quasitriangular Hopf algebras, which we call a quantum group, and see how this relates to the structure found in the previous chapter. This chapter is based on a number of texts on algebraic structures and relations to knot theory, in particular [13], [17], [18], [19], and [20].

## 5.1 Hopf algebra

A Hopf algebra is defined to be a bialgebra with a map known as the antipode [17].

We first review the definition of an algebra:

**Definition** : A linear algebra $A$ consists of a collection $v_1, v_2, \cdots \in V$ called vectors (where $V$ is a vector space), and a field $F$ with elements $f_1, f_2, \ldots$ (equivalently we may replace the field $F$ with a commutative ring, $K$ say). With three operations:

- vector addition $+$
- scalar multiplication $\circ$
- vector multiplication $\square$.

These must satisfy:

- Closure, associativity, identity and bilinearity hold for the vector space $V$ defined by the vector addition $+$ and scalar multiplication $\circ$.
- Closure and bilinearity must also hold for the vector multiplication $\square$, that is:

- $v_1, v_2 \epsilon V \Rightarrow v_1 \Box v_2 \in V$ (closure).
- $(v_1 + v_2)\Box v_3 = v_1 \Box v_3 + v_2 \Box v_3$,
  $v_1 \Box (v_2 + v_3) = v_1 \Box v_2 + v_1 \Box v_3$ (bilinearity) [18].

A bialgebra is then an algebra with additional structure, defined as follows:

**Definition (bialgebra)** : A bialgebra over a commutative ring $K$ with unit, is a quintuple $(A, \mu, \eta, \Delta, \varepsilon)$, where $(A, \mu, \eta)$ is a unital algebra (where $A$ is a vector space, $\mu$ is vector multiplication and $\eta$ is the unit map), and $(A, \Delta, \varepsilon)$ is a coalgebra, with $A$ a vector space, $\Delta$ a linear map $\Delta : A \to A \otimes A$ called the comultiplication (or coproduct), and $\varepsilon$ a linear map $\varepsilon : A \to K$ called the counit. These must satify:

- $(id_A \otimes \Delta)\Delta = (\Delta \otimes id_A)\Delta$,
- $(\varepsilon \otimes id_A)\Delta = (id_A \otimes \varepsilon)\Delta = id_A$.

Where $id_A$ is the identity map, and we identify:
$(A \otimes A) \otimes A = A \otimes (A \otimes A)$ via $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, where $a, b, c \in A$ [17], [19].

In order to define a Hopf algebra, we are now required to add more structure to this definition of a bialgebra, by defining a map called the antipode. This general definition of the antipode is intended to be the analog of an inverse.

**Definition (Hopf algebra)** : A Hopf algebra is a bialgebra with a $K$-linear homomorphism, $s : A \to A$ called the antipode, which must satisfy:

- $\mu(s \otimes id_A)\Delta = \mu(id_A \otimes s)\Delta = \varepsilon \cdot \eta$

(Where $\mu, \eta, \Delta, \varepsilon$ and $id_A$ are defined as above) [19].

Now that we have the definition of a Hopf algebra, we may return to $SL(2)_q$ and see that it does indeed satisfy this definition.

## 5.2 $SL(2)_q$ as a Hopf algebra

We look again at the set of equations that define $SL(2)_q$, as found at the end of section 4.4:

$$\begin{cases} ba = qab \\ db = qbd \\ dc = qcd \\ ca = qac \\ bc = cb \\ ad - da = (q^{-1} - q)bc \\ ad - q^{-1}bc = 1. \end{cases}$$

By defining the maps required in the above definition, we can then verify that this set of equations may define a Hopf algebra. We first define the maps as follows (where $A$ denotes the algebra):

- Coproduct, $\Delta : A \to A \otimes A$

$$\begin{aligned}\Delta(a) &= a \otimes a + b \otimes c \\ \Delta(b) &= a \otimes b + b \otimes d \\ \Delta(c) &= c \otimes a + d \otimes c \\ \Delta(d) &= c \otimes b + d \otimes d\end{aligned}$$

  which we may also write in a compact matrix form (where multiplication in this context is understood to be matrix multipliction):

$$\Delta(P) = \Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

  That is, we define multiplication, $\mu$, here as:

- Multiplication, $\mu : A \otimes A \to A$

$$\mu\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

  Where we must allow $a, b, c, d$ to commute with $a', b', c', d'$.

- Unit, $\eta : K \to A$

$$\eta(\delta^i_j) = \delta^i_j.$$

- Counit, $\varepsilon : A \to K$

$$\varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Antipode, $\gamma : A \to A$

$$\gamma(P) = \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}.$$

We must now check that this forms a bialgebra, and that the antipode is well defined.

To check that this is a bialgebra, we must check that $\Delta$ and $\varepsilon$ satisfy:

- $(id_A \otimes \Delta)\Delta = (\Delta \otimes id_A)\Delta$,
- $(\varepsilon \otimes id_A)\Delta = (id_A \otimes \varepsilon)\Delta = id_A$.

(The fact that $A$, with $\mu$ and $\eta$ is an algebra follows from the bilinearity of matrix multiplication, closure under this multiplication with the given set of equations is verified in appendix D).

Checking the first of these two equations, the left hand side (LHS) gives:

$$\begin{aligned}(id_A \otimes \Delta)\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (id_A \otimes \Delta)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right),\end{aligned}$$

and the right hand side gives:

$$\begin{aligned}(\Delta \otimes id_A)\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (\Delta \otimes id_A)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \text{LHS}.\end{aligned}$$

We now check the second equation:

$$\begin{aligned}(\varepsilon \otimes id_A)\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (\varepsilon \otimes id_A)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= id_A\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right), \\ \text{and, } (id_A \otimes \varepsilon)\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (id_A \otimes \varepsilon)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= id_A\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).\end{aligned}$$

This verifies that, with the given maps, this set of equations is a bialgebra. To establish this as a Hopf algebra, we must now check that the antipode given above is well defined, that is we must check that:

- $\mu(s \otimes id_A)\Delta = \mu(id_A \otimes s)\Delta = \varepsilon \cdot \eta$.

We first look at:

$$\begin{aligned}
\mu(s \otimes id_A)\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \mu(s \otimes id_A)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= \mu\left(\begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= \begin{bmatrix} da - qbc & db - qbd \\ -q^{-1}ca + ac & -q^{-1}cb + ad \end{bmatrix} \\
&= \begin{bmatrix} ad - (q^{-1} - q)bc - qbc & qbd - qbd \\ -q^{-1}qac + ac & -q^{-1}cb + 1 + q^{-1}bc \end{bmatrix} \\
&= \begin{bmatrix} ad - q^{-1}bc & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \varepsilon \cdot \eta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).
\end{aligned}$$

Similarly we also find:

$$\begin{aligned}
\mu(id_A \otimes s)\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \mu(id_A \otimes s)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= \mu\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}\right) \\
&= \begin{bmatrix} ad - q^{-1}bc & -qab + ba \\ cd - q^{-1}dc & -qcb + da \end{bmatrix} \\
&= \begin{bmatrix} ad - ad + 1 & 0 \\ 0 & -qcb + ad - q^{-1}bc + qbc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \varepsilon \cdot \eta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).
\end{aligned}$$

Thus $SL(2)_q$ is indeed a Hopf algebra, with the relavent maps as defined here.

We also stated in section 4.4 that $SL(2)_q$ defines a particular type of Hopf algebra, which is also known as a quantum group. We will now look at these special types, known as quasitriangular Hopf algebras.

## 5.3 Quasitriangular Hopf algebras

A Hopf algebra is quasitriangular if there exists a special element called a universal $R$-matrix. However, before stating the formal definition, we first require some notation.

Suppose we have a bialgebra $A$, then let $R \in A \otimes A$. We may now define three new elements, $R_{12}, R_{13}, R_{23} \in A \otimes A \otimes A$ as such:

$$\text{If } R = \sum_s e_s \otimes e^s \in A \otimes A, \text{ then we define,}$$

$$R_{12} = \sum_s e_s \otimes e^s \otimes 1$$
$$R_{13} = \sum_s e_s \otimes 1 \otimes e^s$$
$$R_{23} = \sum_s 1 \otimes e_s \otimes e^s.$$

We also define the operation $\Delta' : A \to A \otimes A$, known as opposite comultiplication, which is the composition of the comultiplication $\Delta$ with the permutation map on $A \otimes A$.

That is, for $a \in A$, set $\Delta'(a) = P_A(\Delta(a)) \in A \otimes A$. Where $P_A$ is the permutation, flipping elements, $a \otimes b \mapsto b \otimes a$.

We now use this notation in the definition.

**Definition (quasitriangular Hopf algebra)** : A Hopf algebra $(A, \mu, \eta, \Delta, \varepsilon, s)$ over a commutative ring $K$ with unit, is said to be *quasitriangluar* if there exists an invertible element $R$ of the algebra $A \otimes A$, which satisfies, for any $a \in A$:

- $\Delta'(a) = R\Delta(a)R^{-1}$
- $(id_A \otimes \Delta)(R) = R_{13}R_{12}$
- $(\Delta \otimes id_A)(R) = R_{13}R_{23}$.

The element $R$ is called a universal $R$-matrix of $A$ [19].

The definition of a quasitriangular Hopf algebra, and therefore the existence of a universal $R$-matrix, has a direct relation to the Yang-Baxter equation, which we will look at in the next section.

## 5.4 The Yang-Baxter equation

As stated previously, a quasitriangular Hopf algebra must contain a universal $R$-matrix. What is of particular interest about this universal $R$-matrix is that, by the definition, it must also be a solution to the Yang-Baxter equation.

In this context the Yang-Baxter equation must be stated slightly differently to the version we saw earlier. Here, the appropriate version is (using the notation from the previous section):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

This may be derived from the equations in the definition of a quasitriangular Hopf algebra as such:

Starting with the LHS,

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id_A)(R), \text{ (from the third equation)} \\ &= (\Delta' \otimes id_A)(R)R_{12}, \text{ (by the first equation)} \\ &= (P_A \otimes id_A)(\Delta \otimes id_A)(R)R_{12}, \text{ (definition of } \Delta') \\ &= (P_A \otimes id_A)(R_{13}R_{23})R_{12} \\ &= R_{23}R_{13}R_{12} = RHS. \end{aligned}$$

Therefore, if we have a Hopf algebra with a universal $R$-matrix, then this $R$-matrix must satisfy the Yang-Baxter equation.

In our construction of the Hopf algebra $SL(2)_q$, we obtained a solution to the Yang-Baxter equation in the form of the matrix $R^{ab}_{cd}$ (as found due to the third Reidemeister move III, in figure 4.7). It is because of the existence of this $R$-matrix, that the Hopf algebra $SL(2)_q$ is in fact quasitriangular - a quantum group.

Indeed, there is a method due to Faddeev, Reshetikhin and Takhadjian, called the 'FRT construction', that produces cobraided biaglebras from any solution of the Yang-Baxter equation. It can be shown that the quantum group $SL(2)_q$ can be obtained by this method [17].

## 5.5 Summary

There is much more to say about the relationship between knot theory and quantum groups, which can be seen in more detail in texts such as [13], [17] and [19]. What we have looked at here, over the last two chapters, is a direct construction of the quantum group $SL(2)_q$, starting from the Reidemeister moves and the topological invariance of the bracket polynomial.

We have seen in this chapter the formal definitions of Hopf algebras and quasitriangular Hopf algebras, and seen how a complicated algebraic expression such as the Yang-Baxter equation can appear in a much more simple way when viewed as a topological move in knot theory.

# Chapter 6

# Conclusion

In this report we have seen some of the motivation, results and applications of knot theory. We have seen how the oldest of problems in this area - that of comparing different knots - has led to the development of a number of topological and algebraic techniques. We have seen how computer programs are currently being used to help in this research, and how results in knot theory are bringing new insights into other areas of mathematics and physics.

As ever, we are left with more questions than we started with. Although computers may be helping us to distinguish different knots, the fundamental problem of comparing knots remains open - we still have no complete invariant. Indeed, this is an area of current research. Much hope appears to be pinned on what are called *Vassiliev invariants*, but it is still an open question as to whether these Vassiliev invariants distinguish knots [21]. This would be a natural area to look at next, as it hopes to answer this simple to state, yet deeply complex question.

It would also be of interest to continue our look at Hopf algebras, and the Yang-Baxter equation. This is a developed area, and in this report we have only skimmed the surface; with greater insight we can hope to unlock further secrets and relationships to knot theory. So far we have only seen how knot theory can fit into previous work done in other areas of research, but the hope is that as we look further, we may be able to use knot theory to predict results in other fields.

As well as these applications of knot theory, it is still an ongoing area of research to try and construct an efficient program to calculate and enumerate as many knots as possible [22]. We have seen some techniques in this report, but it would be interesting to not only construct a fully functioning program from our algorithm, but to also look at further techniques for calculating different invariants of knots.

Although we have seen a number of different ways that knot theory relates to other areas, there are many more that we have not been able to look at. From a biological perspective, the discovery that DNA molecules

(with the help of enzymes called *topoisomerases*) knot and unknot themselves is fascinating [2]. In terms of mathematics, it would be interesting to look at the idea of generalising the notion of a knot to higher dimensions - instead of looking at 1-dimensional objects knotted in 3-dimensional space, how about 2-dimensional objects knotted in 4-dimensional space, [13]? We could look further at ties to physics through the idea of topological quantum field theory and its relationship to knot theory - an area still in its infancy [8]. Indeed, it would even be of interest to look at this due only to the fact that there have been so many Fields Medals awarded for work in this area (Donaldson, Jones, Witten and Kontsevich all having won such an award for their work [23]) - although perhaps after a lot more work on our part.

There are clearly a large number of areas that we could explore now, relating to a number of different subjects. It is this relation to such a wide range of subjects that, in my opinion, makes knot theory so interesting.

## Acknowledgements

I would like to thank my supervisor, Prof. Chong-Sun Chu for his help over the course of this project.

This report was prepared in LaTeX, using MiKTeX [24], the flowchart was created using Microsoft Excel [25], and diagrams were all drawn by the author in Inkscape [26].

## Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

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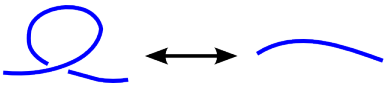
# Appendix A

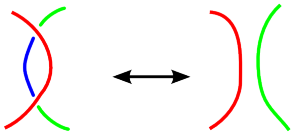
# Proof of Theorem 2

Part of this proof can be found in [7]. It is sufficient to show that if a diagram is 3-colourable then applying any Reidemeister move will leave the resulting diagram 3-colourable.

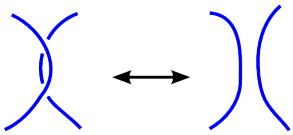
**Reidemeister move 1**
In this case the solution is trivial: both a loop and a line may only be one colour, and so the colouring of the diagram is unaffected.



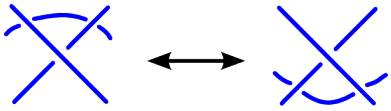
**Reidemeister move 2**
Here we must look at two cases. First we look at the case where the 'free' arcs are coloured with two different colours. The resulting diagram may then be coloured as before, but with the altered section coloured as shown here:



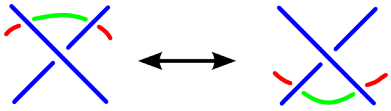
The second case has the arcs coloured with the same colour. After performing an R2 move the arcs remain coloured in the same way and the diagram remains coloured as it was before.



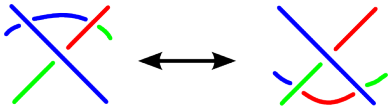
**Reidemeister move 3**
There are 5 cases we must look at. First, where all arcs are of the same colour.



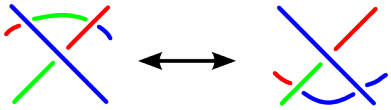
Second where the ‘central crossing’ has the same colours, but the ‘horizontal arcs’ are of a different colour to the central crossing.



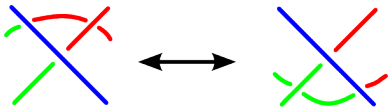
Third, where the central crossing is of different colours, but the horizontal arc on the left is the same colour as the arc it is crossing.



Fourth, where the central crossing is of different colours, but the horizontal arc on the left is one of the different colours.



Fifth, where the central crossing is again of different colours, but the horizontal arc on the left is coloured with the final colour.



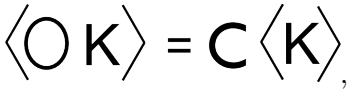
All of the above cases leave the colouring of the rest of the diagram unaffected, and all satisfy the requirements of 3-colourability.

# Appendix B

# Calculating the values $B$ and $C$ for the bracket polynomial

Using the skein relation:

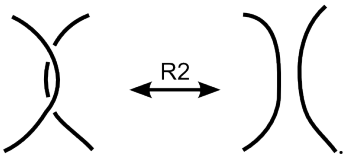
and the relation:



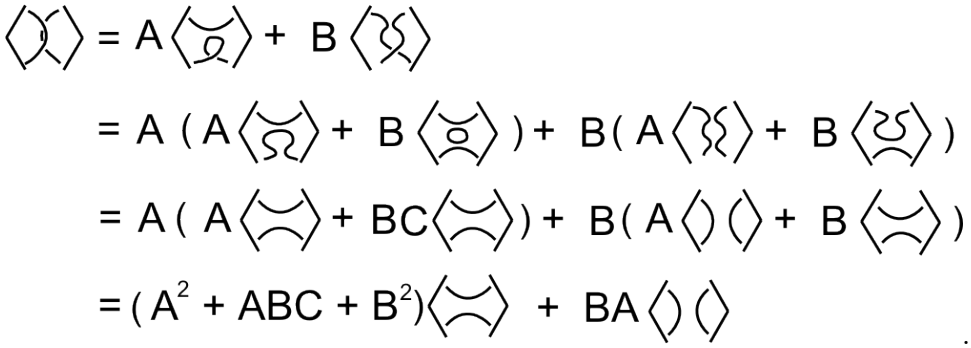
as stated in section 2.4.

We apply this to the second Reidemeister move, and by demanding that this be invariant we establish the required values.

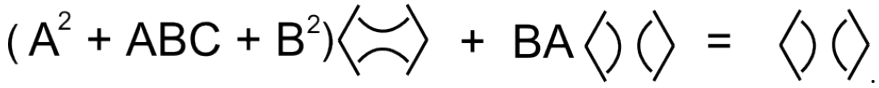
The second Reidemeister move is:




We start by calculating the bracket polynomial of the left hand side:



For this to satisfy the second Reidemeister move, we require that this is the same as the bracket polynomial of the right hand side. That is:




This may be acheived by taking $BA = 1$, and $A^2 + ABC + B^2 = 0$. Therefore:

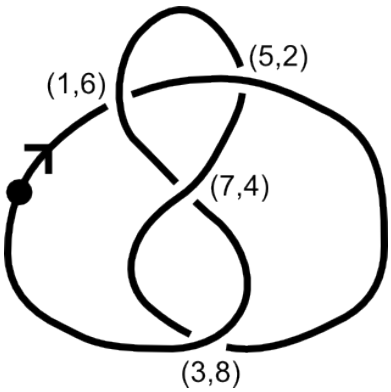
$$\begin{aligned} B &= A^{-1} \\ \text{and,}\ \ C &= -A^2 - A^{-2} \end{aligned}$$

gives the result.

# Appendix C

# Homfly polynomial of the figure eight knot

Here, we will show an example calculation of the Homfly polynomial of the figure eight knot, using the algorithm constructed in section 3.3. The diagram of the figure eight knot that we will use is shown below.




A Dowker notation for this diagram is: $K = \{(1,6),(5,2),(3,8),(7,4)\}$, with corresponding crossing orientations: $\{+1,+1,-1,-1\}$.

We can now use this information in the algorithm to calculate the Homfly polynomial. A table of how this is calculated is given below, laid out in the same way as in section 3.4.

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $K = \{(1,6),(5,2),(3,8),(7,4)\}$, $\{+1,+1,-1,-1\}$ |
| Calculate number of crossings $= k$. | 4 crossings. | $k = 4$ |
| Does $k = 0$? | $k = 4$ | No. |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Calculate components. | $1 \to 2 \to 3 \to 4 \to$ $5 \to 6 \to 7 \to 8$ | Components: $\{1, 2, 3, 4, 5, 6, 7, 8\}$. |
| Remove loops. | No loops. | $k = 4$ |
| Does $k = 0$ ? | $k = 4$ | No. |
| Recalculate components, let $n =$ number of components. | $1 \to 2 \to 3 \to 4 \to$ $5 \to 6 \to 7 \to 8$ | $n = 1$, components: $\{1, 2, 3, 4, 5, 6, 7, 8\}$. |
| Does $n = 1$? | $n = 1$ | Yes. |
| Is $k < 3$ ? | $k = 4$ | No. |
| For each crossing $x$, is $U_x < O_x$? | Crossing 1: $1 < 6$ Yes. Crossing 2: $5 > 2$ No | No for $x = 2$. |
| Read orientation of crossing. | Crossing 2 is positive. | $+1$ |
| Apply (1) | | $P(K) = v^2 P(K_-) + zvP(K_0)$ |
| For $K_0$ change $(U_x, O_x)$ to $\{U_x, O_x\}$ | $(5, 2) \to \{5, 2\}$. | $K_0 = \{(1, 6), \{5, 2\}, (3, 8), (7, 4)\}$ Orientation: $\{+1, -1, -1\}$. |
| For $K_-$, change $(U_x, O_x)$ to $(O_x, U_x)$ and change orientation. | $(5, 2) \to (2, 5)$ $+1 \to -1$ | $K_- = \{(1, 6), (2, 5), (3, 8), (7, 4)\}$ Orientation: $\{+1, -1, -1, -1\}$. |

We now have that the Homfly polynomial of the figure eight knot is:

$$P(K) = v^2 P(K_-) + zvP(K_0),$$

and we have the Dowker notation of $K_-$ and $K_0$. The program must now run through the algorithm again for both $K_-$ and $K_0$. We will first evaluate $K_-$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{(1, 6), (2, 5), (3, 8), (7, 4)\}$, $\{+1, -1, -1, -1\}$ |
| Calculate number of crossings $= k$. | 4 crossings | $k = 4$ |
| Does $k = 0$? | $k = 4$ | No. |
| Calculate components. | $1 \to 2 \to 3 \to 4 \to$ $5 \to 6 \to 7 \to 8$ | Components: $\{1, 2, 3, 4, 5, 6, 7, 8\}$. |
| Remove loops. | No loops. | $k = 4$ |
| Recalculate components, let $n =$ number of components. | $1 \to 2 \to 3 \to 4 \to$ $5 \to 6 \to 7 \to 8$ | $n = 1$, components: $\{1, 2, 3, 4, 5, 6, 7, 8\}$. |
| Does $n = 1$? | $n = 1$ | Yes. |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Is $k < 3$ ? | $k = 4$ | No. |
| For each crossing $x$, is $U_x < O_x$? | Crossing 1: $1 < 6$ Yes. Crossing 2: $2 < 5$ Yes. Crossing 3: $3 < 8$ Yes. Crossing 4: $7 > 4$ No. | No for $x = 4$. |
| Read orientation of crossing. | Crossing 4 is negative. | $-1$ |
| Apply (2). | | $P(L) = v^{-2}P(L_+) - zv^{-1}P(L_0)$ |
| For $L_0$ change $(U_x, O_x)$ to $\{U_x, O_x\}$. | $(7,4) \to \{7,4\}$. | $L_0 = \{(1,6),(2,5),(3,8),\{7,4\}\}$ Orientation: $\{+1,-1,-1\}$. |
| For $L_+$, change $(U_x, O_x)$ to $(O_x, U_x)$ and change orientation. | $(7,4) \to (4,7)$ $-1 \to +1$ | $L_+ = \{(1,6),(2,5),(3,8),(4,7)\}$ Orientation: $\{+1,-1,-1,+1\}$. |

The Homlfy polynomial of the figure eight knot is now:

$$P(K) = v^2(v^{-2}P(L_+) - zv^{-1}P(L_0)) + zvP(K_0).$$

We must now evaluate $P(L_+)$, $P(L_0)$ and $P(K_0)$. We start with $L_+$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{(1,6),(2,5),(3,8),(4,7)\}$ $\{+1,-1,-1,+1\}$ |
| Calculate number of crossings $= k$. | 4 crossings. | $k = 4$ |
| Does $k = 0$? | $k = 4$ | No. |
| Calculate components. | $1 \to 2 \to 3 \to 4 \to$ $5 \to 6 \to 7 \to 8$ | Components: $\{1,2,3,4,5,6,7,8\}$. |
| Remove loops. | No loops. | $k = 4$ |
| Recalculate components, let $n =$ number of components. | $1 \to 2 \to 3 \to 4 \to$ $5 \to 6 \to 7 \to 8$ | $n = 1$, components: $\{1,2,3,4,5,6,7,8\}$. |
| Does $n = 1$? | $n = 1$ | Yes. |
| Is $k < 3$ ? | $k = 4$ | No. |
| For each crossing $x$, is $U_x < O_x$? | Crossing 1: $1 < 6$, yes. 2: $2 < 5$, yes. 3: $3 < 8$, yes. 4: $4 < 7$, yes. | Yes for all $x$. |
| $P(L) = 1$ | | $P(L) = 1$. |

The Homfly polynomial of the figure eight knot is now:

$$\begin{aligned} P(K) &= v^2(v^{-2}(1) - zv^{-1}P(L_0)) + zvP(K_0) \\ &= 1 - zvP(L_0) + zvP(K_0). \end{aligned}$$

We must now evaluate $P(L_0)$ and $P(K_0)$. We start with $L_0$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{(1,6),(2,5),(3,8),\{7,4\}\}$ $\{+1,-1,-1\}$. |
| Calculate number of crossings $= k$. | 3 crossings. | $k = 3$. |
| Does $k = 0$? | $k = 3$ | No. |
| Calculate components. | $1 \to 2 \to 3 \to (7) \to 8$. $5 \to 6 \to (4)$. | Components: $C_1 = \{1,2,3,8\}$, $C_2 = \{5,6\}$. |
| Remove loops. | $3 \to 8$, Remove $(3,8)$. | $L = \{(1,6),(2,5),\{3,8\},\{7,4\}\}$ $k = 2$ |
| Does $k = 0$ ? | $k = 2$ | No. |
| Recalculate components, let $n =$ number of components. | $1 \to 2 \to (8)$. $5 \to 6 \to (4)$. | $n = 2$, components: $C_1 = \{1,2\}$, $C_2 = \{5,6\}$ |
| Does $n = 1$? | $n = 2$. | No. |
| Let $i = 1$. | | $i = 1$. |
| Is $i > n$? | $i = 1$, $n = 2$. | No. |
| Look only at crossings involving elements from both $C_1$ and some other component. | $1 \in C_1$ and $6 \in C_2$, so look at $(1,6)$, $2 \in C_1$ and $5 \in C_2$, so look at $(2,5)$. | Look at crossings: $\{(1,6),(2,5)\}$. |
| Of these, are all elements of $C_1$ under, or all over? | 1 is under, 2 is under. | Yes. |
| $P(L) = \delta P(L - C_1)P(C_1)$ | $L - C_1 = \{\{3,8\},\{7,4\}\}$ $C_1 = \emptyset$ | $P(L) = \delta P(L - C_1)P(C_1)$ |

The Homfly polynomial of the figure eight knot is now:

$$\begin{aligned} P(K) &= 1 - zv(\delta P(L - C_1)P(C_1)) + zvP(K_0) \\ &= 1 - zv(v^{-1} - v)z^{-1}P(L - C_1)P(C_1) + zvP(K_0) \\ &= 1 - (1 - v^2)P(L - C_1)P(C_1) + zvP(K_0). \end{aligned}$$

Where we must evaluate $P(L - C_1)$, $P(C_1)$ and $P(K_0)$. We now run through the algorithm again for $L - C_1$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{\{3,8\},\{7,4\}\}$ |
| Calculate number of crossings $= k$. | No crossings. | $k = 0$ |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

Now, the Homfly polynomial of the figure eight knot is:

$$P(K) = 1 - (1 - v^2)(1)P(C_1) + zvP(K_0).$$

We now calculate $P(C_1)$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \emptyset$ |
| Calculate number of crossings $= k$. | No crossings. | $k = 0$ |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

So the Homfly polynomial of the figure eight knot is:

$$P(K) = 1 - (1 - v^2)(1) + zvP(K_0).$$

We now calculate the Homfly polynomial of $K_0$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{(1,6),\{5,2\},(3,8),(7,4)\}$ $\{+1,-1,-1\}$. |
| Calculate number of crossings $= k$ | 3 crossings. | $k = 3$. |
| Does $k = 0$? | $k = 3$ | No. |
| Calculate components. | $1 \to (5) \to 6 \to 7 \to 8$. $3 \to 4 \to (2)$. | Components: $C_1 = \{1,6,7,8\}$, $C_2 = \{3,4\}$. |
| Remove loops. | $1 \to 6$, Remove $(1,6)$ | $L = \{\{1,6\},\{5,2\},(3,8),(7,4)\}$ $k = 2$ |
| Does $k = 0$ ? | $k = 2$ | No. |
| Recalculate components, let $n =$ number of components. | $3 \to 4 \to (2)$. $7 \to 8 \to (6)$. | $n = 2$, components: $C_1 = \{3,4\}$, $C_2 = \{7,8\}$ |
| Does $n = 1$? | $n = 2$. | No. |
| Let $i = 1$. | | $i = 1$. |
| Is $i > n$? | $i = 1$, $n = 2$. | No. |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Look only at crossings involving elements from both $C_1$ and some other component. | $3 \in C_1$ and $8 \in C_2$, so look at $(3,8)$, $7 \in C_2$ and $4 \in C_1$, so look at $(7,4)$. | Look at crossings: $\{(3,8),(7,4)\}$. |
| Of these, are all elements of $C_1$ under, or all over? | 3 is under, 4 is over. | No. |
| Set $i = i + 1$ | $i = 1 + 1$ | $i = 2$ |
| Is $i > n$? | $i = 2$, $n = 2$. | No. |
| Look only at crossings involving elements from both $C_2$ and some other component. | $3 \in C_1$ and $8 \in C_2$, so look at $(3,8)$, $7 \in C_2$ and $4 \in C_1$, so look at $(7,4)$. | Look at crossings: $\{(3,8),(7,4)\}$. |
| Of these, are all elements of $C_2$ under, or all over? | 8 is over, 7 is under. | No. |
| Set $i = i + 1$ | $i = 2 + 1$ | $i = 3$ |
| Is $i > n$ ? | $i = 3$, $n = 2$ | Yes. |
| Choose a crossing involving two different components. | First crossing: $3, 8$ involves $C_1$ and $C_2$. | (3,8) |
| Read orientation. | (3,8) is negative. | $-1$. |
| Is crossing positive? | Negative. | No. |
| Apply (2). | | $P(L) = v^{-2}P(L_+) - zv^{-1}P(L_0)$ |
| For $L_0$, change $(U_x, O_x)$ to $\{U_x, O_x\}$ | $(3,8) \to \{3,8\}$ | $L_0 = \{\{1,6\},\{5,2\},\{3,8\},(7,4)\}$, Orientation: $\{-1\}$. |
| For $L_+$, change $(U_x, O_x)$ to $(O_x, U_x)$ and change orientation. | $(3,8) \to (8,3)$ $-1 \to +1$ | $L_+ = \{\{1,6\},\{5,2\},(8,3),(7,4)\}$ $\{+1,-1\}$ |

The Homfly polynomial of the figure eight knot is now:

$$\begin{aligned} P(K) &= 1 - (1 - v^2) + zv(v^{-2}P(L_+) - zv^{-1}P(L_0)) \\ &= v^2 + zv^{-1}P(L_+) - z^2P(L_0). \end{aligned}$$

Where $L_+$ and $L_0$ are given in the table above. We first calculate $P(L_0)$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{\{1,6\},\{5,2\},\{3,8\},(7,4)\}$, $\{-1\}$. |
| Calculate number of crossings $= k$. | 1 crossing. | $k = 1$. |

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Does $k = 0$? | $k = 1$ | No. |
| Calculate components. | $4 \to (2) \to (8) \to$ $(6) \to 7$ | Components: $C_1 = \{4, 7\}$. |
| Remove loops. | $4 \to 7$, Remove $(7, 4)$. | $L = \{\{1, 6\}, \{5, 2\}, \{3, 8\}, \{7, 4\}\}$, $k = 0$ |
| Does $k = 0$ ? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

So we have that the Homfly polynomial of the figure eight knot is:

$$P(K) = v^2 + zv^{-1}P(L_+) - z^2(1).$$

We now calculate $P(L_+)$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{\{1, 6\}, \{5, 2\}, (8, 3), (7, 4)\}$ { +1, -1 } |
| Calculate number of crossings $= k$. | 2 crossings. | $k = 2$. |
| Does $k = 0$? | $k = 2$ | No. |
| Calculate components. | $3 \to 4 \to (2)$. $7 \to 8 \to (6)$. | Components: $C_1 = \{3, 4\}$ $C_2 = \{7, 8\}$ |
| Remove loops. | No loops. | $k = 2$ |
| Does $k = 0$? | $k = 2$ | No. |
| Recalculate components, let $n =$ number of components. | $3 \to 4 \to (2)$. $7 \to 8 \to (6)$. | $n = 2$, components: $C_1 = \{3, 4\}$ $C_2 = \{7, 8\}$ |
| Does $n = 1$? | $n = 2$ | No. |
| Let $i = 1$ | | $i = 1$ |
| Is $i > n$? | $i = 1$, $n = 2$ | No. |
| Look only at crossings involving elements from both $C_1$ and some other component. | $3 \in C_1$, and $8 \in C_2$, so look at $(3, 8)$. $4 \in C_1$ and $7 \in C_2$, so look at $(7, 4)$. | Look at crossings: $\{(8, 3), (7, 4)\}$. |
| Of these, are all elements of $C_1$ under, or all over? | 3 is over, 4 is over. | Yes. |
| $P(L) = \delta P(L - C_1)P(C_1)$ | $L - C_1 = \{\{1, 6\}, \{5, 2\}\}$ $C_1 = \emptyset$ | $P(L) = \delta P(L - C_1)P(C_1)$ |

The Homfly polynomial of the figure eight knot is now:

$$
\begin{aligned}
P(K) &= v^2 + zv^{-1}(\delta P(L - C_1)P(C_1)) - z^2 \\
&= v^2 + zv^{-1}(v^{-1} - v)z^{-1}P(L - C_1)P(C_1) - z^2 \\
&= v^2 + (v^{-2} - 1)P(L - C_1)P(C_1) - z^2.
\end{aligned}
$$

Where, $L - C_1$ and $C_1$ are given in the table above. We now calculate $P(L - C_1)$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \{\{1, 6\}, \{5, 2\}\}$ |
| Calculate number of crossings $= k$. | No crossings. | $k = 0$ |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

So the Homfly polynomial of the figure eight knot is now:

$$P(K) = v^2 + (v^{-2} - 1)(1)P(C_1) - z^2,$$

and we calculate $P(C_1)$:

| **Step** | **Evaluation** | **Result** |
|---|---|---|
| Read Dowker notation and crossing orientation. | | $L = \emptyset$ |
| Calculate number of crossings $= k$. | No crossings. | $k = 0$ |
| Does $k = 0$? | $k = 0$ | Yes. |
| $P(L) = 1$ | | $P(L) = 1$ |

So the Homfly polynomial of the figure eight knot is:

$$P(K) = v^2 + v^{-2} - 1 - z^2,$$

which is indeed the correct value.

# Appendix D

# Verifying multiplication $\mu$ in $SL(2)_q$ is well defined

Assuming $A$ is the algebra of $SL(2)_q$, and we have elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in A$, with multiplication defined as:

$$\mu : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \to \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}$$

Where elements $a, b, c, d$ commute with $a', b', c', d'$.
To verify that this is well defined we must ensure that the elements of this matrix satisfy the defining equations for $SL(2)_q$:

$$\begin{cases} ba = qab \\ db = qbd \\ dc = qcd \\ ca = qac \\ bc = cb \\ ad - da = (q^{-1} - q)bc \\ ad - q^{-1}bc = 1. \end{cases}$$

So in this case we require:

$$\begin{cases} (ab' + bd')(aa' + bc') = q(aa' + bc')(ab' + bd') \\ (cb' + dd')(ab' + bd') = q(ab' + bd')(cb' + dd') \\ (cb' + dd')(ca' + dc') = q(ca' + dc')(cb' + dd') \\ (ca' + dc')(aa' + bc') = q(aa' + bc')(ca' + dc') \\ (ab' + bd')(ca' + dc') = (ca' + dc')(ab' + bd') \\ (aa' + bc')(cb' + dd') - (cb' + dd')(aa' + bc') = (q^{-1} - q)(ab' + bd')(ca' + dc') \\ (aa' + bc')(cb' + dd') - q^{-1}(ab' + bd')(ca' + dc') = 1. \end{cases}$$

We do so by evaluating the left hand side of each of the seven equations, and checking that they are the same as the right hand side.

**Equation 1:**

$$\begin{aligned}
(ab' + bd')(aa' + bc') &= ab'aa' + ab'bc' + bd'aa' + bd'bc' \\
&= aab'a' + abb'c' + bad'a' + bbd'c' \\
&= qaaa'b' + abb'c' + qabd'a' + bbd'c' \\
&= q(aaa'b' + bbc'd') + ab(b'c' + q(a'd' - q^{-1}b'c' + qb'c')) \\
&= q(aaa'b' + bbc'd') + ab(b'c' - b'c' + qa'd' + q^2b'c') \\
&= q(aaa'b' + bbc'd') + qaba'd' + q^2abb'c' \\
&= q(aaa'b' + bbc'd' + aba'd' + bac'b') \\
&= q(aa' + bc')(ab' + bd') = RHS.
\end{aligned}$$

**Equation 2:**

$$\begin{aligned}
(cb' + dd')(ab' + bd') &= cabb' + cbb'd' + dad'b' + dbd'd' \\
&= qacb'b' + (q^{-1}bc + da)d'b' + qbdd'd' \\
&= qacb'b' + (q^{-1}bc + ad - q^{-1}bc + qbc)d'b' + qbdd'd' \\
&= qacb'b' + add'b' + qbcd'b' + qbdd'd' \\
&= qacb'b' + qadb'd' + qbcd'b' + qbdd'd' \\
&= q(ab' + bd')(cb' + dd') = RHS.
\end{aligned}$$

**Equation 3:**

$$\begin{aligned}
(cb' + dd')(ca' + dc') &= cb'ca' + cb'dc' + dd'ca' + dd'dc' \\
&= qcca'b' + cd(b'c' + qd'a') + qddc'd' \\
&= qcca'b' + cd(b'c' + q(a'd' - q^{-1}b'c' + qb'c')) + qddc'd' \\
&= qcca'b' + qcda'd' + q^2cdb'c' + qddc'd' \\
&= qcca'b' + qcda'd' + qdcc'b' + qddc'd' \\
&= q(ca' + dc')(cb' + dd') = RHS.
\end{aligned}$$

**Equation 4:**

$$\begin{aligned}
(ca' + dc')(aa' + bc') &= ca'aa' + ca'bc' + dc'aa' + dc'bc' \\
&= qaca'a' + bca'c' + dac'a' + qbdc'c' \\
&= q(aca'a' + bdc'c') + bca'c' + (ad - q^{-1}bc + qbc)c'a' \\
&= q(aca'a' + bdc'c') + bca'a' - bca'c' + qada'c' + qbcc'a' \\
&= q(aca'a' + ada'c' + bcc'a' + bdc'c') \\
&= q(aa' + bc')(ca' + dc') = RHS.
\end{aligned}$$

**Equation 5:**

$$
\begin{aligned}
&(ab' + bd')(ca' + dc') = ab'ca' + ab'dc' + bd'ca' + bd'dc' \\
&= q^{-1}qcaa'b' + q^{-1}qdbc'd' + adc'b' + bcd'a' \\
&= caa'b' + dbc'd' + (da + q^{-1}bc - qbc)c'b' + cb(a'd' - q^{-1}b'c' + qb'c') \\
&= caa'b' + dbc'd' + dac'b' + cba'd' + (q^{-1}bc' - qbc)c'b' - (q^{-1}cb - qcb)b'c' \\
&= caa'b' + dbc'd' + dac'b' + cba'd' \\
&= (ca' + dc')(ab' + bd') = RHS.
\end{aligned}
$$

**Equation 6:**

$$
\begin{aligned}
&(aa' + bc')(cb' + dd') - (cb' + dd')(aa' + bc') \\
&= aa'cb' + aa'dd' + bc'cb' + bc'dd' - cb'aa' - cb'bc' - dd'aa' - dd'bc' \\
&= q^{-1}acb'a' + ada'd' + bcc'b' + q^{-1}bdd'c' - qacb'a' - cbb'c' - dad'a' - qbdd'c' \\
&= q^{-1}acb'a' + ada'd' + q^{-1}bdd'c' - qacb'a' - (ad - q^{-1}bc + qbc)d'a' - qbdd'c' \\
&= q^{-1}acb'a' + ada'd' + q^{-1}bdd'c' - qacb'a' - add'a' + q^{-1}bcd'a' - qbcd'a' - qbdd'c' \\
&= q^{-1}acb'a' + ad(a'd' - d'a') + q^{-1}bdd'c' - qacb'a' + q^{-1}bcd'a' - qbcd'a' - qbdd'c' \\
&= q^{-1}acb'a' + ad(q^{-1} - q)b'c' + q^{-1}bdd'c' - qacb'a' + q^{-1}bcd'a' - qbcd'a' - qbdd'c' \\
&= q^{-1}acb'a' + q^{-1}adb'c' - qadb'c' + q^{-1}bdd'c' - qacb'a' + q^{-1}bcd'a' - qbcd'a' - qbdd'c' \\
&= q^{-1}(acb'a' + adb'c' + bcd'a' + bdd'c') - q(acb'a' + adb'c' + bcd'a' + bdd'c') \\
&= q^{-1}(ab' + bd')(ca' + dc') - q(ab' + bd')(ca' + dc') \\
&= (q^{-1} - q)(ab' + bd')(ca' + dc') = RHS.
\end{aligned}
$$

**Equation 7:**

$$
\begin{aligned}
&(aa' + bc')(cb' + dd') - q^{-1}(ab' + bd')(ca' + dc') \\
&= aa'cb' + aa'dd' + bc'cb' + bc'dd' - q^{-1}(ab'ca' + ab'dc' + bd'ca' + bd'dc') \\
&= acq^{-1}b'a' + ada'd' + bcc'b' + bdq^{-1}d'c' - q^{-1}acb'a' - q^{-1}adb'c' - q^{-1}bcd'a' - q^{-1}bdd'c' \\
&= ad(a'd' - q^{-1}b'c') + bc(c'b' - q^{-1}d'a') \\
&= ad + bc(c'b' - q^{-1}(a'd' - (q^{-1} - q)b'c')) \\
&= ad + bc(c'b' - q^{-1}a'd' + q^{-2}b'c' - b'c') \\
&= ad + bc(-q^{-1}(a'd' - q^{-1}b'c')) \\
&= ad - q^{-1}bc \\
&= 1 = RHS.
\end{aligned}
$$

Hence all equations are satisfied.