# Knot Theory and Quantum Groups

Dominic Goulding

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# Introduction

1. What is a Knot?

2. Abstract Tensor Diagrams

3. Topological Invariance - Applying Knot Theory

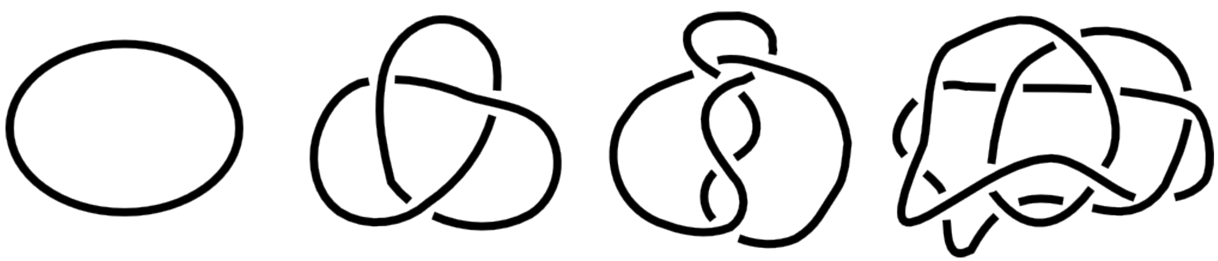
4. The Quantum Group $SL(2)_q$

# 1. What is a Knot?

- A closed curve in $\mathbb{R}^3$, with no self-intersections.

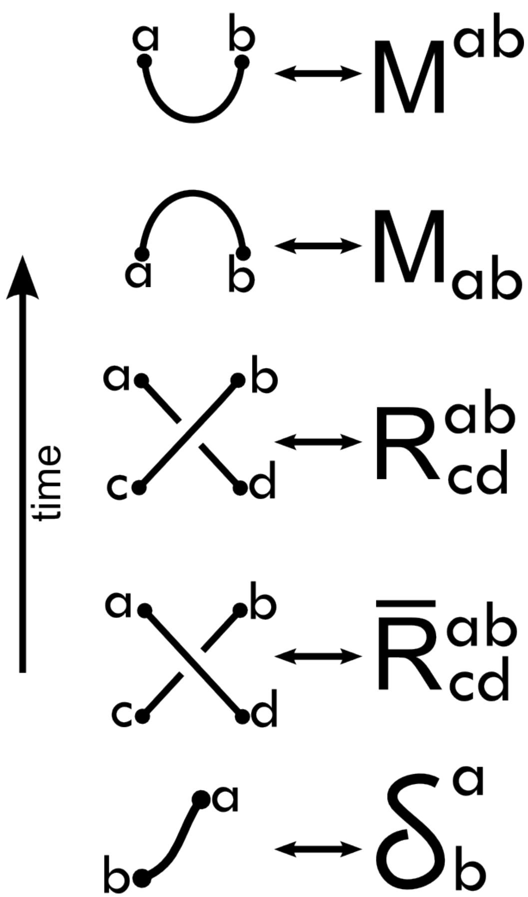
- Like a piece of string, but with the ends fused together.

- We can just look at their projections onto the plane - knot diagrams



# 2. Abstract Tensor Diagrams

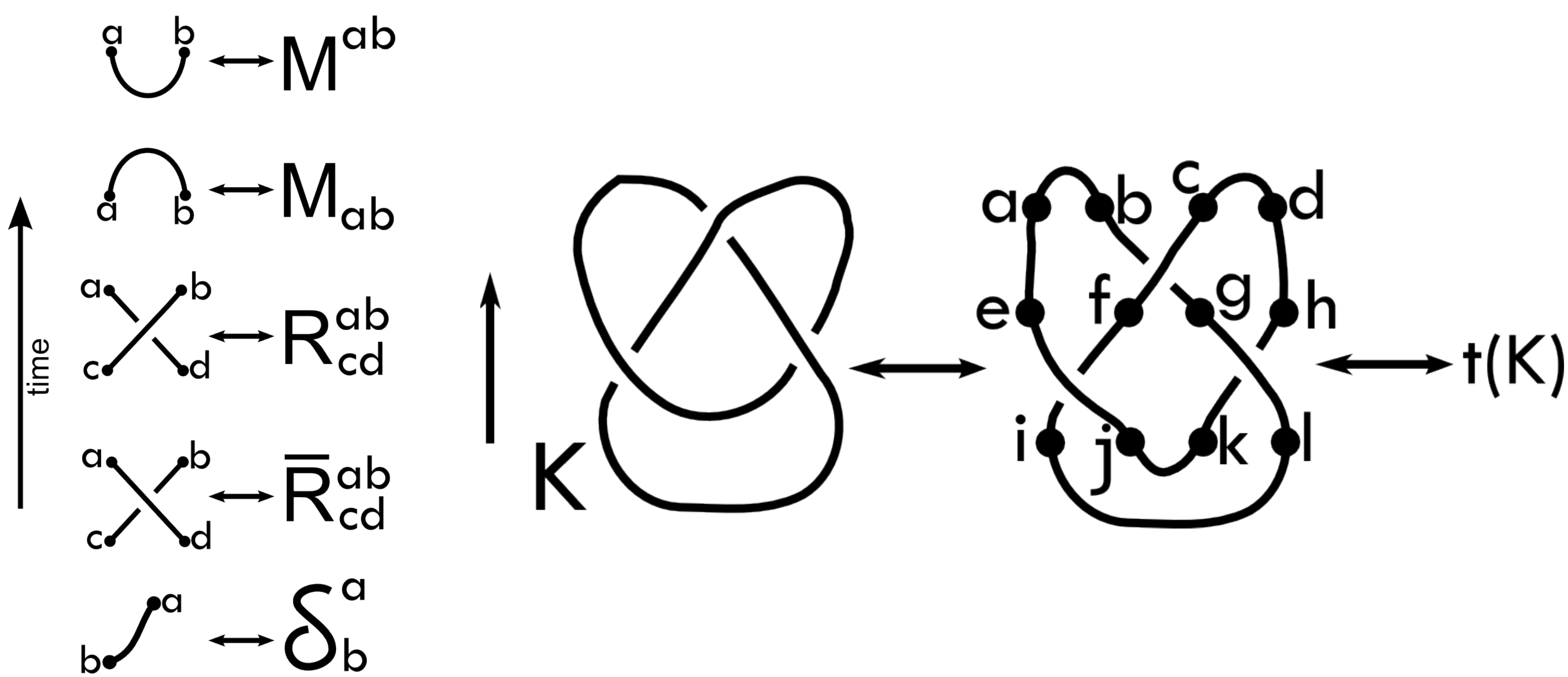
A way to interpret knot diagrams:




- Assign a 'time' direction.
- Associate matrices (tensors).
- Connected strands denote summation over the specified index.

Now, any knot diagram is mapped to a specific contracted tensor t(K).

# Example - the Trefoil Knot

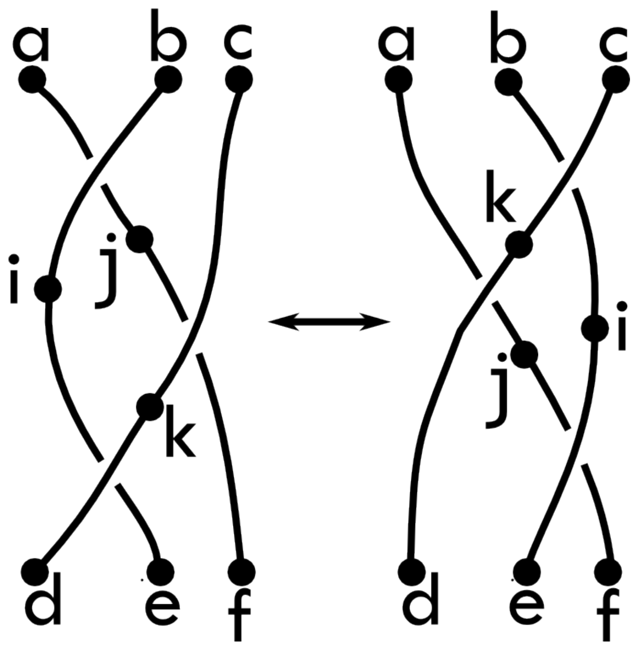



$$t(K) = M_{ab} M_{cd} \delta^{a}_{e} \delta^{d}_{h} R^{bc}_{fg} \bar{R}^{ef}_{ij} \bar{R}^{gh}_{kl} M^{jk} M^{il}$$

## 3. Topological Invariance

We obtain a number of constraints for our matrices.

The most interesting is:

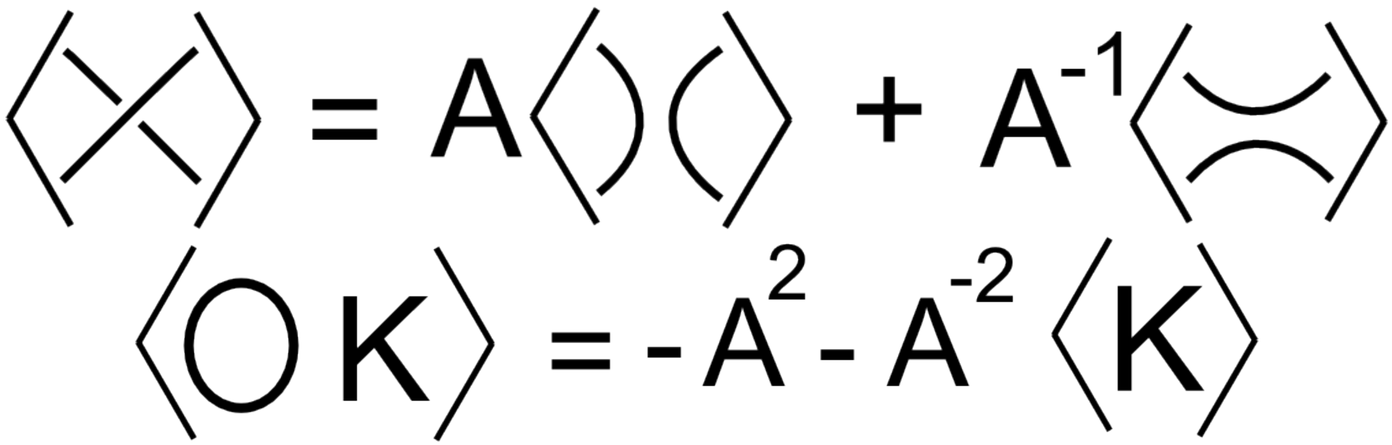



The third Reidemeister move.

$$R_{ij}^{ab} R_{kf}^{jc} R_{de}^{ik} = R_{ki}^{bc} R_{dj}^{ak} R_{ef}^{ji}$$

This is the **Yang-Baxter Equation** - first appeared in the field of statistical mechanics.

# The Bracket Polynomial




So take: $$R^{ab}_{cd} = A\delta^a_c\delta^b_d + A^{-1}M^{ab}M_{cd}$$

$$M_{ab}M^{ab} = -A^2 - A^{-2}$$

$$M_{ak}M^{kb} = \delta^b_a$$

# A Solution

$$M = \begin{bmatrix} 0 & \sqrt{-1}A \\ -\sqrt{-1}A^{-1} & 0 \end{bmatrix}$$

So $M = \sqrt{-1}\tilde{\epsilon}$, with $\tilde{\epsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}$.

## Special case: A=1

$\tilde{\epsilon} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \epsilon$

This is the defining invariant for $SL(2)$ - ie. If P is an element of $SL(2)$, then $P\epsilon P^T = \epsilon$.

Can we find a genealisation of $SL(2)$ that leaves $\tilde{\epsilon}$ invariant?

## 4. The Quantum Group $SL(2)_q$

What matrices will satisfy $P\tilde{\epsilon}P^T = \tilde{\epsilon}$ and $P^T\tilde{\epsilon}P = \tilde{\epsilon}$ ? With:

$$\tilde{\epsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}$$

If $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $q = \sqrt{A}$ then we end up with:

$$\begin{aligned} ba &= qab, & db &= qbd \\ dc &= qcd, & ca &= qac \\ bc &= cb, & ad - da &= (q^{-1} - q)bc \\ ad - q^{-1}bc &= 1 \end{aligned}$$

$a, b, c, d$ are elements of an associative, **non-commutative** algebra.

This (with a few extra things) defines a *quasi-triangular* Hopf Algebra, also known as the **Quantum Group** $SL(2)_q$.

## What is a Hopf Algebra?

A Hopf Algebra is a **bialgebra** with an **antipode**.
(Let $A$ be our algebra $SL(2)_q$), then define:

- Coproduct $\Delta : A \to A \otimes A$

  $$\Delta(P) = \begin{bmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{bmatrix}$$

- Co-unit $\varepsilon : A \to K$ ($K$ is the field for our algebra, such as $\mathbb{C}$)

  $$\varepsilon(P) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Antipode $\gamma : A \to A$

  $$\gamma(P) = \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}$$

End.