How do you Make a Drum Harmonic?

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1. Introducing harmonicity

If a note is played on an instrument then additional vibrations at higher frequencies are produced alongside the fundamental frequency (the pitch of the note). The human ear perceives these as one note instead of hearing the harmonics, constructing the tone quality of the instrument. If an instrument is harmonic, like a violin or a flute, the harmonics are in integer ratios of the fundamental frequency, creating a "tuneful" note. If an instrument is inharmonic, like an ordinary drum, then the harmonics bear no relation to the integer ratio.

2. The Tabla

In northern India a technique for making a drum harmonic evolved in the form of the tabla.

The tabla. The right-handed tabla is on the left.

The tabla has a loaded central circular portion of its membrane that produces a region of greater density. In the case of the right-handed tabla, the centre of the loaded portion is the same as the centre of the whole membrane. The left-handed tabla has a loaded portion that is off-centre.

The loaded region of the tabla

Background: Plot of ψ_{23} mode of the 2 density drum

3. Solution of the ordinary drum

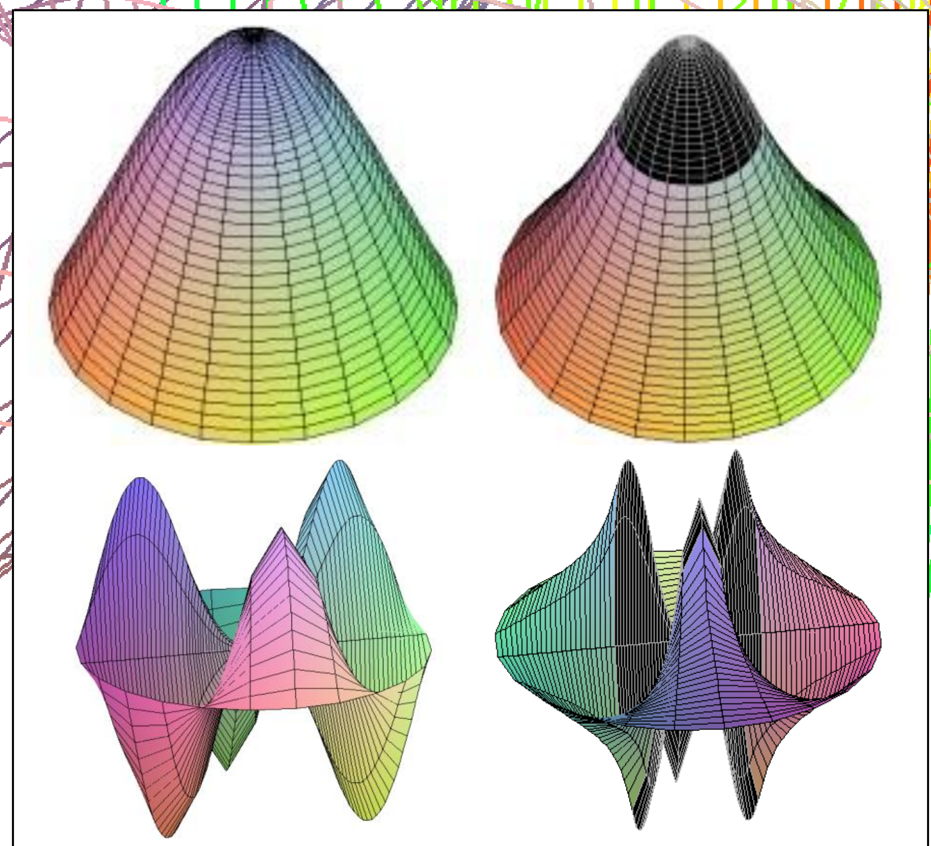
The wave equation describes the displacement of the oscillator (***z***) as a function of its position (***r***,***θ***) and time (***t***)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z$$

For the ordinary drum the imposed boundary conditions are that ***z*** is 2π periodic, finite within $0 \le r \le a$ and at $r = a$ (the edge of the drum) the displacement is zero: ***z***(***a***) = 0. The wave equation is solved through a method of separation of variables and produces the solution

$$z = A_{nj} J_n\left(\frac{\omega r}{c}\right)\sin(\omega t + \phi)\sin(n\theta + \varphi)$$

If the substitution is made that $\lambda = \frac{\omega}{c}$ and $x = \lambda a$ and solved for the boundary condition that ***z***(***a***) = 0 the permitted frequencies of vibration for a given λ and ***a*** can be found for the integers ***n***.



Comparison of drum membrane displacement at a moment in time. The top graphs are the ψ_{01} mode and the bottom are the ψ_{31} mode. The left-hand graphs are the ordinary drum and the right-hand are the 2 density drum.

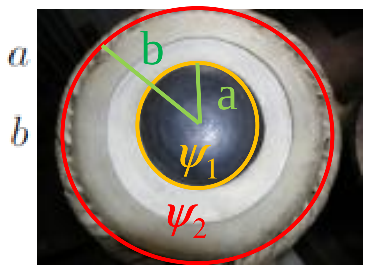
4. Solution of the two density drum

To model the right-handed tabla an approximation is needed. The drum membrane can be modelled as two regions with different densities that are constant in each region. If the variables are separated and ***t*** solved first we can write

$$z = \psi(r,\theta)\sin(\omega t + \phi)$$

The different regions have wave equations

$$\nabla^2\psi_1 + \lambda_1^2\psi_1 = 0 \qquad 0 \le r \le a$$
$$\nabla^2\psi_2 + \lambda_2^2\psi_2 = 0 \qquad a \le r \le b$$
$$\text{where} \quad \lambda_i = \frac{\omega}{c_i}$$



The boundary conditions now imposed are

$$\psi_2(b,\theta) = 0 \qquad \frac{\partial}{\partial r}\psi_1(a,\theta) = \frac{\partial}{\partial r}\psi_2(a,\theta)$$
$$\psi_1(a,\theta) = \psi_2(a,\theta) \qquad \psi(r,\theta) = \psi(r,\theta + 2\pi n)$$

and the function must again be finite within the boundary of the drum. To solve this a substitution is made to get a single variable which relates to the boundary of the drum. The following variables are used.

$$x = \lambda_2 b = \frac{\omega b}{c_2} \qquad k = \frac{a}{b} \qquad \sigma^2 = \frac{\rho_1}{\rho_2} = \frac{\lambda_1^2}{\lambda_2^2}$$

The equation obtained from these conditions is

$$\sigma\frac{J_{n-1}(\sigma k x)}{J_n(\sigma k x)} = \frac{Y_n(x)J_{n-1}(kx) - J_n(x)Y_{n-1}(kx)}{Y_n(x)J_n(kx) - J_n(x)Y_n(kx)}$$

This equation can then be solved for ***x*** to find the permitted frequencies of the drum at each integer ***n***. The time-independent displacement equations for the different regions of the membrane are

$$\psi_1(r,\theta) = A_{nj} J_n\left(\frac{\sigma k x_{nj}}{a} r\right)\sin(n\theta + \phi_n)$$

$$\psi_2(r,\theta) = A_{nj}\frac{J_n(\sigma k x_{nj})}{J_n(k x_{nj})Y_n(x_{nj}) - J_n(x_{nj})Y_n(k x_{nj})}.$$
$$\left[Y_n(x_{nj})J_n\left(\frac{x_{nj}}{b} r\right) - J_n(x_{nj})Y_n\left(\frac{x_{nj}}{b} r\right)\right]\sin(n\theta + \phi_n)$$

5. Comparison of errors

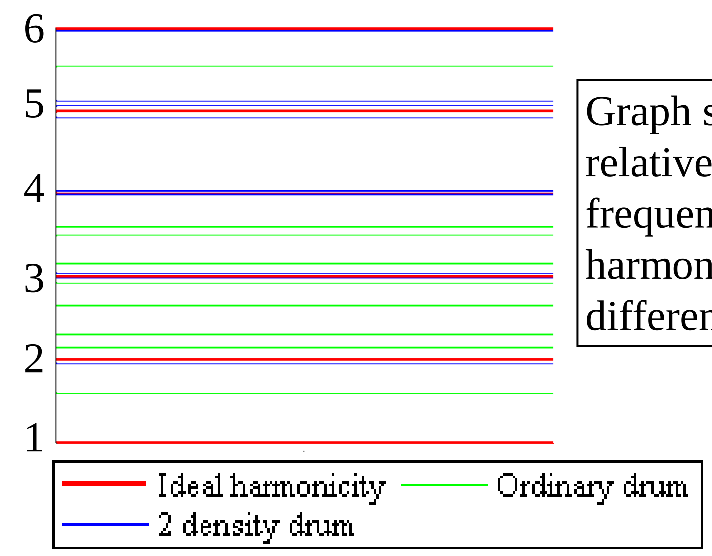
This is a comparative table of the relative frequencies for the different drum models and the error for each. The values of the constants have been optimized to produce the lowest error and the fundamental frequency is defined as frequency 1.000.

Mode of vibration	Relative frequencies of ordinary drum	Relative frequencies of 2 density drum
ψ_{01}	1.000	1.000
ψ_{02}	2.295	3.038
ψ_{11}	1.593	1.951
ψ_{12}	2.917	3.999
ψ_{13}	5.540	5.120
ψ_{21}	2.136	2.984
ψ_{22}	3.500	4.917
ψ_{31}	2.653	4.034
ψ_{41}	3.155	5.066
Harmonic error in the drum for these modes	0.1035	0.001891

The error is found over all the degenerate modes up to the 5th harmonic for the two density drum

$$\text{Harmonic Error} = \sum_{h=2}^{5}\sum_{d=1}^{D}\left(\frac{x_{nj}/x_{01} - h}{h}\right)^2$$

The error here for the ordinary drum is taken over the same modes.



Graph showing the relative frequencies of harmonics for the different drums.

6. Conclusion

The two-density model is shown to be a good approximation for the right-handed tabla and produces an almost harmonic note. The model does not achieve perfect harmonicity and fewer approximations could produce a superior result.

References: Ramakrishna, B. S. and Sondhi, M. M.; Vibrations of Indian Musical Drums Regarded as Composite Membranes, *Journal of the Acoustical Society of America*, **26**, 2, 523-529, (1954). Gaudet, S., et al.; The evolution of harmonic Indian musical Drums: A mathematical perspective, *Journal of Sound and Vibration*, **291**, 388-394, (2006). Benson, D. J.; Music: A Mathematical Offering, *Cambridge University Press*, (2007).