Hydrodynamic Stability of Newtonian and Non-Newtonian fluids

Project IV Presentation

Julian Mak

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$$\underline{\tau} = \mu\underline{\dot{\gamma}}, \quad \mu = constant$$

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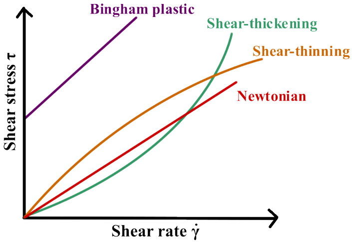
$$\underline{\tau} = \mu \underline{\dot{\gamma}}, \quad \mu = \textit{constant}$$

Non-Newtonian fluid

$$\underline{\tau} = \mu(\dot{\gamma})\underline{\dot{\gamma}}, \quad \dot{\gamma} = \textit{second invariant of } \underline{\dot{\gamma}}$$

Stress-dependent viscosity



some examples:

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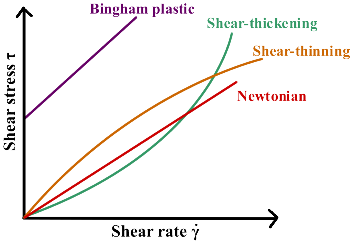

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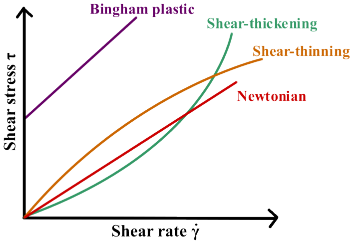

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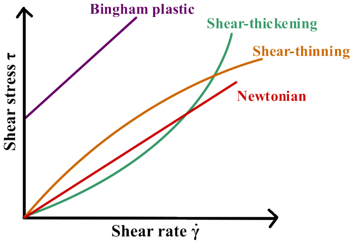

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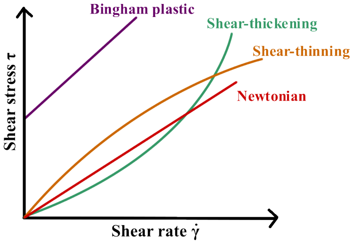

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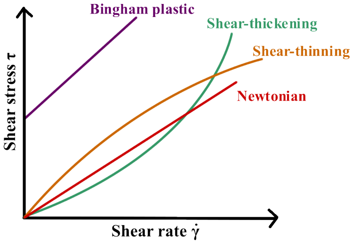

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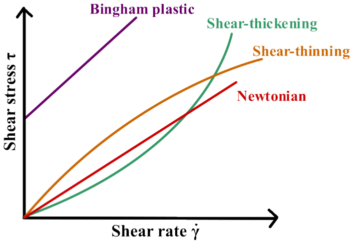

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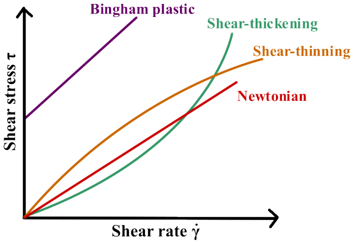

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- custard

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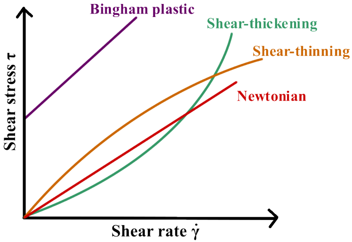

some examples:

- gel
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- egg white
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- “silly putty”

Carreau law

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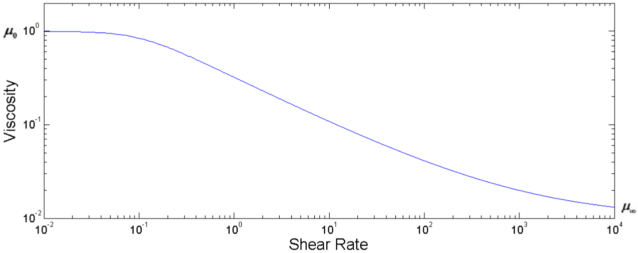
- *Carreau law* ($n = 1$ or $\lambda = 0$ is the Newtonian case):

$$\mu = \frac{\hat{\mu}_\infty}{\hat{\mu}_0} + \left[1 - \frac{\hat{\mu}_\infty}{\hat{\mu}_0}\right] [1 + (\lambda\dot{\gamma})^2]^{(n-1)/2} \tag{1}$$



Stability Analysis

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- *General momentum equation* for incompressible fluid is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} = -\nabla p + \frac{1}{Re} \nabla \cdot \underline{\tau}$$

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- $\mu = const$ for Newtonian fluids, reducing above to the (non-dimensional, unforced) *Navier-Stokes equation*:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} = -\nabla p + \frac{1}{Re} \Delta u,$$

  where the Reynolds number is given by

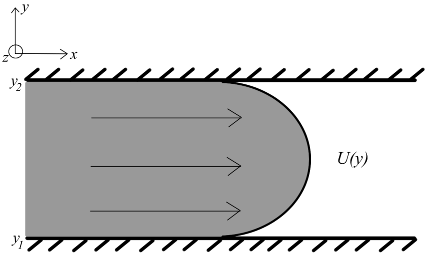
$$Re = \rho L V / \mu_0.$$

  L = length scale, V = velocity scale
  ρ = fluid density (assumed constant due to incompressibility)

- Base solution (parallel shear flows):

$$\boldsymbol{U} = U(y)\boldsymbol{e}_x, \qquad -1 \leq y \leq 1$$

with no slip boundary conditions.



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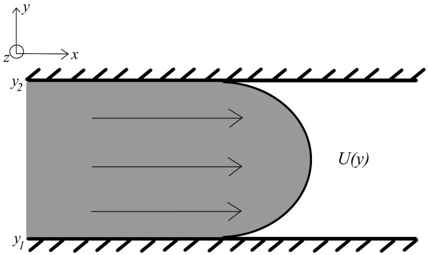

- This reduces the momentum equations to the following:

Newtonian

$$Re\frac{\partial p}{\partial x} = \frac{\partial^2 U}{\partial^2 y}$$

non-Newtonian

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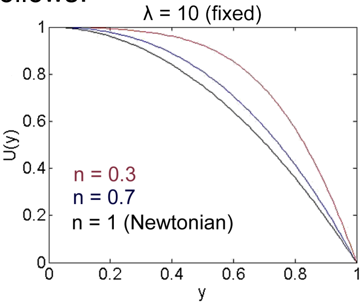
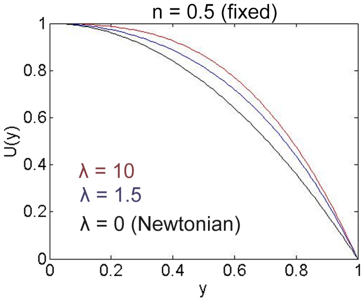
- We solve for U. The plots of the base solutions are as follows:





- We perturb the base flow by letting:

$$\mathbf{u}(x, y, z, t) = \mathbf{U} + \tilde{\mathbf{u}}(x, y, z, t)$$
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- Temporal analysis: spatial wave numbers real, but $\omega = \omega_r + i\omega_i$.
- Unstable is when $\omega_i > 0$, stable otherwise.
- Substitute into linearised momentum equation, eliminate $\tilde{p}$, tidy up and get...

Orr-Sommerfeld and Squire equation

$$\begin{pmatrix} \mathcal{L}_{OS} & 0 \\ \beta(DU) & \mathcal{L}_{SQ} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} = \omega \begin{pmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix}$$

$$\mathcal{L}_{OS} = \alpha U(D^2 - k^2) - \alpha(D^2 U) - \frac{1}{iRe}(D^2 - k^2)^2$$

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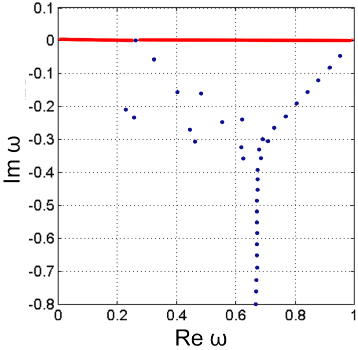
- Along with boundary conditions $\hat{v} = D\hat{v} = 0$, this is a *generalised eigenvalue problem* with eigenvalue ω.

Results



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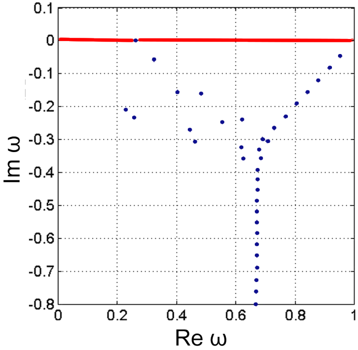

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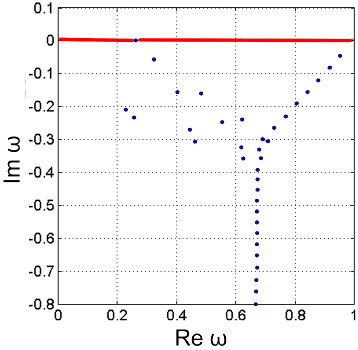

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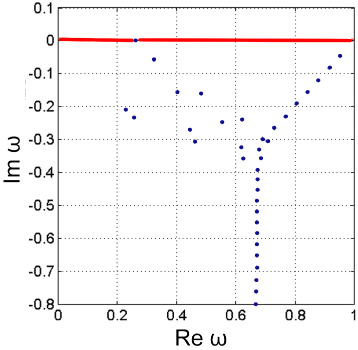

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- We used α = 1, β = 0, and $Re = 5772$.
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- Red line is the line of marginal stability, $\omega_i = 0$.

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$$\begin{aligned}
\mathcal{L} &= \alpha U \Delta - \alpha (D^2 U) \\
&\quad - \frac{1}{iRe}[\mu \Delta^2 + 2 D\mu D^3 + D^2\mu D^2 - 2k^2 D\mu D + k^2 D^2 \mu] \\
&\quad - \frac{\alpha^2}{iRe\, k^2}(D^2 + k^2)[(\mu_t - \mu)(D^2 + k^2)] \\
\mathcal{E}_1 &= \frac{\alpha\beta}{iRe\, k^2}(D^2 + k^2)[(\mu_t - \mu) D] \\
\mathcal{E}_2 &= \beta(DU) + \frac{\alpha\beta}{iRe\, k^2} D[(\mu_t - \mu)(D^2 + k^2)] \\
\mathcal{S} &= \alpha U - \frac{1}{iRe}[\mu\Delta + D\mu D] + \frac{1}{iRe}\frac{\beta^2}{k^2} D[(\mu_t - \mu) D]
\end{aligned}$$

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- Question time?