Hydrodynamic Stability of Newtonian and Non-Newtonian fluids

Project IV Presentation

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Non-Newtonian fluid?

What is a fluid?

Stress tensor
\[ \sigma = -pI + \tau \]

Rate of strain tensor
\[ \dot{\gamma} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right) \]

\( p \) = pressure,
\( I \) = identity tensor,
\( \tau \) = extra stress tensor
\( \nabla \) = gradient operator,
\( u \) = velocity field

Newtonian fluid
\[ \tau = \mu \dot{\gamma} \]
\( \mu \) = constant

Non-Newtonian fluid
\[ \tau = \mu(\dot{\gamma}) \dot{\gamma} , \dot{\gamma} = \text{second invariant of} \ \dot{\gamma} \]
Non-Newtonian fluid?

- What is a fluid?
Non-Newtonian fluid?

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Non-Newtonian fluid?

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Non-Newtonian fluid?

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- Stress tensor - $\sigma = -pI + \tau$
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- $\rho = \text{pressure}$, $I = \text{identity tensor}$, $\tau = \text{extra stress tensor}$
- $\nabla = \text{gradient operator}$, $u = \text{velocity field}$
Non-Newtonian fluid?

- What is a fluid?
- **Stress tensor** - $\sigma = -pl + \tau$
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**Newtonian fluid**

$$\tau = \mu \dot{\gamma}, \quad \mu = constant$$
Non-Newtonian fluid?

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Introduction
Stability analysis
Newtonian fluid
non-Newtonian fluids

Stress-dependent viscosity

some examples:

[Graph showing different types of viscosity behavior: Newtonian, Shear-thinning, Shear-thickening, Bingham plastic]
Stability analysis

Newtonian fluid

non-Newtonian fluids

Stress-dependent viscosity

some examples:

- gel
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Stress-dependent viscosity

some examples:
- gel
- oil paint

![Graph showing stress vs shear rate for different fluid types: Bingham plastic, Shear-thickening, Shear-thinning, Newtonian.](image)
Stress-dependent viscosity

some examples:
- gel
- oil paint
- blood
Stress-dependent viscosity

some examples:

- gel
- oil paint
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- mayonnaise
Stress-dependent viscosity

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- oil paint
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- egg white
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Stress-dependent viscosity

some examples:
- gel
- oil paint
- blood
- mayonnaise
- magma
- egg white
- custard
- “silly putty”
Carreau law

For shear thinning fluids, we require a model for the viscosity function.

\[
\mu = \hat{\mu}_\infty \hat{\mu}_0 + \left[1 - \frac{\hat{\mu}_\infty \hat{\mu}_0}{1 + \left(\frac{\lambda \dot{\gamma}}{n}\right)^2 \left(\frac{n-1}{2}\right)}\right]
\]
For shear thinning fluids, we require a model for the viscosity function.

Carreau law ($n = 1$ or $\lambda = 0$ is the Newtonian case):

$$\mu = \frac{\mu_\infty}{\mu_0} + \left[1 - \frac{\mu_\infty}{\mu_0}\right] \left[1 + (\lambda \dot{\gamma})^2\right]^{(n-1)/2}$$

(1)
Stability Analysis

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General momentum equation for incompressible fluid is

\[
\frac{\partial}{\partial t} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla \cdot \mathbf{\tau}
\]

\[
\mu = \text{const}
\]

for Newtonian fluids, reducing above to the (non-dimensional, unforced) Navier-Stokes equation:

\[
\frac{\partial}{\partial t} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u},
\]

where the Reynolds number is given by

\[
\text{Re} = \frac{\rho L V}{\mu_0}
\]

\(L\) = length scale,
\(V\) = velocity scale
\(\rho\) = fluid density (assumed constant due to incompressibility)
Stability Analysis

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Re = \rho LV / \mu_0.
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\( L = \) length scale, \( V = \) velocity scale
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Base solution (parallel shear flows):

\[ U = U(y)e_x, \quad -1 \leq y \leq 1 \]

with no slip boundary conditions.
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\mathbf{U} = U(y) \mathbf{e}_x, \quad -1 \leq y \leq 1
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with no slip boundary conditions.

This reduces the momentum equations to the following:
Newtonian

\[
Re \frac{\partial p}{\partial x} = \frac{\partial^2 U}{\partial^2 y}
\]

non-Newtonian

\[
Re \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \mu(\dot{\gamma}) \frac{\partial U}{\partial y} \right)
\]
Stability analysis

Newtonian fluid

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non-Newtonian fluids

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- We solve for \( U \). The plots of the base solutions are as follows:
We perturb the base flow by letting:

\[ u(x, y, z, t) = U + \tilde{u}(x, y, z, t) \]

\[ p = p_0 + \tilde{p}. \]
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Sub this new \( u \) in to momentum equation, assume \( \tilde{u} \) small and linearise.
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Plane geometry so consider a Fourier expansion of the disturbance:

\[ \tilde{u}(x, y, z, t) = \hat{u}(y) \ e^{i(\alpha x + \beta z - \omega t)}. \]
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Unstable is when \( \omega_i > 0 \), stable otherwise.
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Substitute into linearised momentum equation, eliminate \( \tilde{p} \), tidy up and get...
Orr-Sommerfeld and Squire equation

\[
\begin{pmatrix}
\mathcal{L}_{OS} & 0 \\
\beta(DU) & \mathcal{L}_{SQ}
\end{pmatrix}
\begin{pmatrix}
\hat{v} \\
\hat{\eta}
\end{pmatrix}
= \omega
\begin{pmatrix}
D^2 - k^2 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{v} \\
\hat{\eta}
\end{pmatrix}
\]

\[
\mathcal{L}_{OS} = \alpha U(D^2 - k^2) - \alpha (D^2 U) - \frac{1}{iRe}(D^2 - k^2)^2
\]

\[
\mathcal{L}_{SQ} = \alpha U - \frac{1}{iRe}(D^2 - k^2)
\]

\[
k^2 = \alpha^2 + \beta^2,
\]

\[
D = \frac{d}{dy}
\]

\[
\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.
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\[
\mathcal{L}_{OS} = \alpha U(D^2 - k^2) - \alpha(D^2 U) - \frac{1}{i\text{Re}}(D^2 - k^2)\]

\[
\mathcal{L}_{SQ} = \alpha U - \frac{1}{i\text{Re}}(D^2 - k^2)
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\[
k^2 = \alpha^2 + \beta^2, \quad D = \frac{d}{dy}
\]

\[
\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.
\]

Along with boundary conditions \(\hat{\nu} = D\hat{\nu} = 0\), this is a \textit{generalised eigenvalue problem} with eigenvalue \(\omega\).
Results

- Eigenvalue spectrum for Poiseuille flow, $U(y) = 1 - y^2$. 

$\alpha = 1$, $\beta = 0$, and $Re = 5772$. Done using Chebyshev spectral collocation method in MATLAB. 

![Graph showing eigenvalue spectrum](Image)
Results

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- We used \( \alpha = 1 \), \( \beta = 0 \), and \( Re = 5772 \).
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- Red line is the line of marginal stability, \( \omega_i = 0 \).
non-Newtonian fluids

\begin{align*}
\begin{pmatrix}
\mathcal{L} & \mathcal{E}_1 \\
\mathcal{E}_2 & S
\end{pmatrix}
\begin{pmatrix}
\hat{v} \\
\hat{\eta}
\end{pmatrix}
= \omega
\begin{pmatrix}
D^2 - k^2 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{v} \\
\hat{\eta}
\end{pmatrix},
\end{align*}

\mathcal{L} = \alpha U \Delta - \alpha (D^2 U)
- \frac{1}{iRe} \left[ \mu \Delta^2 + 2D \mu D^3 + D^2 \mu D^2 - 2k^2 D \mu D + k^2 D^2 \mu \right]
- \frac{\alpha^2}{iRe k^2} (D^2 + k^2) [\mu_t - \mu] (D^2 + k^2)

\mathcal{E}_1 = \frac{\alpha \beta}{iRe k^2} (D^2 + k^2) [\mu_t - \mu] D

\mathcal{E}_2 = \beta (DU) + \frac{\alpha \beta}{iRe k^2} D [\mu_t - \mu] (D^2 + k^2)

S = \alpha U - \frac{1}{iRe} \left[ \mu \Delta + D \mu D \right] + \frac{1}{iRe k^2} \beta^2 D [\mu_t - \mu] D
We take the base profile before as data points and work out the corresponding values of $\mu$, $D\mu$, $\mu_t$ etc.
The work to be done

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- Code it into MATLAB and compute eigenvalues.
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- Nouar et al. (2007) reported that shear thinning viscosity stabilises the flow.
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NOTE: All the above is a linear theory.
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Question time?