Durham University

REGULATORS, DEDEKIND ZETA VALUES AND POLYLOGARITHMS

John Rhodes
St Chad's College
February 20, 2008

Supervisor: Dr. Herbert Gangl

Geometric Representation of an Algebraic Number Field

Definition: An element $\alpha \in \mathbb{C}$ is called an *algebraic number* if it satisfies $f(\alpha) = 0$ for some $f(x) \in \mathbb{Q}[x]$. A field K with $\mathbb{C} \supset K \supset \mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$ is called an *algebraic number field*. Typically, an algebraic number field is of the form:

$$K = \mathbb{Q}(\theta) = \frac{\mathbb{Q}[x]}{p_\theta(x)}$$

where $p_\theta(x)$ is the minimum polynomial for θ.
It is possible to geometrically represent an algebraic number field in a logarithmic subspace of $\mathbb{R}^{r_1+r_2}$, where r_1 and r_2 are, respectively, the number of real and pairs of complex embeddings of the number field (when the number field is given using a minimum polynomial these relate to the number of real and pairs of complex roots). For $\alpha \in K$, we construct co-ordinates:

$$l(\alpha) = (l_1(\alpha), \ldots, l_{r_1+r_2}(\alpha))$$

where:

- $l_i(\alpha) = \begin{cases} ln|\sigma_i(\alpha)| & \text{if } i = 1, \ldots, r_1 \\ ln|\sigma_i(\alpha)|^2 & \text{if } i = r_1+1, \ldots, r_1+r_2 \end{cases}$
- $\sigma_i(\alpha)$ is a real embedding of alpha for $i = 1, \ldots, r_1$ and a complex embedding for $i = r_1+1, \ldots, r_1+r_2$.

The geometric representation only really becomes useful however, when we look at the representation of the units. For this we first turn to a famous theorem by Johann Dirichlet:

Dirichlet's Unit Theorem:

For an algebraic number field K of degree $n = r_1 + 2r_2$, there exists units $\varepsilon_1, \ldots, \varepsilon_r$ with $r = r_1 + r_2 - 1$ such that every unit $u \in \mathcal{O}_K$ has a unique representation in the form:

$$u = \zeta\varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r}$$

where $a_1, \ldots, a_r \in \mathbb{Z}$ and ζ is some root of unity contained in $\mathcal{O}_K$, the ring of integers.

Proposition: For a unit $u = \zeta\varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r} \in \mathcal{O}_K$, with $a_1, \ldots, a_r \in \mathbb{Z}$ as in Dirichlet's Theorem. Then:

$$l(u) = a_1 l(\varepsilon_1) + \ldots + a_r l(\varepsilon_r)$$

i.e. the units map to a lattice of dimension $r = r_1 + r_2 - 1$ under the representation. This lattice will have fundamental parallelepipeds of volume v, which define the Dirichlet Regulator, R:

$$R = \frac{v}{\sqrt{r_1 + r_2}}$$

This important invariant gives us an idea of the 'size of the units' of the algebraic number field.

The Analytic Class Number Formula

This can be written in several forms, the one easiest to understand is the following:

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{(r_1+r_2-1)}} = -\frac{h_K R}{w_K}$$

where:

- $\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$ is the Dedekind Zeta function for the field K, where p runs over all the ideals in $\mathcal{O}_K$ and $N(\mathfrak{a})$ is the ideal norm of $\mathfrak{a}$. See [1] for a good description. A result by Dirichlet shows it has a zero at $s = 0$ of order $r_1 + r_2 - 1$, meaning the left-hand side of the above equation is a finite real value.
- h_K is the class number of K, an important invariant which measures to what extent unique factorisation fails in K.
- w_K is the number of roots of unity contained in K.
- R is the Dirichlet Regulator previously described.

The Dirichlet Class Number formula provides quite an astonishing link between very important invariants of an algebraic number field and the Dedekind Zeta function.

A specific example

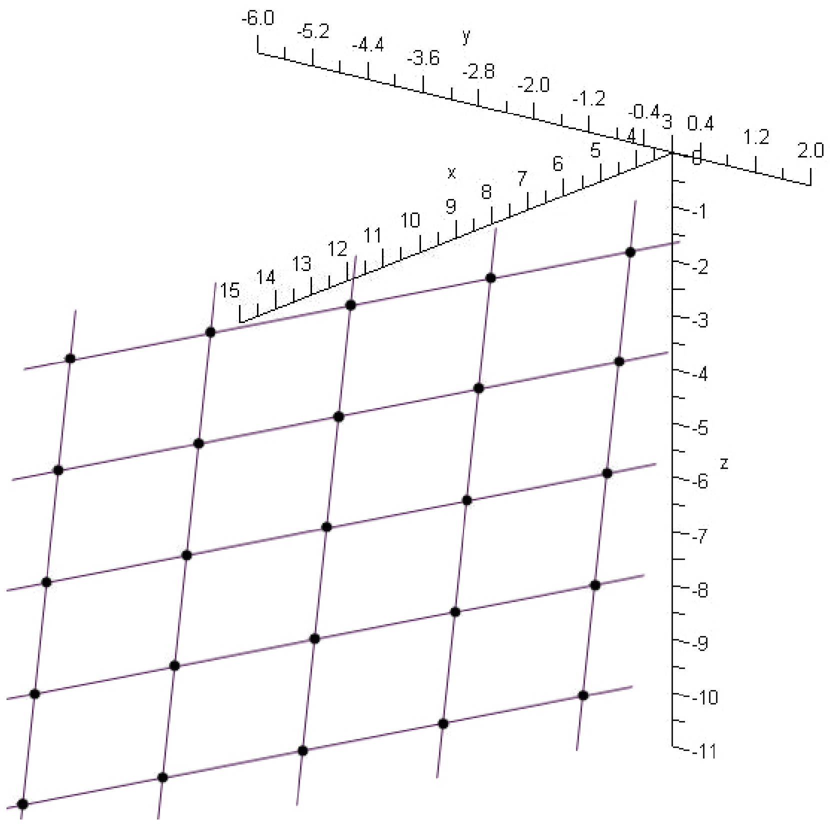
For the algebraic number field, $K = \mathbb{Q}(\theta)$ with

$$\theta^6 + \theta^5 + \theta^4 + \theta^3 + \theta^2 + \theta + 1 = 0$$

we find that, since $r_1 = 0$ and $r_2 = 3$, that we get a 2-dimensional lattice. The units $u_1 = 1 + \theta$ and $u_2 = 1 + \theta + \theta^2$ can be used as a system of fundamental units of this field.
We get:

$$\begin{aligned} l(u_1) &= [1.177725212, 0.4414486199, -1.619173833] \\ l(u_2) &= [1.619173832, -1.177725212, -0.4414486206] \end{aligned}$$

Our specific example, $K = \mathbb{Q}(\theta)$ then gives the following 2-dimensional lattice in $\mathbb{R}^3$, with units represented by lattice points:



Verification of the Analytic Class Number Formula

We can now verify the Analytic Class Number Formula for the specific example above. We also have that $h_K = 1$, and the roots of unity in the field are generated by -1 and the primitive seventh root of unity, giving $w_K = 14$. Using Maple we find that the volume of a fundamental parallelepiped is $v = 3.64056828$, giving $R = 2.101818729$. So we get

$$\lim_{s\to 0} \frac{\zeta_{\mathbb{Q}(\theta)}(s)}{s^2} = -\frac{h_K R}{w_K} = -\frac{2.101818729}{14} = -0.1501299092$$

Using GP/PARI we can calculate Dedekind Zeta values directly, by making s very small we can find an evaluation very near to the limit at $s = 0$. For $s = 10^{-21}$ we get

$$\frac{\zeta_K(s)}{s^2} = -0.1501299092,$$

which corroborates our result to 10 decimal places.

Going higher with Polylogarithms

We define the *Polylogarithm* $Li_m(z)$ by:

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad z \in \mathbb{C}, |z| < 1, m \in \mathbb{N}$$

We also note that the monologarithm, $Li_1(z)$ equals $-ln(1-z)$ with $z \in \mathbb{C}$, $|z| < 1$. Polylogarithms have been found to appear in areas of mathematics as varied as as Feynman integrals, volumes of hyperbolic manifolds and algebraic geometry. Don Zagier remarked:

"The dilogarithm is perhaps the only mathematical function with a sense of humor"

—in relation to the variety and oddness of the appearance of the dilogarithm.

So where do Polylogarithms fit in?

You will notice that the Geometric representation we used was a logarithmic representation. From this, we managed to reach an evaluation of the Dedekind Zeta function. There is a strong relation between $\zeta_K(s)$ and $\zeta_K(1-s)$ (as with the Riemann Zeta Function, a specific Dedekind Zeta Function), so we can view our calculation in the Analytic Class Number Formula as an evaluation of the Dedekind Zeta Function at the value $s = 1$.

Motivational idea: If we can use the natural logarithm to calculate the Dedekind Zeta Function at $s = 1$, can we use the dilogarithm to calculate evaluations at $s = 2$?

Through the work of Don Zagier, Spencer Bloch and others, we can answer yes to this question. Where before we used the logarithm of units of the field, it is possible to take a 'higher logarithm' of 'higher units' to obtain 'higher regulators' and 'higher Dedekind Zeta values'. Below is a specific evaluation of a Dedekind zeta value at $s = -1$ (which is, in a sense, at $s = 2$), using the dilogarithm, for a simple quadratic extension of $\mathbb{Q}$.

A specific example

An example, taken from [3], for the algebraic number field $K = \mathbb{Q}(\sqrt{-7})$:

$$\lim_{s\to -1} \frac{6\pi \cdot \zeta_K(s)}{s} = 2D\left(\frac{1+\sqrt{-7}}{2}\right) + D\left(\frac{-1+\sqrt{-7}}{4}\right)$$

The function $D(x)$ is the Bloch-Wigner function, a modification of the dilogarithm:

$$D(x) = \Im(\mathrm{Li}_2(x)) + \arg(1-x)\ln|x| \quad x \in \mathbb{C}, x \notin \{0, 1\}$$

It is possible to see the relation between this evaluation and the analytic class number formula, here we obtain a 'higher regulator' by taking a modified dilogarithm of the 'higher unit':

$$2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{2}\right]$$

Even higher?

There are also results concerning the calculation of the Dedekind zeta value at $s = 3$ using the trilogarithm. There is much research to be done on taking this fascinating idea to higher and higher values.

References

[1] Z.I. Borevich and I. R. Shafarevich. *Number Theory*. Academic Press, 1966.

[2] Don Zagier. The remarkable dilogarithm. *J. Math and Phys. Soc. 22*, pages 131–145, 1988.

[3] Don Zagier. The Bloch-Wigner-Ramakrishnan polylogarithm function. *Math-Annalen*, pages 612–624, 1990.