Polylogarithms

and The Geometric Representation of Algebraic Number Fields

John Rhodes

Durham University

Overview

- Geometric Representation of an Algebraic Number Field
- Specific Example
- The Analytic Class Number Formula
- What is a Polylogarithm?
- Going Higher

Geometric Representation of an Algebraic Number Field

- An element $\alpha \in \mathbb{C}$ is called an *algebraic number* if it satisfies $f(\alpha) = 0$ for some $f(x) \in \mathbb{Q}[x]$
- A field K with $\mathbb{C} \supset K \supset \mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$ is called an *algebraic number field*.
- Typically an algebraic number field is of the form:

$$K = \mathbb{Q}(\theta) = \frac{\mathbb{Q}[x]}{p_\theta(x)}$$

  Where $p_\theta(x)$ is the minimum polynomial for θ.
- Example: $\mathbb{Q}(\theta)$ where $\theta^3 = 2$

Durham University

Geometric Representation of an Algebraic Number Field

- An element $\alpha \in \mathbb{C}$ is called an *algebraic number* if it satisfies $f(\alpha) = 0$ for some $f(x) \in \mathbb{Q}[x]$
- A field K with $\mathbb{C} \supset K \supset \mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$ is called an *algebraic number field*.
- Typically an algebraic number field is of the form:

$$K = \mathbb{Q}(\theta) = \frac{\mathbb{Q}[x]}{p_\theta(x)}$$

  Where $p_\theta(x)$ is the minimum polynomial for θ.
- Example: $\mathbb{Q}(\theta)$ where $\theta^3 = 2$
- Idea: Can we represent an algebraic number field geometrically, say, in $\mathbb{R}^n$?

Durham University

A Specific Example

Introduce the algebraic number field $\mathbb{Q}(\theta)$ with θ a root of:

$$\theta^7 = 1$$

A Specific Example

Introduce the algebraic number field $\mathbb{Q}(\theta)$ with θ a root of:

$$\theta^6 + \theta^5 + \theta^4 + \theta^3 + \theta^2 + \theta + 1 = 0$$

A Specific Example

Introduce the algebraic number field $\mathbb{Q}(\theta)$ with θ a root of:

$$\theta^6 + \theta^5 + \theta^4 + \theta^3 + \theta^2 + \theta + 1 = 0$$

We shall denote the primitive 7^{th} root of unity $\zeta_7 \in \mathbb{C}$

Properties:

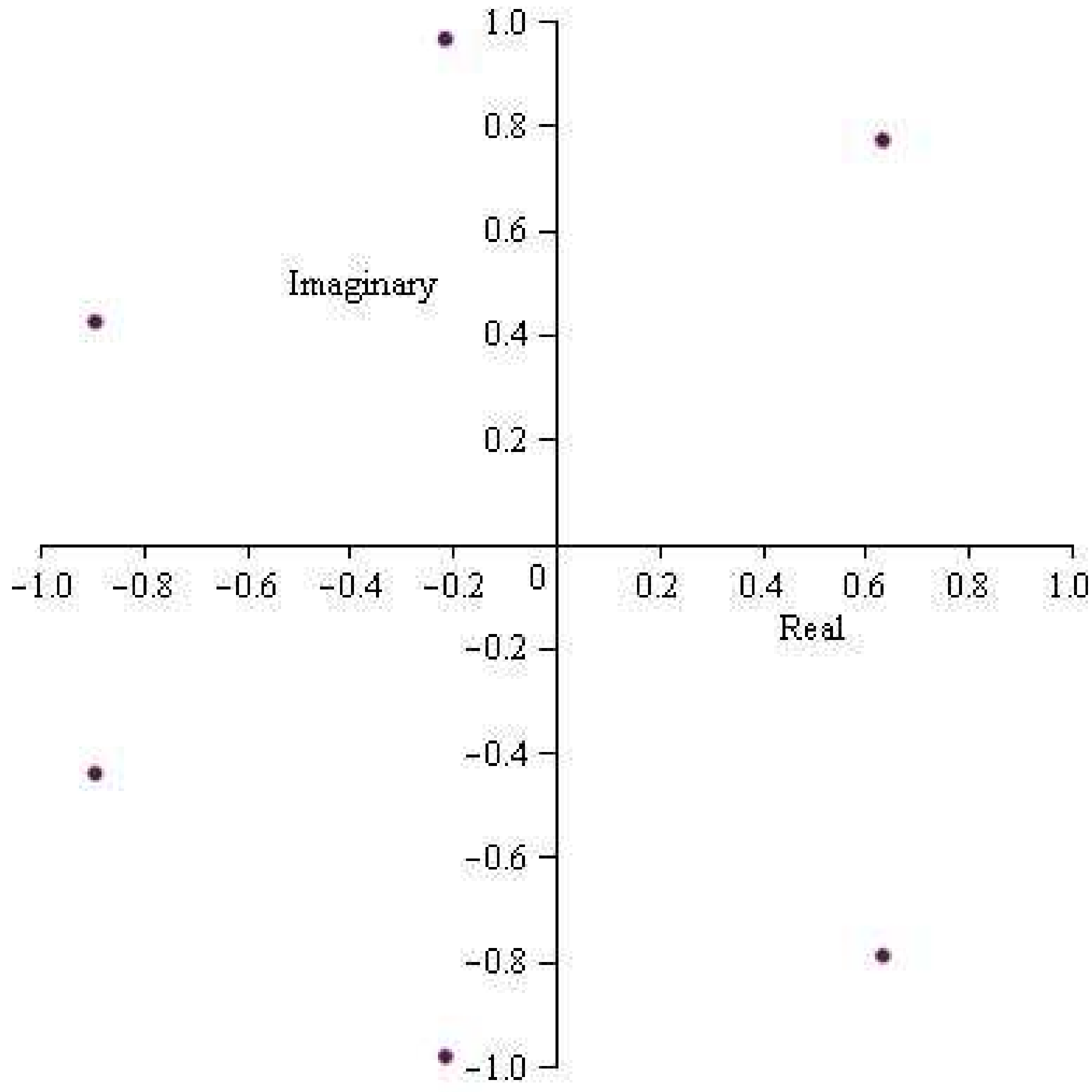
- $r_1 = 0$, $r_2 = 3$, $[\mathbb{Q}(\theta) : \mathbb{Q}] = 6$
- Six embeddings into $\mathbb{C}$:

$$\sigma_1, \overline{\sigma_1}, \sigma_2, \overline{\sigma_2}, \sigma_3, \overline{\sigma_3} : \mathbb{Q}(\theta) \hookrightarrow \mathbb{C}$$

- $\sigma_1(\theta) = \zeta_7$, $\sigma_2(\theta) = \zeta_7^2$, $\sigma_3(\theta) = \zeta_7^3$

Durham University

Plot of Seventh Roots of Unity



The Logarithmic Space

For $\alpha \in \mathbb{Q}(\theta)$, define a vector $l(\alpha) \in \mathbb{R}^3$ as:

$$l(\alpha) := (ln|(\sigma_1(\alpha))|^2, ln|(\sigma_2(\alpha))|^2, ln|(\sigma_3(\alpha))|^2)$$

Durham University

The Logarithmic Space

For $\alpha \in \mathbb{Q}(\theta)$, define a vector $l(\alpha) \in \mathbb{R}^3$ as:

$$l(\alpha) := (ln|(\sigma_1(\alpha))|^2, ln|(\sigma_2(\alpha))|^2, ln|(\sigma_3(\alpha))|^2)$$

Properties:

- $l(\alpha\beta) = l(\alpha) + l(\beta)$
- $l(\alpha^a) = al(\alpha)$
- $\sum_{i=1}^{r_1+r_2} l_i(\alpha) = ln|N(\alpha)|$

Durham University

Representation of Units

Obvious units in $\mathbb{Q}(\theta)$:

$$\pm 1, \pm\zeta_7, \pm\zeta_7^2, \pm\zeta_7^3, \pm\zeta_7^4, \pm\zeta_7^5, \pm\zeta_7^6$$

Other units:

$$u_1 = 1 + \theta \quad \text{and} \quad u_2 = 1 + \theta + \theta^2$$

For a generic unit $u = \pm\zeta_7^a u_1^b u_2^c$ with $a, b, c \in \mathbb{Z}$:

Durham University

Representation of Units

Obvious units in $\mathbb{Q}(\theta)$:

$$\pm 1, \pm\zeta_7, \pm\zeta_7^2, \pm\zeta_7^3, \pm\zeta_7^4, \pm\zeta_7^5, \pm\zeta_7^6$$

Other units:

$$u_1 = 1 + \theta \quad \text{and} \quad u_2 = 1 + \theta + \theta^2$$

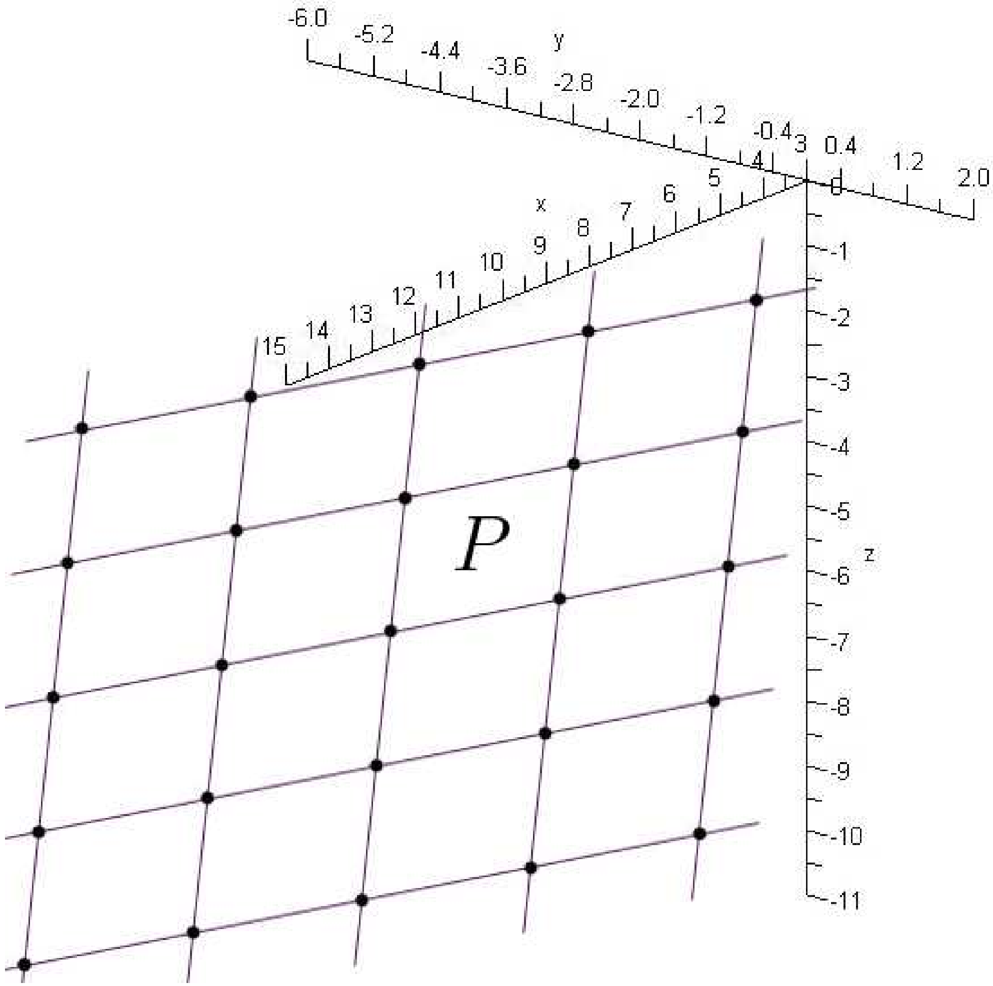
For a generic unit $u = \pm\zeta_7^a u_1^b u_2^c$ with $a, b, c \in \mathbb{Z}$:

$$l(u) = b.l(u_1) + c.l(u_2)$$

The units are the points of a lattice of dimension 2.

Durham University

The Dirichlet Regulator



The Dirichlet Regulator, R_K:

$$R_K = \frac{vol(P)}{\sqrt{r_1 + r_2}}$$

Durham University

The Analytic Class Number Formula

Theorem:

$$\lim_{s \to 0} \frac{\zeta_K(s)}{s^{(r_1+r_2-1)}} = -\frac{h_K R_K}{w_K}$$

where:

- $\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$ is the Dedekind Zeta function for the field K, where $\mathfrak{a}$ runs over all the ideals in $\mathcal{O}_K$, the ring of integers.
- h_K is the class number of K.
- w_K is the number of roots of unity contained in K.
- R_K is the Dirichlet Regulator previously described.

Durham University

What is a Polylogarithm?

- The natural logarithm $ln(x)$
- $$-ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad z \in \mathbb{C},\ |z| < 1$$

Durham University

What is a Polylogarithm?

- The natural logarithm $ln(x)$
- $$-ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad z \in \mathbb{C},\ |z| < 1$$
- Definition of a *Polylogarithm*:

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad z \in \mathbb{C},\ |z| < 1,\ m \in \mathbb{N}$$

Durham University

Going Higher

- $\zeta_K(s) \sim \zeta_K(1-s)$
- Zagier Conjecture says that $\zeta_K(2)$ can be expressed in terms of $Li_2(x)$
- Higher Regulators
- Higher Class Numbers
- Higher Dedekind Zeta Values

Durham University

Questions?