Hamilton-Jacobi-Bellman Equations

Viscosity Solutions of Partial Differential Equations

Iain Smears

March 2^{nd}, 2011

Outline

Hamilton-Jacobi-Bellman (HJB) equations

HJB equation

$$\sup_{\alpha \in \Lambda} \left[-\sum_{i,j=1}^{n} a_{ij}(x,\alpha) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{n} b_i(x,\alpha) \frac{\partial u}{\partial x_i}(x) - f(x,\alpha) \right] = 0$$

Example

Consider

$$\left| \frac{\mathrm{d}u}{\mathrm{d}x} \right| = 1 \text{ on } (0,2), \qquad u(0) = u(2) = 0.$$

Equivalently

$$\max_{\alpha \in [-1,1]} \left[\alpha \frac{\mathrm{d}u}{\mathrm{d}x} - 1 \right] = 0 \text{ on } (0,2), \qquad u(0) = u(2) = 0.$$

A problem with the classical meaning of PDE

Classical meaning of a solution

Find $u \in C^1(0,2)$ such that

$$\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| = 1 \text{ on } (0,2) \qquad u(0) = u(2) = 0,$$

Impossible to find a C^1 solution to this problem.

Rolle's theorem

$u(0) = u(2)$ and $u \in C^1$ implies

$$\exists \xi \in (0,2) \text{ s.t. } \frac{\mathrm{d}u}{\mathrm{d}x}(\xi) = 0 \neq \pm 1.$$

The Problem

Classical concept of solutions is often too restrictive.

Generalised solutions in Mathematics

Purpose

Give a definition of a generalised solution that doesn't require C^1 or C^2 differentiability.

Criteria:

Useful

A unique generalised solution exists and it is the relevant one.

Consistent

Classical solutions are generalised solutions, if they exist.

Selective

A smooth generalised solution is then a classical solution.

Structure of HJB Operator

Set

$$H\left(x, Du(x), D^2u(x)\right) = \sup_{\alpha\in\Lambda}\left[-\sum_{i,j=1}^{n} a_{ij}(x,\alpha)u_{x_ix_j}(x) - \sum_{i=1}^{n} b_i(x,\alpha)u_{x_i}(x) - f(x,\alpha)\right]$$

Ordering Structure

If $u, v \in C^2$, s.t. $u - v$ has a local maximum at x

$$\Rightarrow H\left(x, Dv(x), D^2v(x)\right) \leq H\left(x, Du(x), D^2u(x)\right).$$

Reversed inequalities if $u - v$ has a minimum at x.

Viscosity Solutions

Definition (Viscosity Solutions)

A function u is a viscosity solution if:

- $v \in C^2$ with $u - v$ has a maximum at x implies
$$H\left(x, Dv(x), D^2 v(x)\right) \leq 0,$$
- $v \in C^2$ with $u - v$ has a minimum at x implies
$$H\left(x, Dv(x), D^2 v(x)\right) \geq 0.$$

Differentiability not required for viscosity solutions.

Results

Theorem (Comparison Property)

If u, v are viscosity solutions, then

$$\sup_{U}(u - v) \leq \sup_{\partial U}(u - v)$$

Dirichlet boundary conditions $\Rightarrow$ solutions are unique.

Theorem

- *Classical solutions are viscosity solutions.*
- *C^2 viscosity solutions are classical solutions.*

Summary

Generalised Solutions

Non-differentiable functions can be "solutions" to PDE.
Requirements:

1. **Existence**, **Uniqueness** and **Relevance**
2. **Consistency**
3. **Selectivity**

Viscosity Solutions

Viscosity solutions use ordering structure of HJB.
Checked requirements:

1. **Comparison property** implies uniqueness.
2. Consistency and selectivity verified from definition.