D-BRANES AND NONCOMMUTATIVE GEOMETRY IN STRING THEORY

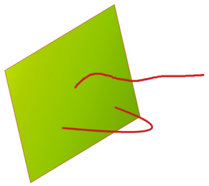
Durham University

Pichet Vanichchapongjaroen
Supervisor: Prof. Chu

Introduction

Quantisation of gravity is an interesting area in physics. In terms of spacetime, general relativity and quantum theory do not fit together. Noncommutative geometry (NCG) might provide a framework to combine the two theories. It turns out that NCG arises very naturally in string theory. Spacetime as well as momentum space can become noncommutative. Physical consequences of NCG are interesting to study.

Strings and D-branes



Strings are one dimensional objects. They come in 2 kinds: closed and open. Closed strings can move anywhere in spacetime, but in general open strings cannot. An open string's endpoint can be attached to an extended object called D-brane or Dp-brane if it has p spatial dimensions.

D-branes impose boundary conditions on strings. A string's endpoint can move freely in the coordinates along a Dp-brane. After quantising, the coordinates of the endpoint satisfy the commutation relation

$$[\hat{x}^i,\hat{p}^j]=i\hbar\delta^{ij},[\hat{x}^i,\hat{x}^j]=[\hat{p}^i,\hat{p}^j]=0 \text{ for } i,j=0,\dots,p.$$

Here, $\hat{x}^i$ and $\hat{p}^j$ are operators associated with coordinate x^i and momentum p^i of the string endpoint.

D-brane with constant NS-NS B-field

NS-NS B-fields are antisymmetric spin 2 massless fields. They arise when quantising closed strings. Charged strings can couple to the constant B-field living on the D-brane. The boundary conditions and the commutation relation of the string are modified by the B-field. We have

$$[\hat{x}^i,\hat{p}^j]=i\hbar\delta^{ij},[\hat{x}^i,\hat{x}^j]=i\theta^{ij},[\hat{p}^i,\hat{p}^j]=0 \text{ for } i,j=0,\dots,p,$$

where θ^{ij} is related to the B-field.

This leads to the study of quantum field theory in noncommutative spacetime. Some interesting topics are UV/IR mixing, noncommutative gauge field theory, Morita equivalence, etc.

Generalisation

A D-brane in pp-wave backgrond in the presence of constant B-field has the commutation relations [2]

$$[\hat{x}^i,\hat{p}^j]=i\hbar\delta^{ij},[\hat{x}^i,\hat{x}^j]=i\theta^{ij},[\hat{p}^i,\hat{p}^j]=i\phi^{ij} \text{ for } i,j=0,\dots,p.$$

It is now interesting to study quantum theories in this noncommutative phase space.

Phase Space Quantisation

Phase space quantisation is the quantisation method that allows us to view quantum operators and states as functions in phase space. Functions describing states are called *Wigner distribution functions*.

Consider a Poisson bracket of any two phase space functions $g(\vec{\eta})$ and $h(\vec{\eta})$:

$$\{g(\vec{\eta}),h(\vec{\eta})\}=\Lambda^{ij}\partial_i g(\vec{\eta})\partial_j h(\vec{\eta}),$$

where $\vec{\eta}$ is a phase space vector: $\vec{\eta}=(x^1,x^2,\dots,x^d,p^1,p^2,\dots,p^d)$, and Λ^{ij} is antisymmetric. To quantise, we define a star product so that

$$g(\vec{\eta})*h(\vec{\eta})=g(\vec{\eta})\exp\left(\frac{i}{2}\overleftarrow{\partial_i}\Theta^{ij}\overrightarrow{\partial_j}\right)h(\vec{\eta}),$$

where $\Theta^{ij}=\hbar\Lambda^{ij}$. We replace the multiplication between any two functions by a symmetrised star product (parameters omitted for brevity): $gh\to(g*h+h*g)/2$. Poisson bracket is replaced by Moyal bracket: $\{g,h\}\to g*h-h*g$.

Two Dimensional Simple Harmonic Oscillator

Consider a two dimensional simple harmonic oscillator (SHO) with the Hamiltonian

$$H=\frac{x^2+y^2+p_x^2+p_y^2}{2},$$

where we scale the phase space coordinates to be dimensionless and $\hbar=1$. The Poisson brackets are given by

$$\{x,p_x\}=\{y,p_y\}=1,\quad \{x^1,x^2\}=\theta,\quad \{p_x,p_y\}=\phi.$$

Quantum SHO

Let us first study the case $\theta=\phi=0$. Bayen et al.[1] used spectral theory to get the Wigner distribution function as

$$\pi_{n,m}(H,L_3)=4(-1)^n e^{-2H}L_{(n+m)/2}(2(H+L_3))L_{(n-m)/2}(2(H-L_3)),$$

where $L_r(a)$ is Laguerre polynomial, $L_3=xp_y-yp_x$ (not to be confused with Laguerre polynomial), $n=0,1,2,\dots$ and $m=-n,-n-2,\dots,n$. This function is the star eigenfunction of H and L_3, i.e.

$$H*\pi_{n,m}(H,L_3)=\pi_{n,m}(H,L_3)*H=E_{n,m}\pi_{n,m}(H,L_3).$$

$$L_3*\pi_{n,m}(H,L_3)=\pi_{n,m}(H,L_3)*L_3=m\pi_{n,m}(H,L_3).$$

The energy is $E_{n,m}=n+1$. It is degenerate as expected.

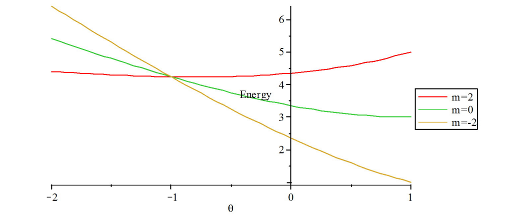
Alternatively, it is useful to study group theory. We notice that $J_z=H/2, J_x=(xp_x+yp_y)/2$, and $J_y=(x^2+y^2-p_x^2-p_y^2)/4$ form a closed $SO(1,2)$ Moyal bracket with Casimir function

$$C=J_z*J_z-J_x*J_x-J_y*J_y=\left(\frac{L_3}{2}-\frac{1}{2}\right)*\left(\frac{L_3}{2}+\frac{1}{2}\right).$$

Quantum SHO in a noncommutative phase space

For the general case, the Moyal brackets of the set $\{J_x,J_y,J_z,C\}$ are complicated. Nevertheless, we can make a shift transformation of J_x,J_y,J_z, and C to get $SO(1,2)$ group. This is true only when $\theta\phi<1$. The functions $\pi_{n,m}$ with the parameters shifted are the star eigenfunction of Hamiltonian. The energy is $E_{n,m}=(n+1)\sqrt{4+(\theta-\phi)^2}/2-(\theta+\phi)m/2$. It is nondegenerate except when $\theta=-\phi$.

Naïvely, when $\theta\phi=1$, the vacuum energy is degenerate with the value of $(\theta+\phi)/2$ for $m=n$. When $\theta\phi>1$, the energy is not bounded from below: The vacuum is then unstable.



The graph shows energy of two dimensional SHO in noncommutative phase space with $\phi=1$ and $n=2$. Energy is degenerate when $\theta=-1$. The values in the region $\theta\phi\geq 1$ are not included in the graph.

What's Next?

It is interesting to study the region $\theta\phi\geq 1$ properly. We would like to see if the vacuum actually becomes degenerate when $\theta\phi=1$, and if the vacuum becomes unstable when $\theta\phi>1$.

It is also interesting to see the relationship between the $SO(1,2)$ group and the star eigenfunctions.

When we have the full picture of two dimensional SHO, the next task would be to study higher dimensional SHO. It might also be possible to study quantum field theory in noncommutative phase space. This is because SHO was used in order to study quantum field theory. So we hope that noncommutative SHO would allow us to study quantum field theory in noncommutative phase space.

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. II. Physical applications. *Annals of Physics*, 111:111–151, Mar. 1978.

[2] C.-S. Chu and P.-M. Ho. Noncommutative D-brane and open string in pp-wave background with B-field. *Nucl. Phys.*, B636:141–158, 2002.