D-Branes and Noncommutative Geometry in String Theory

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Abstract

We discuss the simplest systems in string theory that noncommutative geometry arises. We discuss the cases of noncommutative spacetime and totally noncommutative phase space. Physical consequences of noncommutativity are discussed. The work in progress on physics in totally noncommutative phase space is also discussed. We constructed phase space quantisation to study quantum theory in totally noncommutative phase space. We also discuss two dimensional simple harmonic oscillator in totally noncommutative phase space and find that the energy spectrum is generally nondegenerate. This is because the $Sp(4)$ algebra is deformed. We also find two separate sets of ladder operators.

Contents

Chapter 1

Introduction

1.1 Motivations

In physics, there are two big theories that describe motions and physical properties of particles. Quantum theory is used to describe particles in a very small region while general relativity is used to describe a system with highly gravitating objects. Normally, quantum theory is used to describe atoms or smaller objects while general relativity is used to describe big stars. So it seems that we do not need to combine these two theories. However, there exists objects that are small but highly gravitating. The first example is black hole. Its singularity is believed to contain all the matters of the black hole. So we have to use quantum theory. Additionally, black hole is highly gravitating. Another example is big bang. Around the time of big bang, the matters of universe are contained in a very small region.

So in order to describe black hole singularity or big bang, we will need to combine quantum theory with general relativity. However the problem arises if we try to directly combine quantum theory with general relativity. The key idea in general relativity is that spacetime is a smooth manifold. So in order to quantise gravity we may need a new model of spacetime.

String theory suggests a new model that is described by noncommutative geometry. When a D-brane is put in a background of constant B-field, its worldvolume becomes noncommutative. This suggests a model called noncommutative spacetime. In this spacetime we cannot simultaneously measure the position of a particle along any two axes. In other words, we cannot determine an exact position of a particle. Some physical consequences of noncommutative spacetime are discussed in literatures. For example noncommutative quantum field theory is constructed and is found that, unlike the usual quantum field theory, the interaction becomes nonlocal.

There is another suggestion from string theory. When a D-brane is put in a pp-wave background with a constant B-field, the whole spacetime of D-brane becomes noncommutative. In this case, we are also unable to determine an exact position of a particle. We now want to see the physical consequences of this kind of noncommutative phase space. We may try to construct quantum field theory in totally noncommutative phase space. However, this is difficult. Let us briefly discuss this. Quantum field is a quantisation of classical function of spacetime. In smooth spacetime, we are allowed to construct such functions. We are also allowed to use momentum space representation. In fact, the two representations are Fourier transform of each other. In noncommutative spacetime, the situation is more difficult. However, even if the spacetime is noncommutative we are still allowed to use momentum space representation. However, in the totally noncommutative phase space, we are not allowed to use either coordinate space or momentum space representation.

Nevertheless, it might be possible to deal with this issue by considering phase space quantisation. Its first construction was to study quantum phase space which is already noncommutative. So the generalisation to phase space quantisation might allow us to describe the totally noncommutative phase space.

We will see in this report that it is possible to generalisation phase space quantisation. We also try to study a simple quantum system in totally noncommutative phase space to get some ideas. We hope that this might be a good first step for the construction of quantum field theory in totally noncommutative phase space.

1.2 Contents

This report is divided into two parts. In Part I, we discuss necessary basics to the report. The readers are expected to know most of the stuffs in this part. In part II we discuss some works in scientific papers. We also discess some works in progress.

In Part I, we begin with the discussion of classical and quantum theories of point particles in chapter 2. This chapter gives the basic for the rest of the report. We now move on to discuss basics of string theory in chapter 3.

In Part II, we start by discussing how noncommutative geometry arises in string theory in chapter 4. We then generalise phase space quantisation in chapter 5 to prepare for the study of physical systems with noncommutative geometry. In chapter 6, we review some features of quantum field theory in noncommutative spacetime. Phase space quantisation was adapted to construct the theory. In chapter 7, we discuss two dimensional simple harmonic oscillator in totally noncommutative phase space.

1.3 Conventions

- In this report, some Greek indices for example μ, ν are spacetime indices. The exceptions are the two indices α and β which are worldsheet indices. Roman indices for example i, j, k are multi-purposed. In each context, we normally define what they represent.
- We use Einstein summation convention. Repeated indices are summed over.
- The signature of spacetime is $(-, +, \ldots, +)$.
- When the limit of integration is not put on an integral sign $\int$, we mean that the integration is carried out over the whole domain.

Part I

Basics

Chapter 2

Theories of Point Particles

In this chapter we present a quick reminder for classical and quantum theory of point particles. The materials are standard. However, they are important basics for the rest of the report. In particular, the material on canonical transformation and simple harmonic oscillator will be related to the main results.

The materials in this chapter are based on and inspired by [1], [2], [3], [4], [5], [6], [7], [8].

2.1 Classical Mechanics

2.1.1 Lagrangian Formalism

A system of point particles with d degrees of freedom is described by d coordinates $q^1, q^2, \ldots, q^d$. The differentiate with respect to time of the coordinates is called velocity $\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^d$. The function called Lagrangian $L(q^1, q^2, \ldots, q^d, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^d, t) \equiv L(\vec{q}, \dot{\vec{q}}, t)$ is used to describe the motion of the system. Hamilton's principle states that, between two given positions $\vec{q}^{(1)}, \vec{q}^{(2)}$ and times t_1, t_2, the action

$$S = \int_{t_1}^{t_2} L(\vec{q}, \dot{\vec{q}}, t) \tag{2.1}$$

for physical motion is extremised. This requirement gives Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \ldots, d, \tag{2.2}$$

which are equations of motion.

Let us consider a nonrelativistic point particle of mass m moving in one dimension under a potential $V(q)$. The Lagrangian is given by

$$L = \frac{1}{2} m\dot{q}^2 - V(q). \tag{2.3}$$

The equation of motion is then

$$m\ddot{q} = -V'(q), \tag{2.4}$$

which is Newton's second law of motion.

2.1.2 Hamiltonian Formalism

Given a Lagrangian $L(\vec{q}, \dot{\vec{q}}; t)$, we define a conjugate momentum $\vec{p}$ by

$$p^i := \frac{\partial L}{\partial \dot{q}^i}. \tag{2.5}$$

We define Hamiltonain $H(\vec{p}, \vec{q}; t)$ as a Legendre transformation of the Lagrangian:

$$H(\vec{p}, \vec{q}; t) \equiv \vec{p} \cdot \dot{\vec{q}} - L. \tag{2.6}$$

The equations of motion are given by

$$\dot{q}^i = \frac{\partial H}{\partial p^i}, \quad \dot{p}^i = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \ldots, d. \tag{2.7}$$

In Hamiltonian formalism, we have, as equations of motion, $2d$ first order differential equations for d degrees of freedom. This is simpler than Lagrangian formalism which has d second order differential equations.

Consider a phase space function $g(\vec{p}, \vec{q}, t)$. Its total time derivative is

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_{i=1}^{d} \left(\frac{\partial g}{\partial q^i} \dot{q}^i + \frac{\partial g}{\partial p^i} \dot{p}^i \right). \tag{2.8}$$

Using the equations of motion, we have

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \{g, H\}, \tag{2.9}$$

where

$$\{g, H\} \equiv \sum_{i=1}^{d} \left(\frac{\partial g}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial g}{\partial p^i} \frac{\partial H}{\partial q^i} \right) \tag{2.10}$$

is the Poisson bracket of g and H.

From definition, we have the following properties for Poisson bracket for any phase space functions g, h, l and a number a:

1. Antisymmetry $\{g, h\} = -\{h, g\}$
2. Linearity $\{g + h, l\} = \{g, l\} + \{h, l\}$
3. Multiplication by a number $\{ag, h\} = a\{g, h\}$
4. Jacobi identity $\{g, \{h, l\} + \{h, \{l, g\} + \{l, \{g, h\} = 0$

If a function g is a constant of motion, then $dg/dt = 0$. Additionally, if the constant of motion does not depend explicitly on time, then we have

$$\{g, H\} = 0. \tag{2.11}$$

For example, consider a point particle of mass m moving in a plane under central force:

$$H = \frac{p_1^2 + p_2^2}{2m} + V(x_1^2 + x_2^2). \tag{2.12}$$

We can check that the angular momentum $L = m(x_1p_2 - x_2p_1)$ is a constant of motion.

One of the important results in this report is relating to two dimensional simple harmonic oscillator (SHO). So it is useful to get some visualisations. The Hamiltonian of two dimensional SHO is given by

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}m\omega^2(x_1^2 + x_2^2), \tag{2.13}$$

where m is the mass, and ω is the angular velocity of the SHO. Obviously, as discussed earlier, $L_z = m(x_1p_2 - x_2p_1)$ is a constant of motion. This will be more important in quantum case. For now, let us consider a hypersurface of constant energy $H = E$ in phase space. From the equation, this hypersurface is the 3-dimensional hypersurface of hyper-ellipsoid centred at the origin. The larger the hyper-ellipsoid, the more energy the system has. The lowest energy of the system is $E = 0$, when the hypersurface shrinks into the point at the origin. We will see that in quantum case, not all the values of energy are allowed.

2.1.3 Canonical Transformation

Canonical transformation is the transformation on phase space coordinate so that the Poisson bracket is preserved. This transformation is also useful in quantum mechanics and in one of the main results of this report. We first start by writing formalisms compactly.

We define $\vec{\xi}$ as a vector in phase space. It is given by

$$\vec{\xi} = (q^1, q^2, \ldots, q^d, p^1, p^2, \ldots, p^d), \tag{2.14}$$

where d is the dimension of space. We will call the dimension of phase space as $n = 2d$. We define Poisson bracket via

$$\{\xi^i, \xi^j\} = \Lambda^{ij}, \tag{2.15}$$

so that

$$\{g(\vec{\xi}), h(\vec{\xi})\} = \partial_i g(\vec{\xi})\Lambda^{ij}\partial_j h(\vec{\xi}). \tag{2.16}$$

Let us now discuss canonical transformation. Consider a coordinate transformation

$$\xi^i \to \xi'^i = \xi'^i(\vec{\xi}). \tag{2.17}$$

The Poisson bracket is given by

$$\{\xi'^i(\vec{\xi}), \xi'^j(\vec{\xi})\} = \partial_k\xi^i\Lambda^{kl}\partial_l\xi^j. \tag{2.18}$$

For canonical transformation, we require $\{\xi'^i(\vec{\xi}), \xi'^j(\vec{\xi})\} = \Lambda^{ij}$. Therefore

$$\underline{\underline{M}}\boldsymbol{\Lambda}\underline{\underline{M}}^T = \boldsymbol{\Lambda}, \tag{2.19}$$

where $\underline{\underline{M}}$ is the matrix with elements $\partial_j\xi^i$, and $\underline{\underline{M}}^T$ is the transpose of $\underline{\underline{M}}$, and $\boldsymbol{\Lambda}$ is a matrix with elements Λ^{ij}. We require that $\boldsymbol{\Lambda}$ is non-singular.

Let us consider an infinitesimal transformation

$$\xi^i \to \xi^i + \eta^i. \tag{2.20}$$

We are interested in the transformation on a phase space function $g(\vec{\xi})$. We take the active view: the transformed function evaluated at the new coordinates is given by the old function evaluated at the old coordinates. i.e.

$$\begin{aligned} g^{new}(\vec{\xi}) &= g(\vec{\xi} - \vec{\eta}) \\ &\approx g(\vec{\xi}) - \eta^i \partial_i g(\vec{\xi}) \\ &= g(\vec{\xi}) - \epsilon \Lambda^{ij} \partial_j G(\vec{\xi}) \partial_i g(\vec{\xi}) \\ &= g(\vec{\xi}) - \epsilon \{ g(\vec{\xi}), G(\vec{\xi}) \}, \end{aligned}$$

where $G(\vec{\xi})$ is defined via $\eta^i = \epsilon \Lambda^{ij} \partial_j G(\vec{\xi})$. We see that the generator $G(\vec{\xi})$ defined here generates the infinitesimal transformation on $g(\vec{\xi})$ via Poisson bracket. Now consider a transformation of the form

$$\eta^i = -i\epsilon \tilde{G}^i{}_j \xi^j. \tag{2.21}$$

Now we have

$$-i\epsilon \tilde{G}^i{}_j \xi^j = \epsilon \Lambda^{ij} \partial_j G(\vec{\xi}). \tag{2.22}$$

Therefore, the generator $G(\vec{\xi})$ is of the form

$$G(\vec{\xi}) = \frac{1}{2} \xi^i G_{ij} \xi^j. \tag{2.23}$$

Now we define $\underline{\underline{G}}$ as a matrix with elements $\tilde{G}^i{}_j$, and define $\mathbf{G}$ as a matrix with elements G_{ij}. Then the generator in matrix form $\underline{\underline{G}}$ is related to the generator in function form $G(\vec{\xi})$ via

$$\underline{\underline{G}} = i \boldsymbol{\Lambda} \mathbf{G}. \tag{2.24}$$

Let us now discuss an example which will also be used in our main results.

Example 1 (Homogeneous Linear Canonical Transformation in Four-Dimensional Phase Space)**.** *The infinitesimal homogeneous linear canonical transformation is the transformation of the form*

$$\eta^i = -i\epsilon \tilde{G}^i{}_j \xi^j. \tag{2.25}$$

We relate the infinitesimal transformation with the matrix $\underline{\underline{M}}$ *by*

$$\underline{\underline{M}} = \exp(-i\alpha \underline{\underline{G}}). \tag{2.26}$$

For a four dimensional phase space with $\vec{\xi} = (x, y, p_x, p_y)$ *and*

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.27}$$

there are ten generators forming the symplectic group $Sp(4)$*. The generators are [1]*

$$\underline{\underline{J_1}} = \frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \underline{\underline{J_2}} = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \underline{\underline{J_3}} = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \underline{\underline{J_0}} = \frac{i}{2} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.28}$$

$$\underline{\underline{K_1}} = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \underline{\underline{K_2}} = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \underline{\underline{K_3}} = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \tag{2.29}$$

$$\underline{\underline{Q_1}} = -\frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \underline{\underline{Q_2}} = \frac{i}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \underline{\underline{Q_3}} = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \tag{2.30}$$

where I is the 2×2 identity matrix, and $\sigma_1, \sigma_2, \sigma_3$ are Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.31}$$

The generators satisfy the commutation relations:

$$[\underline{\underline{J_i}}, \underline{\underline{J_j}}] = i\epsilon_{ijk}\underline{\underline{J_k}}, [\underline{\underline{J_i}}, \underline{\underline{J_0}}] = 0, \tag{2.32}$$

$$[\underline{\underline{J_i}}, \underline{\underline{K_j}}] = i\epsilon_{ijk}\underline{\underline{K_k}}, [\underline{\underline{J_i}}, \underline{\underline{Q_j}}] = i\epsilon_{ijk}\underline{\underline{Q_k}}, \tag{2.33}$$

$$[\underline{\underline{K_i}}, \underline{\underline{K_j}}] = [\underline{\underline{Q_i}}, \underline{\underline{Q_j}}] = -i\epsilon_{ijk}\underline{\underline{J_k}}, \tag{2.34}$$

$$[\underline{\underline{K_i}}, \underline{\underline{Q_j}}] = i\delta_{ij}\underline{\underline{J_0}}, \tag{2.35}$$

$$[\underline{\underline{K_i}}, \underline{\underline{J_0}}] = i\underline{\underline{Q_i}}, [\underline{\underline{Q_i}}, \underline{\underline{J_0}}] = -i\underline{\underline{K_i}}. \tag{2.36}$$

The corresponding generators in function form are

$$J_1 = \frac{1}{2}(xy+p_xp_y), \; J_2 = \frac{1}{2}(xp_y-yp_x), \; J_3 = \frac{1}{4}(x^2+p_x^2-y^2-p_y^2), \; J_0 = \frac{1}{4}(x^2+y^2+p_x^2+p_y^2), \tag{2.37}$$

$$K_1 = -\frac{1}{4}(x^2 - p_x^2 - y^2 + p_y^2), \quad K_2 = \frac{1}{2}(xp_x + yp_y), \quad K_3 = \frac{1}{2}(xy - p_xp_y), \tag{2.38}$$

$$Q_1 = -\frac{1}{2}(xp_x - yp_y), \quad Q_2 = -\frac{1}{4}(x^2 - p_x^2 + y^2 - p_y^2), \quad Q_3 = \frac{1}{2}(xp_y + yp_x). \tag{2.39}$$

2.1.4 Classical Fields

So far we have discussed the systems of finite degrees of freedom. However, there are many systems that are continuous. These systems are described by fields. Fields are usually written as a function $\phi_a(t, \vec{x})$ of spacetime. Here a, and $(t, \vec{x})$ are labels of the field. For each a, the field at each point in spacetime is treated as coordinates. Therefore the system has uncountably many degrees of freedom.

The Lagrangian of field is usually written in the form

$$L(t) = \int d^d\vec{x}\mathcal{L}(\phi_a, \partial_\mu\phi_a), \tag{2.40}$$

where $x^\mu \in (t, \vec{x}) = (t, x^1, x^2, \ldots, x^d)$ is the spacetime coordinates, and $\mathcal{L}$ is called Lagrangian density. The action integral now is in the form

$$S = \int d^{d+1}x\mathcal{L}. \tag{2.41}$$

Applying Hamilton's principle, we have the Euler-Lagrange's equation for fields:

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \tag{2.42}$$

As an example, consider a free real scalar field $\phi(t, \vec{x})$ with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2, \tag{2.43}$$

where $\eta_{\mu\nu}$ is the flat metric defined by $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$, and its inverse is $\eta^{\mu\nu} = \eta_{\mu\nu}$. Solving Euler-Lagrange's equation gives Klein-Gordon equation

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = 0. \tag{2.44}$$

A plane wave $e^{\pm i\eta_{\mu\nu}k^\mu x^\nu}$ is the solution if for $k^\mu = (k^0, \vec{k})$, we have $k^0 = \pm\sqrt{\vec{k}^2 + m^2}$. Let us denote $k^0 = E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. The general solution can be obtained by superposition:

$$\phi(t, \vec{x}) = \int \frac{d^d\vec{k}}{(2\pi)^d\sqrt{2E_{\vec{k}}}}(a_{\vec{k}}e^{ik_\mu x^\mu} + a^*_{\vec{k}}e^{-ik_\mu x^\mu})\Big|_{k^0=E_{\vec{k}}}. \tag{2.45}$$

We see that the positive energy modes are given by $a_{\vec{k}}$ while the negative energy modes are given by $a^*_{\vec{k}}$. This is to ensure that the field is real.

Given a Lagrangian density, we can also define Hamiltonian density by

$$\mathcal{H} = \sum_a \Pi_{\phi_a}\partial_0\phi_a - \mathcal{L}, \tag{2.46}$$

where the conjugate momentum Π_{ϕ_a} of the field ϕ_a is given by

$$\Pi_{\phi_a} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}. \tag{2.47}$$

The Hamiltonian is given by

$$H = \int d^d\vec{x}\mathcal{H}. \tag{2.48}$$

For the case of free real scalar field, the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}(\Pi_\phi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2), \tag{2.49}$$

where the conjugate momentum is

$$\Pi_\phi = \partial_0\phi = -i\int \frac{d^d\vec{k}}{(2\pi)^d}\sqrt{\frac{E_{\vec{k}}}{2}}(a_{\vec{k}}e^{ik_\mu x^\mu} - a^*_{\vec{k}}e^{-ik_\mu x^\mu})\Big|_{k^0=E_{\vec{k}}}. \tag{2.50}$$

After a long calculation, we get the Hamiltonian

$$H = \int \frac{d^d\vec{k}}{(2\pi)^d}E_{\vec{k}}|a_{\vec{k}}|^2. \tag{2.51}$$

It is also useful to discuss Poisson bracket for the fields. The Poisson bracket of two functions $G(\phi_a, \Pi_a, t), K(\phi_a, \Pi_a, t)$ is defined at equal time by

$$\{G, K\} \equiv \sum_a \int d^d\vec{x}\left(\frac{\delta G}{\delta\phi_a}\frac{\delta K}{\delta\Pi_a} - \frac{\delta K}{\delta\phi_a}\frac{\delta G}{\delta\Pi_a}\right) \tag{2.52}$$

For scalar field we have

$$\{\phi(t, \vec{x}), \Pi(t, \vec{y})\} = \delta^{(d)}(\vec{x} - \vec{y}), \quad \{\phi(t, \vec{x}), \phi(t, \vec{y})\} = 0, \quad \{\Pi(t, \vec{x}), \Pi(t, \vec{y})\} = 0. \tag{2.53}$$

where $\delta^{(d)}(\vec{x}-\vec{y})$ is Dirac delta function defined via

$$\int d^d\vec{x} g(\vec{x})\delta^{(d)}(\vec{x}-\vec{y}) = g(\vec{y}), \tag{2.54}$$

for any function $g(\vec{x})$.

Let us now discuss about symmetry. Noether's theorem states that for each transformation that leaves the action invariant, there is always a conserved current $j_i^\mu(t,\vec{x})$ which satisfies

$$\partial_\mu j_i^\mu = 0. \tag{2.55}$$

For a conserved current, we have conserved charge Q_i defined by

$$Q_i \equiv \int d^d\vec{x} j_i^0(t,\vec{x}). \tag{2.56}$$

It is conserved in the sense that $\partial_0 Q_i = 0$.

When constructing relativistic field theories, we want the action to be invariant because it is a scalar. The conserved current for this symmetry is the energy-momentum tensor $j^\mu_{(\nu)} = T^\mu_{\ \nu}$ which is given by

$$T^{\mu\nu} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - \eta^{\mu\nu}\mathcal{L}. \tag{2.57}$$

The conserved charge is called momentum

$$P^\nu \equiv \int d^d\vec{x} T^\nu_{\ 0}. \tag{2.58}$$

The free scalar field is also a relativistic field. Its energy-momentum tensor is given by

$$T^{\mu\nu} = -\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}. \tag{2.59}$$

The zeroth component of the momentum is just the Hamiltonian H. The spatial component of the momentum is

$$P^l = \int \frac{d^n\vec{k}}{(2\pi)^n} k^l |a_{\vec{k}}|^2. \tag{2.60}$$

2.2 Quantum Mechanics

2.2.1 States and Observables

A state in quantum mechanics can be described by a ket $|\psi\rangle$ living in a vector space V. For the given vector space V, there is a dual vector space $\bar{V}$ such that its element $\langle\chi|$ maps the vector space into complex number. i.e.

$$\langle\chi| : V \to \mathbb{C} \tag{2.61}$$

$$|\psi\rangle \mapsto \langle\chi|\psi\rangle. \tag{2.62}$$

This element $\langle\chi|$ is called a bra. Given a ket $|u\rangle$, there is always an associated bra $\langle u|$. For example, if $|\psi\rangle$ is a column vector of complex numbers: $|\psi\rangle = (a_1, a_2, \dots)^T$, then the bra is given by $\langle\psi| = (a_1^*, a_2^*, \ldots)$. The bra acts on the ket by the matrix multiplication. So

$$\langle\psi|\psi\rangle = |a_1|^2 + |a_2|^2 + \ldots. \tag{2.63}$$

A linear operator $\hat{A}$ maps the vector space into itself. i.e.

$$\hat{A} : V \to V. \tag{2.64}$$

As an example, we can see that an outer product $|u\rangle\langle v|$ can be treated as an operator. For any linear operators $\hat{A}, \hat{B}, \hat{C}$ any ket $|\psi\rangle, |\phi\rangle$, and any (complex) number a, the properties

1. Linearity in operators: $(\hat{A} + \hat{B})|\psi\rangle = \hat{A}|\psi\rangle + \hat{B}|\psi\rangle$ and $(a\hat{A})|\psi\rangle = a(\hat{A}|\psi\rangle)$
2. Linearity in kets: $\hat{A}(|\psi\rangle + |\phi\rangle) = \hat{A}|\psi\rangle + \hat{A}|\phi\rangle$
3. Associativity: $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$

are satisfied. However, note that the operators do not commute. For any linear operators $\hat{A}, \hat{B}$ we define the commutator $[\hat{A}, \hat{B}]$ as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{2.65}$$

Given an operator $\hat{A}$, we define its hermitian conjugate $\hat{A}^\dagger$ via

$$\langle u|\hat{A}^\dagger|v\rangle = \langle u|\hat{A}|v\rangle^*. \tag{2.66}$$

We have the properties for hermitian conjugate:

- $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$,
- $(a\hat{A})^\dagger = a^*\hat{A}^\dagger$,
- $(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$,
- $(|u\rangle\langle v|)^\dagger = |v\rangle\langle u|$.

An operator $\hat{A}$ is called hermitian if $\hat{A} = \hat{A}^\dagger$.

Given an operator $\hat{A}$, we are interested in the states $|a\rangle$ such that

$$\hat{A}|a\rangle = a|a\rangle, \tag{2.67}$$

where $a \in \mathbb{C}$. This equation is called an eigenvalue equation. The ket $|a\rangle$ is called an eigenket, and the number a is called an eigenvalue. If two operators $\hat{A}, \hat{B}$ commutes, then they share the same eigenket $|a, b\rangle$. Then there is a degeneracy, e.g. the kets $|a, b\rangle$, and $|a, b'\rangle$ shares the same eigenvalue a of $\hat{A}$.

In quantum mechanics, operators are used to measure states. Hermitian operators $\hat{A}$ are important in quantum mechanics because the average $\langle u|\hat{A}|u\rangle$ is real. When we make a measurement in a state $|u\rangle$, we expect to get the average value to be real. Therefore the hermitian operators can be used as observables in quantum mechanics.

A state that can be described by a single ket $|u\rangle$ is called a pure state. However, it is also the case that we want to measure a system, called mixed state, containing many pure states $|\psi_i\rangle$ with probability p_i of getting each state. In this situation, we cannot define the average as above. Instead, we define a density matrix

$$\hat{\rho} = \sum_i p_i|\psi_i\rangle\langle\psi_i|. \tag{2.68}$$

Now the average can be written as

$$\langle \hat{A} \rangle = Tr(\hat{\rho}\hat{A}) = Tr\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\hat{A}\right) \equiv \sum_i p_i \langle\psi_i|\hat{A}|\psi_i\rangle. \tag{2.69}$$

Note that the density matrix can also be used in a pure state $|\psi\rangle$ where $\hat{\rho} = |\psi\rangle\langle\psi|$.

2.2.2 Canonical Quantisation

The very common measurement in quantum mechanics is the measurement of position and momentum. Therefore we want the phase space position $\vec{\xi}$ to become hermitian operators $\hat{\vec{\xi}} = (\hat{x}^1, \hat{x}^2, \ldots, \hat{p}^1, \hat{p}^2, \ldots)$. The Poisson bracket is made to be a commutator. i.e.

$$\{\xi^i, \xi^j\} \to \frac{[\hat{\xi}^i, \hat{\xi}^j]}{i\hbar}, \tag{2.70}$$

where $\hbar$ is a Planck constant. We then have the following commutation relations

$$[\hat{x}^i, \hat{p}^j] = i\hbar\delta^{ij}, [\hat{x}^i, \hat{x}^j] = [\hat{p}^i, \hat{p}^j] = 0. \tag{2.71}$$

From the commutation relation, since $\hat{x}^i$ commutes with $\hat{x}^j$, there is a shared eigenket $|x^1, x^2, \ldots\rangle$. The eigenket satisfies the completeness relation. i.e. any ket $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \int d^d\vec{x}|\vec{x}\rangle\langle\vec{x}|\psi\rangle. \tag{2.72}$$

In this way, we can represent $|\psi\rangle$ as a wavefunction $\psi(\vec{x}) \equiv \langle\vec{x}|\psi\rangle$. We call this coordinate space representation. Similarly, we also have a shared eigenket $|p^1, p^2, \ldots\rangle$ for momentum operators, and we can define momentum space representation accordingly.

In coordinate space representation, the momentum operator is represented as a differential operator. Let us focus on the case of two dimensional phase space. We consider the operation $e^{-i\alpha\hat{p}/\hbar}$ which translates $|x\rangle$ to a different eigenket:

$$\begin{aligned} \hat{x}e^{-i\alpha\hat{p}/\hbar}|x\rangle &= (e^{-i\alpha\hat{p}/\hbar}\hat{x} + [\hat{x}, e^{-i\alpha\hat{p}/\hbar}])|x\rangle \\ &= (x+\alpha)e^{-i\alpha\hat{p}/\hbar}|x\rangle, \end{aligned}$$

where we used the fact that $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$, so that, iteratively, $[\hat{x}, (\hat{p})^n] = ni\hbar(\hat{p})^{n-1}$, and this implies $[\hat{x}, e^{-i\alpha\hat{p}/\hbar}] = \alpha e^{-i\alpha\hat{p}/\hbar}$. Therefore

$$e^{-i\alpha\hat{p}/\hbar}|x\rangle = |x+\alpha\rangle, \tag{2.73}$$

since $(e^{-i\alpha\hat{p}/\hbar})^\dagger e^{-i\alpha\hat{p}/\hbar} = 1$, the norm of the ket does not change. Now we consider

$$\begin{aligned} \langle x|e^{i\alpha\hat{p}/\hbar}|\psi\rangle &= \langle x+\alpha|\psi\rangle \\ &= \psi(x+\alpha) \\ &= \sum_{i=0}^{\infty} \frac{\alpha^n}{n!}\frac{d^n}{dx^n}\psi(x) \\ &= e^{\alpha\frac{d}{dx}}\psi(x). \end{aligned}$$

By comparison, we conclude that the coordinate space representation of $\hat{p}$ is $-i\hbar d/dx$. Generally, the coordinate space representation of a momentum operator $\hat{p}^i$ is $-i\hbar\partial/\partial x^i$.

The classical Hamiltonian which is a phase space function is quantised to become an operator. Let us now discuss the time evolution of states. A time dependent ket $|\psi, t\rangle$ satisfies the Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}|\psi, t\rangle = \hat{H}|\psi, t\rangle. \tag{2.74}$$

Since the Hamiltonian $\hat{H}$ is hermitian, we can obtain the time dependence of density matrix as

$$i\hbar\frac{\partial}{\partial t}\hat{\rho} = -[\hat{\rho}, \hat{H}]. \tag{2.75}$$

In practice, the Planck's constant is normally scaled down to 1. Therefore, from this point on, we will take $\hbar = 1$.

2.2.3 Simple Harmonic Oscillator

As an illustration of quantum systems, we quantise a system of simple harmonic oscillator. The energy spectrum of quantum simple harmonic oscillator is discrete. This is not what we would predict from classical mechanics.

We consider the Hamiltonian of a one dimensional simple harmonic oscillator

$$\hat{H} = \frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m}, \tag{2.76}$$

with commutation relation

$$[\hat{x}, \hat{p}] = i. \tag{2.77}$$

For convenient, we define ladder operators

$$\hat{a} = (\sqrt{\frac{m\omega}{2}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\omega}}), \quad \hat{a}^\dagger = (\sqrt{\frac{m\omega}{2}}\hat{x} - i\frac{\hat{p}}{\sqrt{2m\omega}}). \tag{2.78}$$

The commutation relation is $[\hat{a}, \hat{a}^\dagger] = 1$. We also define a number operator

$$\hat{N} = \hat{a}^\dagger\hat{a}. \tag{2.79}$$

It is easy to show that $\hat{H} = \omega(\hat{N} + 1/2)$. Therefore the number operator shares the eigenket with Hamiltonian. Let us denote the eigenket $|n\rangle$ where

$$\hat{N}|n\rangle = n|n\rangle. \tag{2.80}$$

Let us consider the commutation relations

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger \tag{2.81}$$

which can easily be proved. These commutation relations imply that

$$\hat{N}\hat{a}^\dagger|n\rangle = (n+1)\hat{a}|n\rangle, \tag{2.82}$$

$$\hat{N}\hat{a}|n\rangle = (n-1)\hat{a}|n\rangle. \tag{2.83}$$

Therefore $\hat{a}^\dagger(\hat{a})$ increase (lower) the energy eigenvalue by one. For this reason we call $\hat{a}^\dagger$ and $\hat{a}$ as creation and annihilation operator respectively. We make a normalisation by requiring that the norm of $\hat{a}|n\rangle, \hat{a}^\dagger|n\rangle$, and $|n\rangle$ are the same. Therefore, we have

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.84}$$

Therefore, given an eigenket $|n\rangle$, we can create the other states by applying repeatedly by $\hat{a}$ or $\hat{a}^\dagger$. Actually, the energy is bounded from below because

$$n = \langle n|\hat{N}|n\rangle = (\langle n|\hat{a}^\dagger)(a|n\rangle) \geq 0. \tag{2.85}$$

The ground state must be annihilated by $\hat{a}$. Therefore, the ground state is $|0\rangle$. Its energy is $E_0 = \omega/2$. We create the other states by $\hat{a}^\dagger$. Therefore, the energy eigenvalues are (for $\hat{H}|n\rangle = E_n|n\rangle$)

$$E_n = \omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots. \tag{2.86}$$

Let us now discuss the system of $\mathcal{N}$ noninteracting SHOs with equal mass. The Hamiltonian is written as

$$\hat{H} = \sum_{i=1}^{\mathcal{N}} \omega_i(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}), \tag{2.87}$$

where

$$\hat{a}_i = \left(\sqrt{\frac{m\omega_i}{2}}\hat{x}^i + i\frac{\hat{p}^i}{\sqrt{2m\omega_i}}\right), \quad \hat{a}_i^\dagger = \left(\sqrt{\frac{m\omega_i}{2}}\hat{x}^i - i\frac{\hat{p}^i}{\sqrt{2m\omega_i}}\right), \tag{2.88}$$

with commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. The eigenstates of the Hamiltonian form Fock space. The states in Fock space are given generally in the form

$$|n_1, n_2, \ldots, n_{\mathcal{N}}\rangle, \tag{2.89}$$

which represents the state created from the ground state $|0\rangle = |0, 0, \ldots, 0\rangle$ by acting n_i times with $\hat{a}_i^\dagger$, for $i = 1, 2, \ldots, \mathcal{N}$.

2.2.4 Quantisation of Fields

In canonical quantisation, we make the coordinates and their conjugate momenta to become hermitian operators, and we impose commutation relations. We can also do the canonical quantisation for fields. Given a field $\phi(t, \vec{x})$ and its conjugate momentum $\Pi(t, \vec{x})$, we make them become $\hat{\phi}(t, \vec{x})$, and $\hat{\Pi}(t, \vec{x})$. The equal time commutation relations are

$$[\hat{\phi}(t, \vec{x}), \hat{\Pi}(t, \vec{y})] = i\delta^{(d)}(\vec{x} - \vec{y}), \quad [\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = 0, \quad [\hat{\Pi}(t, \vec{x}), \hat{\Pi}(t, \vec{y})] = 0. \tag{2.90}$$

Let us now quantise the real scalar field given in equation (2.45). We promote the amplitudes $a_{\vec{k}}$ to operators $\hat{a}_{\vec{k}}$, and promote $a_{\vec{k}}^*$ to become $\hat{a}_{\vec{k}}^\dagger$. This ensures that the field operator $\hat{\phi}(t, \vec{x})$ is hermitian. Therefore we have

$$\hat{\phi}(t, \vec{x}) = \int \frac{d^d\vec{k}}{(2\pi)^d\sqrt{2E_{\vec{k}}}}(\hat{a}_{\vec{k}}e^{-ik_\mu x^\mu} + \hat{a}_{\vec{k}}^* e^{ik_\mu x^\mu})\Big|_{k^0=E_{\vec{k}}}.$$

We can obtain conjugate momentum accordingly. From the commutation relation we have

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = (2\pi)^d\delta^{(d)}(\vec{k} - \vec{k}'), \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0, \quad [\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k}'}^\dagger] = 0. \tag{2.91}$$

To quantise the Hamiltonian given in the equation (2.51), we write $|a_{\vec{k}}|^2 = (a_{\vec{k}}a_{\vec{k}}^* + a_{\vec{k}}^* a_{\vec{k}})/2$. The Hamiltonian is then

$$\hat{H} = \int \frac{d^d\vec{k}}{(2\pi)^d} E_{\vec{k}}(\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \frac{1}{2}[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}}^\dagger]). \tag{2.92}$$

From the commutation relations, we realise that the second term is infinite. However, we can just ignore this term because it is the vacuum energy, and in experiments we only measure energy differences from vacuum. Therefore we rewrite the Hamiltonian as

$$\hat{H} = \int \frac{d^d\vec{k}}{(2\pi)^d} E_{\vec{k}} \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}. \tag{2.93}$$

Similarly, the spatial momentum is

$$\hat{\vec{P}} = \int \frac{d^d\vec{k}}{(2\pi)^d} \vec{k} \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}. \tag{2.94}$$

Looking at the Hamiltonian and the commutation relations, we realise that the system can be considered as having infinitely many SHOs. A Fock state

$$|\psi\rangle = \vec{a}^\dagger_{\vec{k}_1} \vec{a}^\dagger_{\vec{k}_2} \cdots \vec{a}^\dagger_{\vec{k}_n} |0\rangle \tag{2.95}$$

is an eigenstate for Hamiltonian and spatial momentum with eigenvalues $\sum_{i=1}^n E_{\vec{k}_i}$ and $\sum_{i=1}^n \vec{k}_i$ respectively. Therefore we may interpret the state $|\psi\rangle$ as the state containing n particles with momentum $(E_{\vec{k}_i}, \vec{k}_i)$ for $i = 1, \ldots, n$.

In the next chapter, free massless vector and tensor fields will emerge from string theory. Fock states can also be constructed similar to the case of free scalar field. However, the creation and annihilation operators will carry vector or tensor index. We also have to be careful about the gauge freedom.

Chapter 3

String Theory

The materials in this chapter are also standard. The purpose of this chapter is to remind the readers about the basic of string theory. We start this chapter by considering classical string. We then quantise the string, and will find that particle states are coming from the quantised string. We will also discuss about the string in background field. We end this chapter by a brief discussion on superstring theory.

The materials in this chapter are based on [9], [10], [11], [12], [5], [13].

3.1 Classical String

Let us first consider the case of relativistic point particle. The coordinates of a particle $x^\mu(\tau)$ are described using one parameter which is proper time τ. By construction we need the action to be invariant under reparameterisation $\tau \to \widetilde{\tau}(\tau)$. The action can be written as

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{3.1}$$

where $\dot{x}^\mu \equiv dx^\mu/d\tau$.

Let us now consider a string in $d+1$ dimensional flat spacetime. We need two parameters to describe string coordinates $X^\mu(\tau,\sigma)$. We need proper time τ and the parameter along the string σ. Let us denote the end points of the string by σ_L and σ_R. If the string is closed, then $\sigma_L = \sigma_R$, otherwise the string is open. The space of the parameters τ and σ is called worldsheet. We require the action to be invariant under reparameterisation. So we have

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu}, \tag{3.2}$$

where α' is a constant; μ is the Lorentz index; α and β are worldsheet indices; $\eta^{\alpha\beta} = \mathrm{diag}(-1,1)$; $d^2\sigma \equiv d\sigma d\tau$; $\partial_\alpha \equiv \partial/\partial\sigma^\alpha$ with $\sigma^\alpha = (\tau,\sigma)$. This action is called Nambu-Goto action. Because of the square root in the action, the calculations will become messy. The trick is to write the action as

$$S_{Poly} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{3.3}$$

where $h^{\alpha\beta}$ is an inverse of the metric $h_{\alpha\beta}$, and $h \equiv \det h_{\alpha\beta}$. This action is called Polyakov action. If we vary the action with respect to $h^{\alpha\beta}$ we get constraint conditions. Solving these conditions for $h^{\alpha\beta}$ and pluging $h^{\alpha\beta}$ into the Polyakov action gives us the Nambu-Goto action.

The Polyakov action has the following symmetries

1. $(d+1)-$dimensional Poincaré invariant

$$X'^{\mu}(\tau,\sigma) = \Lambda^{\mu}{}_{\nu} X^{\nu}(\tau,\sigma) + a^{\mu}, \tag{3.4}$$

$$h'_{\alpha\beta}(\tau,\sigma) = h_{\alpha\beta}(\tau,\sigma), \tag{3.5}$$

where $\Lambda^{\mu}{}_{\nu}$ is the Lorentz transformation matrix and a^{μ} is a constant.

2. Diffeomorphism invariant (reparameterisation)

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma), \tag{3.6}$$

$$\frac{\partial\sigma'^{\alpha}}{\partial\sigma^{\gamma}}\frac{\partial\sigma'^{\beta}}{\partial\sigma^{\delta}} h_{\alpha\beta}(\tau',\sigma') = h_{\gamma\delta}(\tau,\sigma). \tag{3.7}$$

3. Weyl invariant

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma), \tag{3.8}$$

$$h'_{\alpha\beta}(\tau,\sigma) = \exp\left(2\omega(\tau,\sigma)\right) h_{\alpha\beta}(\tau,\sigma). \tag{3.9}$$

Using the diffeomorphism invariant and Weyl invariant, we can choose

$$h_{\alpha\beta} = \eta_{\alpha\beta}. \tag{3.10}$$

This choice is called flat gauge. Varying the Polyakov action with respect to string coordinates gives the equations of motions and boundary conditions. Furthermore, there are constraints obtained from $\delta S/\delta h^{\alpha\beta} = 0$. Let us first consider the equations of motion

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^{\mu}(\tau,\sigma) = 0. \tag{3.11}$$

The equations of motion are just wave equations with general solutions in the form

$$X^{\mu}(\tau,\sigma) = X^{\mu}_{R}(\tau-\sigma) + X^{\mu}_{L}(\tau+\sigma). \tag{3.12}$$

We call $X^{\mu}_{R}(\tau-\sigma)$, and $X^{\mu}_{L}(\tau+\sigma)$ as right-moving part and left-moving part, respectively. Let us now discuss about constraints and boundary conditions. In flat gauge, the constraints read

$$\dot{X}^{\mu}\dot{X}_{\mu} + X'^{\mu}X'_{\mu} = 0, \quad \dot{X}^{\mu}X'_{\mu} = 0. \tag{3.13}$$

The boundary term from the variation of S in flat gauge is

$$-\frac{1}{2\pi\alpha'}\int d\tau \partial_{\sigma}X^{\mu}\delta X_{\mu}\Big|^{\sigma=\sigma_R}_{\sigma=\sigma_L}. \tag{3.14}$$

Setting this term equals to zero gives the boundary condition. Obviously, there is no boundary condition for closed strings. So let us consider an open string. At endpoints, for each Lorentz index μ, we have either $\partial_{\sigma}X^{\mu} = 0$ or $\delta X^{\mu} = 0$. These boundary conditions are called Neumann boundary condition and Dirichlet boundary condition, respectively.

Let us now study the string coordinates. For closed string, we take the interval to be $\sigma \in [0, 2\pi]$. The solutions $X^\mu(\tau,\sigma)$ of equations of motion are in the general form $X^\mu(\tau,\sigma) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$ with

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}(\tau-\sigma)p^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}, \tag{3.15}$$

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}(\tau+\sigma)p^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}, \tag{3.16}$$

where $x^\mu, p^\mu, \alpha^\mu_n$, and $\widetilde{\alpha}^\mu_n$ are constant. We also require

$$(\alpha^\mu_n)^* = \alpha^\mu_{-n}, \quad (\widetilde{\alpha}^\mu_n)^* = \widetilde{\alpha}^\mu_{-n}. \tag{3.17}$$

For open string, we take $\sigma \in [0, \pi]$. If the direction X^a has the Dirichlet boundary condition, the solutions are given by

$$X^a = x^a + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^a_n e^{-in\tau}\sin n\sigma. \tag{3.18}$$

If the direction X^i has the Neumann boundary condition, the solutions are given by

$$X^i = x^i + 2\alpha' p^i\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^i_n e^{-in\tau}\cos n\sigma. \tag{3.19}$$

For now let us focus on the case where the open string satisfies Neumann boundary condition for the whole spacetime.

Having got the string coordinates, we may start quantising the strings using covariant quantisation. However, we can also choose some gauge before start quantising. Let us introduce the coordinates

$$\sigma^\pm \equiv \tau \pm \sigma. \tag{3.20}$$

After imposing the flat gauge, we are still allowed to make a further transformation which preserves the symmetries. The transformation is of the form

$$\sigma^+ \to \widetilde{\sigma}^+(\sigma^+), \quad \sigma^- \to \widetilde{\sigma}^-(\sigma^-), \tag{3.21}$$

or

$$\widetilde{\tau} = \frac{1}{2}(\widetilde{\sigma}^+(\tau+\sigma) + \widetilde{\sigma}^-(\tau-\sigma)), \tag{3.22}$$

$$\widetilde{\sigma} = \frac{1}{2}(\widetilde{\sigma}^+(\tau+\sigma) - \widetilde{\sigma}^-(\tau-\sigma)). \tag{3.23}$$

We see immediately that $\widetilde{\tau}$ satisfies the wave equation (3.11). We can therefore set τ equal to a linear combination of string coordinates. We take $\tau \sim (X^0 + X^1)/\sqrt{2} \equiv X^+$. This choice is called light-cone gauge. Explicitly, for closed strings we choose

$$X^+ = \alpha' p^+ \tau, \tag{3.24}$$

and for open strings (with Neumann boundary condition for X^0 and X^1) we choose

$$X^+ = 2\alpha' p^+ \tau. \tag{3.25}$$

For other components we define $X^- = (X^0 - X^1)/\sqrt{2}$, and the components X^I for $I = 2, \ldots d$ remain the same. We can also write X^- as a mode expansion. Explicitly, for closed string, we have

$$X_R^-(\tau - \sigma) = \frac{1}{2}x^- + \frac{\alpha'}{2}(\tau - \sigma)p^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in(\tau-\sigma)}, \tag{3.26}$$

$$X_L^-(\tau + \sigma) = \frac{1}{2}x^- + \frac{\alpha'}{2}(\tau + \sigma)p^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^- e^{-in(\tau+\sigma)}. \tag{3.27}$$

For open strings (with Neumann boundary condition for X^0 and X^1) we have

$$X^- = x^- + 2\alpha' p^- \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma. \tag{3.28}$$

There is a useful quantity arising from constraint equations (3.13). We study the Fourier components of the constraint equations. We take each component equals zero. The most important condition from this is

$$\int_0^{\sigma_0} d\sigma(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu) = 0, \tag{3.29}$$

where $\sigma_0 = \pi$ for a closed string but $\sigma_0 = 2\pi$ for an open string. We evaluate this equation at $\tau = 0$, and we work in light-cone gauge. For closed string, the condition implies

$$M^2 = -2p^+p^- + \sum_{i=2}^{d} p^I p^I = \frac{2}{\alpha'}\left(N^\perp + \widetilde{N}^\perp\right), \tag{3.30}$$

where

$$N^\perp \equiv \sum_{I=2}^{d}\sum_{n=1}^{\infty}\alpha_n^I\alpha_{-n}^I, \quad \widetilde{N}^\perp \equiv \sum_{I=2}^{d}\sum_{n=1}^{\infty}\widetilde{\alpha}_n^I\widetilde{\alpha}_{-n}^I. \tag{3.31}$$

The quantity M^2 is realised as mass square of a relativistic particle in light-cone guage. There is also another useful condition from constraint equations. From

$$\int_0^{2\pi} d\sigma(\dot{X}^\mu + X'^\mu)(\dot{X}_\mu + X'_\mu) = \int_0^{2\pi} d\sigma(\dot{X}^\mu - X'^\mu)(\dot{X}_\mu - X'_\mu) = 0, \tag{3.32}$$

we have

$$N^\perp = \widetilde{N}^\perp. \tag{3.33}$$

Similarly, for open string (with Neumann boundary condition in the whole spacetime), we have

$$M^2 = \frac{1}{\alpha'}N^\perp, \tag{3.34}$$

where M^2 and $N^\perp$ are defined similarly to closed string's.

We can treat string coordinates $X^\mu(\tau, \sigma)$ as one-dimensional fields with 'time' τ and 'space' σ. For flat gauge the conjugate momentum for $X^\mu(\tau, \sigma)$ is

$$P_\mu(\tau, \sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu(\tau, \sigma)} = \frac{1}{2\pi\alpha'}\dot{X}_\mu(\tau, \sigma), \tag{3.35}$$

where the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4\pi\alpha'}(-\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu). \tag{3.36}$$

We use Poisson bracket as defined in equation (2.52). In this case

$$\{G, K\} \equiv \int_0^{\sigma_0} d\sigma \left(\frac{\delta G}{\delta X^\mu}\frac{\delta K}{\delta P_\mu} - \frac{\delta K}{\delta X^\mu}\frac{\delta G}{\delta P_\mu} \right),$$

for functions $G(X^\mu, P^\mu), K(X^\mu, P^\mu)$. We have

$$\{X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta^(\sigma - \sigma'), \quad \{X^\mu(\tau,\sigma), X^\mu(\tau,\sigma')\} = 0, \quad \{P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\} = 0. \tag{3.37}$$

For usual fields, we are normally not worried about the spatial boundaries because they are at infinity. It is also safe for closed strings, because they have no boundary. For open strings however, the boundaries are not at infinity but at the end points. Therefore we need to be careful[1]. It turns out that, because of the Neumann and Dirichlet boundary conditions, the Poisson bracket for string end points also satisfy the equation (3.37). In Part II, when the string is put in a background, the boundary conditions are changed. The Poisson bracket at the end points will no longer satisfy the equation (3.37).

3.2 Quantum String

We are now ready to quantise strings. The main quantisation methods are canonical quantisation, which we have already encountered, and path integral quantisation, which will be discussed later. In canonical quantisation, we may either quantise the Poisson brackets (3.37) directly or transform the coordinates into light-cone gauge before quantising. In the former case, there are ghost states arising, but they can be decoupled from physical states after fixing spacetime dimensions. In the light-cone gauge, all states are physical but we lose Lorentz symmetry; however, we can recover this after fixing spacetime dimensions. Let us discuss light-cone gauge quantisation in a bit more details.

The Poisson brackets (3.37) are modified in light cone gauge. When quantising, we promote the string coordinates and momenta to be operators. The Poisson brackets become commutators. The non-zero commutation relations are

$$[\hat{X}^-(\tau,\sigma), \hat{P}^+(\tau,\sigma')] = \quad -i\delta(\sigma - \sigma'), \tag{3.38}$$
$$[\hat{X}^+(\tau,\sigma), \hat{P}^-(\tau,\sigma')] = \quad -i\delta(\sigma - \sigma'), \tag{3.39}$$
$$[\hat{X}^I(\tau,\sigma), \hat{P}^J(\tau,\sigma')] = \quad i\delta^{IJ}\delta(\sigma - \sigma'), \tag{3.40}$$

where we label transverse coordinates by $I, J = 2, 3, \ldots, d$. We also promote the mode expansions of string coordinates to operators. Important commutation relations for closed string are

$$[\hat{x}^I, \hat{p}^J] = i\delta^{IJ}, [\hat{x}^-, \hat{p}^+] = -i. \tag{3.41}$$

$$[\hat{\tilde{\alpha}}^I_m, \hat{\tilde{\alpha}}^J_n] = m\delta_{m+n}\delta^{IJ}, [\hat{\alpha}^I_m, \hat{\alpha}^J_n] = m\delta_{m+n}\delta^{IJ}, [\hat{\tilde{\alpha}}^I_m, \hat{\alpha}^J_n] = 0, \tag{3.42}$$

[1]As an aside, some references for example [14] deal formally with Poisson bracket for fields on a manifold with boundaries. Equivalently, we can also use Dirac bracket which is a generalisation to Poisson bracket.

For open string with Neumann boundary condition for whole spacetime the commutation relations are similar. See for example [5] for calculations.

The reality conditions (3.17) are promoted to

$$(\hat{\alpha}_n^\mu)^\dagger = \hat{\alpha}_{-n}^\mu, \quad (\hat{\tilde{\alpha}}_n^\mu)^\dagger = \hat{\tilde{\alpha}}_{-n}^\mu. \tag{3.43}$$

The commutation relations for the modes suggest that for $m = 1, 2, \ldots$

$$\frac{1}{\sqrt{m}}\hat{\alpha}_m^I, \quad \frac{1}{\sqrt{m}}\hat{\tilde{\alpha}}_m^I \tag{3.44}$$

are annihilation operators, and

$$\frac{1}{\sqrt{m}}(\hat{\alpha}_m^I)^\dagger, \quad \frac{1}{\sqrt{m}}(\hat{\tilde{\alpha}}_m^I)^\dagger \tag{3.45}$$

are creation operators. The quantities $N^\perp$, and $\widetilde{N}^\perp$ for classic strings are promoted to be number operators $\hat{N}^\perp$, and $\hat{\tilde{N}}^\perp$. It is natural to write number operators as

$$\hat{N}^\perp = \sum_{I=2}^{d}\sum_{n=1}^{\infty}\left(\hat{\alpha}_{-n}^I\hat{\alpha}_n^I + \frac{1}{2}[\hat{\alpha}_n^I, \hat{\alpha}_{-n}^I]\right) \tag{3.46}$$

and similar for $\hat{\tilde{N}}^\perp$. We see that the second term blows up. This problem is similar to the case of quantising the scalar field. Therefore, we expect that there is a reasonable way to deal with infinity. It turns out that if we require Lorentz symmetry, we get the spacetime dimension $d + 1 = 26$ and the mass square operator

$$\hat{M}^2 = \frac{2}{\alpha'}(\hat{N} + \hat{\tilde{N}} - 2) \tag{3.47}$$

for closed string, and

$$\hat{M}^2 = \frac{1}{\alpha'}(\hat{N} - 1) \tag{3.48}$$

for open string. Here we redefine the number operators as

$$\hat{N}^\perp = \sum_{I=2}^{d}\sum_{n=1}^{\infty}\hat{\alpha}_{-n}^I\hat{\alpha}_n^I, \quad \hat{\tilde{N}}^\perp = \sum_{I=2}^{d}\sum_{n=1}^{\infty}\hat{\tilde{\alpha}}_{-n}^I\hat{\tilde{\alpha}}_n^I. \tag{3.49}$$

Let us consider the states for closed string. From equation (3.33) we have a level matching constraint $\hat{N}^\perp = \hat{\tilde{N}}^\perp$. So we see that the right-moving and left-moving modes are separate, but not completely. The number operators for right-moving and left-moving mode have the same eigenstates and eigenvalues. We denote the ground state by

$$|p^+, \vec{p}_T\rangle. \tag{3.50}$$

This state is annihilated by all annihilation operators. Therefore

$$\hat{N}^\perp|p^+, \vec{p}_T\rangle = \hat{\tilde{N}}^\perp|p^+, \vec{p}_T\rangle = 0. \tag{3.51}$$

The mass-square of this state is then

$$\hat{M}^2|p^+,\vec{p}_T\rangle = -\frac{4}{\alpha'}|p^+,\vec{p}_T\rangle. \tag{3.52}$$

We interpret the particle associated to this state as tachyon which has a negative mass-square. The tachyon travels faster than speed of light. Therefore we do not expect the tachyon to enter into the theory. Another interpretation of tachyon is that it represents the unstable vacuum. The issue about tachyon is solved in superstring theory.

As usual the excited states are generated by creation operators. The first excited state is in the form

$$\hat{\alpha}^I_{-1}\hat{\tilde{\alpha}}^J_{-1}|p^+,\vec{p}_T\rangle \tag{3.53}$$

which satisfies level matching constraint. Explicitly

$$\hat{N}\hat{\alpha}^I_{-1}\hat{\tilde{\alpha}}^J_{-1}|p^+,\vec{p}_T\rangle = \hat{\tilde{N}}^{\perp}\hat{\alpha}^I_{-1}\hat{\tilde{\alpha}}^J_{-1}|p^+,\vec{p}_T\rangle = \hat{\alpha}^I_{-1}\hat{\tilde{\alpha}}^J_{-1}|p^+,\vec{p}_T\rangle. \tag{3.54}$$

These states are massless

$$\hat{M}^2\hat{\alpha}^I_{-1}\hat{\tilde{\alpha}}^J_{-1}|p^+,\vec{p}_T\rangle = 0. \tag{3.55}$$

In order to realise the states as particles, we make linear combinations of these states. The symmetric traceless part is graviton which is massless symmetric spin 2. The antisymmetric part is called B-field (in superstring theory the B-fields are called NS-NS B-fields) which is massless antisymmetric spin 2. The trace part is called dilaton which is a massless scalar. We can also get these massless states in superstring theory. Higher states can also be constructed. They are all massive. However, we can ignore these states if we work in low energy limit.

The states for open string are constructed similarly. The ground state $|p^+,\vec{p}_T\rangle$ is tachyon. The first excited state $\hat{\alpha}^J_{-1}|p^+,\vec{p}_T\rangle$ is a massless spin 1 particle called photon which is a particle corresponding to electromagnetic field. Higher states are all massive.

So far when considering open string, we have only restricted ourselves to open string with Neumann boundary condition in the whole spacetime. We now consider open string with both Neumann and Dirichlet boundary conditions. In classical string we see that the Dirichlet boundary condition requires string position to be fixed in some direction. This is interpreted as open string attached to extended dynamical objects called 'D-branes'.

Let us consider the simplest case where two end points attached to the same D-brane. Let the coordinates $X^i, i = 0, 1, \ldots, p$ satisfy Neumann boundary conditions

$$X'^i(\tau,\sigma)\Big|_{\sigma=0} = X'^i(\tau,\sigma)\Big|_{\sigma=\pi} = 0, \tag{3.56}$$

and the coordinates $X^a, a = p+1, p+2, \ldots, d$ satisfy Dirichlet boundary conditions

$$X^a(\tau,\sigma)\Big|_{\sigma=0} = X^a(\tau,\sigma)\Big|_{\sigma=\pi} = 0. \tag{3.57}$$

This is interpreted as an open string with both end points attached to the same Dp-brane. Here the Dp-brane is an extended object having p spatial dimensions. The coordinates along the Dp-brane are X^i, and the coordinates normal to the Dp-brane are X^a.

When quantising the string attached to D-brane, we do as previous. We change the coordinates into light-cone gauge. Then we quantise, and the Fourier modes become creation and annihilation operators. We then consider the eigenstates of number operator. The ground state is tachyon. The first excited states are massless and consist of two parts. The first part $\hat{\alpha}^m_{-1}|p^+,\vec{p}_T\rangle, m = 2,\ldots,p$ is interpreted as electromagnetic field living on the D-brane. The second part $\hat{\alpha}^a_{-1}|p^+,\vec{p}_T\rangle$ is interpreted as $(d-p)$ massless scalars living on D-brane.

3.3 Strings in Background

Alternative to canonical quantisation, there is another quantisation method which is called 'path integral quantisation' . Let us review the idea of path integral quantisation in the case of point particle. Given two points $\vec{q}_i, t_i$, and $\vec{q}_f, t_f$ in spacetime. We want to find the transition probability from initial state $|\vec{q}_i, t_i\rangle$ at initial position to the final state $|\vec{q}_f, t_f\rangle$ at final position. In order to do so we calculate the transition amplitude $\langle\vec{q}_f, t_f|\vec{q}_i, t_i\rangle$ and find its modulus square.

To find the transition amplitude we note that a quantum particle can choose any path between the two points. There is an amplitude associated to each path, and there is an associated phase which is related to action S of that path. From superposition principle, the total transition amplitude is the sum of amplitudes for all paths. Therefore the transition amplitude is in the form

$$\langle\vec{q}_f, t_f|\vec{q}_i, t_i\rangle = \sum_{\text{all paths}} e^{iS[\vec{q}(t)]}. \tag{3.58}$$

In the case of field theory the idea is similar. We usually consider the transition from vacuum at $x \to -\infty$ to vacuum at $x \to \infty$ (we consider one dimensional space). The amplitude is

$$Z = \int D\phi e^{iS[\phi(x,t)]}, \tag{3.59}$$

where $D\phi$ is the integration measure for integrating the field ϕ over all paths. If there is an interaction, the source $J(x)$ is usually added into the action. The amplitude Z can now be used to generate correlation functions which describe interaction.

This can also be applied to string. We see that in order to describe classical string, we only need Polyakov action S. When we want to path-integral quantise the string we write the amplitude Z and compute important quantities. For free string we have the action

$$S_0 = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{3.60}$$

So

$$Z_0 = \int DXDg e^{-S_0[X,g]}. \tag{3.61}$$

Here we have made Wick rotation $\tau \to i\tau, S \to iS$ in the definitions. Note also that from Wick rotation the worldsheet metric transforms as $h_{\alpha\beta} \to g_{\alpha\beta}$. We have discussed that free string action have 3 symmetries: Poincaré, diffeomorphism, and Weyl. These symmetries also remain after quantisation. Actually, the integral in the amplitude (3.61) is over counted. Because of the symmetry, some different values of X^μ and $h_{\alpha\beta}$ give the same path. There is a way to cure this by using Faddeev-Popov technique. The over-counting terms are factored out at the expense of introducing ghost fields. We will not discuss the details of this issue.

Let us now try to put some backgrounds in. The most obvious generalisation would be

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X), \tag{3.62}$$

where $G_{\mu\nu}$ is metric for curved spacetime. This action actually describes the interaction between string and graviton. Consider a small gravity limit $G_{\mu\nu} \approx \eta_{\mu\nu} + \mathcal{G}_{\mu\nu}$. The integrand of the amplitude Z is approximately

$$e^{-S} \approx e^{-S_0}(1+V), \tag{3.63}$$

where

$$V = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu \mathcal{G}_{\mu\nu}(X) \tag{3.64}$$

is realised as the quantity describing the interaction between string and graviton. We can also include other massless backgrounds by writing the action as

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\left[\left(g^{\alpha\beta}G_{\mu\nu}(X) + i\epsilon^{\alpha\beta}B_{\mu\nu}(X)\right)\partial_\alpha X^\mu\partial_\beta X^\nu + \alpha' R\Phi(X)\right], \tag{3.65}$$

where $\epsilon^{\alpha\beta}$ is antisymmetric with $\epsilon^{01}=1$; $B_{\mu\nu}(X)$ is an antisymmetric tensor; $\Phi(X)$ is a scalar; R is Ricci scalar of string worldsheet. If we expand $G_{\mu\nu}(X), B_{\mu\nu}(X)$, and $\Phi(X)$ around the flat background $G_{\mu\nu}(X)=\eta_{\mu\nu}, B_{\mu\nu}(X)=0, \Phi(X)=0$ we will see the term of interaction between string and massless backgrounds which are graviton, B-field, and dilation. The dilaton term (the term involving $\Phi(X)$) depends on local worldsheet topology. Explicitly, in this term, the dilaton couples to Euler characteristic $\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R$ of the worldsheet. For example, at the tree level of oriented open string, the worldsheet is topologically equivalent to a sheet $[0,1]\times\mathbb{R}$, and the Euler characteristic is $\chi=1$. At one-loop level of oriented open string, the worldsheet is topologically equivalent to $[0,1]\times\mathbb{R}$ with an open disk removed. The Euler characteristic is then $\chi=0$.

We require that the theory is symmeric under Poincaré, diffeomorphism, and Weyl transformation. The Poincaré and diffeomorphism symmetry are easy to get. For Weyl symmetry, it requires $T^\alpha_{\ \alpha}=0$, where

$$T^{\alpha\beta} \equiv -\frac{2\pi}{\sqrt{g}}\frac{\delta S}{\delta g_{\alpha\beta}}. \tag{3.66}$$

The calculation actually requires details and interpretations. However, we will state only the result. For quantum string we have

$$T^\alpha_{\ \alpha} = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}\beta^\Phi R, \tag{3.67}$$

where $\beta^G_{\mu\nu}, \beta^B_{\mu\nu}$, and β^Φ are called $\beta-$functions (not to be confused with index β). In order to have Weyl symmetry, the $\beta-$functions must be set to zero. For small energy limit $\alpha' << 1$, the $\beta-$functions are

$$\beta^G_{\mu\nu} = \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\kappa\sigma}H_\nu^{\ \kappa\sigma}\right) + \mathcal{O}(\alpha'^2), \tag{3.68}$$

$$\beta^B_{\mu\nu} = \alpha'\left(-\frac{1}{2}\nabla^\kappa H_{\kappa\mu\nu} + \nabla^\kappa\Phi H_{\kappa\mu\nu}\right) + \mathcal{O}(\alpha'^2), \tag{3.69}$$

$$\beta^\Phi_{\mu\nu} = \alpha'\left(\frac{d+1-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\kappa\Phi\nabla^\kappa\Phi - \frac{1}{24}H_{\kappa\mu\nu}H^{\kappa\mu\nu}\right) + \mathcal{O}(\alpha'^2), \tag{3.70}$$

where ∇_μ is a covariant derivative in spacetime; $R_{\mu\nu}$ is Ricci tensor; $H_{\mu\nu\kappa} \equiv \nabla_\mu B_{\nu\kappa} + \nabla_\nu B_{\kappa\mu} + \nabla_\kappa B_{\mu\nu}$. In fact, these equations can be derived from the effective action

$$S_{\text{eff}} = \frac{1}{2\kappa_0^2}\int d^{d+1}x\sqrt{-G}e^{-2\Phi}\left[R + 4\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{2(d+1-26)}{3\alpha'} + \mathcal{O}(\alpha')\right], \tag{3.71}$$

where κ_0 is a constant, and G is the determinant of $G_{\mu\nu}$. We can check that this action is invariant since $d^{d+1}x\sqrt{-G}$ is invariant hypervolume, and the integrands are invariant.

This means that, at low energy limit, the string can only move in the massless backgrounds satisfying $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$. Immediately, we see the requirement for spacetime dimension $d+1 = 26$. We can also see that the requirement $\beta^G_{\mu\nu} = 0$ gives the quantum correction to vacuum Einstein equation. The very obvious background is $R_{\mu\nu} = 0, \Phi =$ constant, and $H_{\kappa\mu\nu} = 0$. From Poincaré's lemma, the equation $H_{\kappa\mu\nu} = 0$ implies that $B_{\mu\nu}$ has a gauge freedom: therefore in contractible spacetime manifold $B_{\mu\nu} = \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu$ for any function $\Lambda_\mu(X)$.

3.4 A Very Brief Introduction To Superstring Theory

So far we have been discussing bosonic string theory. This theory requires 26 dimensional spacetime. The particles arising in this theory are all boson. The bosonic states are symmetric under an exchange of any two bosons. There is another kind of particle which is fermion. Fermionic states are antisymmetric under an exchange of any two fermions. Fermionic worldsheet variables are anticommuting. This implies that two fermions cannot have the same state. Bosons and fermions can be produced from superstring theory which is a shorthand for supersymmetric string theory. In superstring theory, there is a symmetry between bosons and fermions.

Since there are also fermionic worldsheet variables in superstring theory, there must be a modification in the calculation of spacetime dimension. It turns out that superstring theory requires 10 dimensional spacetime.

There are two kinds of boundary conditions for fermion: periodic and anti-periodic. A fermionic sector having periodic boundary condition is called Ramond (R) sector while the sector having anti-periodic boundary conditions is called Neveu-Schwarz (NS) sector. For closed strings there are right-moving and left-moving sectors. Each sector can either be Ramond or Neveu-Schwarz sector. All possible combinations are NS-NS, NS-R, R-NS, and R-R sectors. States from NS-NS and R-R sectors are bosonic. There are no tachyonic states in superstring theory. Massless bosonic states are graviton, B-field, dilaton, and potentials coupled to D-brane charge. States from NS-R and R-NS are fermions.

Having briefly reviewed the basics, let us now move on to study more advanced stuffs.

Part II

SOME PHYSICS OF NONCOMMUTATIVE GEOMETRY

Chapter 4

Noncommutative Geometry from String Theory

One of the interesting developments in string theory is the realisation of noncommutative geometry. The simplest situation is the noncommutative worldvolume on D-brane in a constant B-field. Many physical consequences have been studied.

Actually, it is also possible to have noncommutativity in momentum space so that the whole phase space becomes noncommutative. This happens when we consider D-brane in pp-wave background with a constant B-field. Physical consequences have not yet been studied. So we may try this in later chapters

We will discuss the two situations in this chapter. The discussions are based on [15], [16], [17]

4.1 D-branes and Noncommutative Spacetime

We have seen that strings can live in some certain backgrounds, and it is very obvious that the background

$$G_{\mu\nu}(X) = \eta_{\mu\nu}, \quad H_{\mu\nu\kappa}(X) = 0, \quad \Phi(X) = \text{constant} \tag{4.1}$$

is allowed. We consider an open string attached to a Dp-brane living in this background. Let X^i, for $i = 0, \ldots, p$ be the coordinates along the Dp-brane. We have also seen that electromagnetic field can live on Dp-brane. This field couples with charged open string's endpoints as the endpoints behave like point particles. This interaction is given by

$$\frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau A_i(X) \partial_\tau X^i \tag{4.2}$$

where $\partial\Sigma$ is the boundary of the string worldsheet Σ. In this case $\partial\Sigma$ represents string endpoints at any time. Note that we have put the endpoint charge $q = 1$ at $\sigma = \pi$ and $q = -1$ at $\sigma = 0$. The full action in bosonic string theory or the bosonic part in superstring theory is given by (cf. before Wick rotation of equation (3.65))

$$S_B = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(\eta^{\alpha\beta}\eta_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}(X)\right) \partial_\alpha X^\mu \partial_\beta X^\nu + \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau A_i(X) \partial_\tau X^i. \tag{4.3}$$

Note that the dilaton part is just a constant since $\Phi(X) = \text{constant}$, and we only consider one type of worldsheet topology. Therefore the dilaton part makes no contribution to the theory, and we can just ignore them.

There is an important relation between B-field and electromagnetic potential $A_i(X)$. Let $F_{ij}(X) = \partial_i A_j(X) - \partial_j A_i(X)$ be electromagnetic field strength. Consider

$$
\begin{aligned}
\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{\alpha\beta}F_{ij}(X)\partial_\alpha X^i\partial_\beta X^j =&\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{\alpha\beta}(\partial_i A_j(X)-\partial_j A_i(X))\partial_\alpha X^i\partial_\beta X^j\\
=&\frac{2}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{\alpha\beta}\partial_i A_j(X)\partial_\alpha X^i\partial_\beta X^j\\
=&\frac{2}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{\alpha\beta}\partial_\alpha A^j(X)\partial_\beta X^j\\
=&\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma[\partial_\tau A_j(X)\partial_\sigma X^j - \partial_\sigma A_j(X)\partial_\tau X^j]\\
=&\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma[\partial_\tau(A_j(X)\partial_\sigma X^j) - \partial_\sigma(A_j(X)\partial_\tau X^j)]\\
=&0-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau A_j(X)\partial_\tau X^j,
\end{aligned}
\tag{4.4}
$$

where in the third step we used chain rule, and in the final step we assume that $A_j(X)$ vanishes at initial and final time τ. If we set the component of B-field to be parallel to the D-brane, we see that the action becomes

$$
S_B = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\left(\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + \epsilon^{\alpha\beta}\mathcal{F}_{ij}(X)\partial_\alpha X^i\partial_\beta X^j\right), \tag{4.5}
$$

where $\mathcal{F}_{ij} = B_{ij} + F_{ij}$. Let us now interpret this result. The B-field and field strength $F_{ij}(X)$ appears together in the action. The resulting field $\mathcal{F}_{ij}$ is invariant under guage transformation of potential A_i :

$$
A_i \to A_i + \partial_i K, \quad B_{ij} \to B_{ij}, \tag{4.6}
$$

and under gauge transformation of B-field:

$$
A_i \to A_i - \Lambda_i, \quad B_{ij} \to B_{ij} + \partial_i\Lambda_j - \partial_j\Lambda_i, \tag{4.7}
$$

for any functions K and Λ_i. These gauge transformations are allowed because they keep the conditions

$$
H_{ijk} = 0, \quad \text{and} \quad \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \tag{4.8}
$$

satisfied.

Let us now vary the action. The variation of the first part is

$$
\begin{aligned}
\delta\Big(-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\Big) =& -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma[-\partial_\tau X^\mu\partial_\tau\delta X_\mu + \partial_\sigma X^\mu\partial_\sigma\delta X_\mu]\\
=& -\frac{1}{2\pi\alpha'}\Big[\cancelto{0}{\int d\sigma(-\partial_\tau X^\mu\delta X_\mu)\Big|_\tau} + \int_\Sigma d^2\sigma(\partial_\tau^2 - \partial_\sigma^2)X^\mu\delta X_\mu + \int_{\partial\Sigma} d\tau\partial_\sigma X^\mu\delta X_\mu\Big].
\end{aligned}
\tag{4.9}
$$

The first term vanishes due to the requirement that $\delta X^\mu = 0$ at initial and final time τ. Now we want to vary the B-field part of the action. Inspired by the equation (4.4) we instead vary

$$
\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\omega_i(X)\partial_\tau X^i, \tag{4.10}
$$

for $\mathcal{F}_{ij} = \partial_i\omega_j - \partial_j\omega_i$. Applying chain rule and the requirement that $\delta X^\mu = 0$ at initial and final time, we have

$$\delta\left(\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\omega_i(X)\partial_\tau X^i\right) = \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} d\tau\mathcal{F}_{ij}(X)\delta X^i\partial_\tau X^j. \tag{4.11}$$

The requirement $\delta S_B = 0$ gives the equation of motion

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0, \tag{4.12}$$

and boundary conditions at $\sigma = 0, \pi$

$$\partial_\sigma X^i + \partial_\tau X^j \mathcal{F}_j{}^i = 0, \quad i, j = 0, 1, \ldots, p, \tag{4.13}$$

$$X^a = x_0^a, \quad a = p + 1, \ldots d. \tag{4.14}$$

In Part I, we have discussed that the general solution to the equation of motion is of the form

$$X^\mu(\tau, \sigma) = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma). \tag{4.15}$$

We can as well write the solution using Fourier expansion. We therefore have

$$X^\mu = x_0^\mu + (a_0^\mu\tau + b_0^\mu\sigma) + \sum_{n\neq 0}\frac{e^{-in\tau}}{n}(ia_n^\mu\cos n\sigma + b_n^\mu\sin n\sigma). \tag{4.16}$$

Imposing the boundary condition (4.13) along D-brane, we have

$$b_n^k + a_n^j\mathcal{F}_j{}^k = 0, \quad \text{all } n. \tag{4.17}$$

Therefore the solution along the D-brane is

$$X^k = x_0^k + (p_0^k\tau - p_0^j\mathcal{F}_j{}^k\sigma) + \sum_{n\neq 0}\frac{e^{-in\tau}}{n}(ia_n^k\cos n\sigma - a_n^j\mathcal{F}_j{}^k\sin n\sigma). \tag{4.18}$$

Here we denoted $a_0^k = p_0^k$. Imposing the boundary condition (4.14), we have

$$a_n^a = 0, \quad b_0^a = 0. \tag{4.19}$$

Therefore

$$X^a = x_0^a + \sum_{n\neq 0}\frac{e^{-in\tau}}{n}a_n^a\sin n\sigma, \tag{4.20}$$

where we have renamed $b_n^a \to a_n^a$. Notice that in the presence of the field $\mathcal{F}_{ij}$, the coordinates along the D-brane are modified.

As discussed in Part I, we can treat, with care, string theory as a one dimensional quantum field theory. From the action S_B the conjugate momentum is

$$P^k(\tau, \sigma) = \frac{1}{2\pi\alpha'}(\partial_\tau X^k + \partial_\sigma X^j\mathcal{F}_j{}^k), \tag{4.21}$$

$$P^a(\tau, \sigma) = \frac{1}{2\pi\alpha'}(\partial_\tau X^a). \tag{4.22}$$

The mode expansion for momentum is then

$$P^k(\tau,\sigma) = \frac{1}{2\pi\alpha'}(p_0^l + \sum_{n\neq 0} a_n^l e^{-in\tau}\cos n\sigma){M_l}^k, \tag{4.23}$$

$$P^a(\tau,\sigma) = -\frac{i}{2\pi\alpha'}\sum_{n\neq 0} a_n^a e^{-in\tau}\sin n\sigma, \tag{4.24}$$

where $M_{ij} = \eta_{ij} - {\mathcal{F}_i}^k \mathcal{F}_{kj}$. Since the spacetime manifold is different from the case of usual field theory, we cannot use the usual definition of Poisson bracket. The direct calculation is quite involved. Alternatively, we can do this by using symplectic form. We first follow the analysis in [18]. Let $\vec{\xi} = (q^1, q^2, \ldots, q^d, p^1, p^2, \ldots, p^d)$ be a vector in phase space. We consider the action of the form

$$S = \int dt \left\{ \mathcal{A}_i(\vec{\xi})\frac{d\xi^i}{dt} + (\text{terms without time derivatives})\right\}. \tag{4.25}$$

When computing Poisson bracket, we ignore the second term. Consider the change of S under infinitesimal variation $\xi^i \to \xi^i + \delta\xi^i$:

$$\begin{aligned} \delta S &= \int dt \left(\frac{\partial \mathcal{A}_i}{\partial \xi^j}\delta\xi^j \frac{d\xi^i}{dt} + \mathcal{A}_i \frac{d}{dt}\delta\xi^i\right) \\ &= \int dt \left(\frac{\partial \mathcal{A}_i}{\partial \xi^j}\delta\xi^j \frac{d\xi^i}{dt} - \frac{d\mathcal{A}_i}{dt}\delta\xi^i\right) \\ &= \int dt \left(\frac{\partial \mathcal{A}_j}{\partial \xi^i} - \frac{\partial \mathcal{A}_i}{\partial \xi^j}\right)\delta\xi^i \frac{d\xi^j}{dt}. \end{aligned} \tag{4.26}$$

In the second step, we used chain rule and the requirement that $\delta\xi^i$ vanishes at initial and final time. In the last step, we used chain rule. Let us introduce $\Omega_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$, and let Ω^{ij} be the inverse of Ω_{ij}. We define Poisson bracket between functions G and K as

$$\{G, K\} = \Omega^{ij}\frac{\partial G}{\partial \xi^i}\frac{\partial K}{\partial \xi^j}. \tag{4.27}$$

In the case where $\vec{\mathcal{A}} = (\vec{0}, \vec{p})$, the Poisson bracket reduces to the usual definition. Define a symplectic form

$$\Omega \equiv \frac{1}{2}\Omega_{ij} d\xi^i \wedge d\xi^j, \tag{4.28}$$

where $d\xi^i \wedge d\xi^j$ forms a basis for Ω, and $d\xi^i \wedge d\xi^j = -d\xi^j \wedge d\xi^i$. The Poisson bracket is obtained from the inverse of Ω. For usual field theory with fields $\phi_a(t,\vec{x})$, we have

$$\Omega = \int d^d\vec{x}\; d\Pi_a(t,\vec{x}) \wedge d\phi_a(t,\vec{x}). \tag{4.29}$$

The Poisson bracket is found to agree with the equation (2.52). We can also apply this method to our discussion. The symplectic form is given by

$$\Omega = \int_0^\pi d\sigma dP_\mu \wedge dX^\mu. \tag{4.30}$$

This symplectic form gives a correct Poisson bracket as long as we ignore boundaries. However, the modification to Poisson bracket only occurs at end points which have measure zero. Therefore, we can still use the given symplectic form. The modification to the

Poisson bracket at the boundaries will be seen from mode expansions. For consistency, we require that

$$\frac{d\Omega}{d\tau} = 0. \tag{4.31}$$

Using equation of motion and boundary conditions, we see that this is actually the case. Working out explicityly, we have

$$\Omega = \frac{1}{2\alpha'}\left\{M_{ij}dp_0^i \wedge (dx_0^j + \frac{\pi}{2}\mathcal{F}^j{}_k dp_0^k) + \sum_{n>0}\frac{-i}{n}(M_{ij}da_n^i \wedge da_{-n}^j + da_n^a \wedge da_{-n}^a)\right\}. \tag{4.32}$$

Here we have used symmetric property of M_{ij}, antisymmetric property of wedge product $\wedge$, and the following formulas

$$\int_0^\pi d\sigma \cos m\sigma = \pi\delta_{m,0}, \tag{4.33}$$

$$\int_0^\pi d\sigma \cos n\sigma \cos m\sigma = \frac{\pi}{2}(\delta_{n,m} + \delta_{n,-m}), \tag{4.34}$$

$$\int_0^\pi d\sigma \cos n\sigma \sin n\sigma = 0, \tag{4.35}$$

$$\int_0^\pi d\sigma \sin n\sigma \sin m\sigma = \frac{\pi}{2}(\delta_{n,m} - \delta_{n,-m}). \tag{4.36}$$

The Poisson bracket is obtained from inverse of Ω. We now do the canonical quantisation by promoting all modes to operators, and make the Poisson bracket become commutator $\{\ ,\ \} \to [\ ,\ ]/i$. The commutation relations for the modes are therefore

$$[\hat{a}_n^i, \hat{x}_0^j] = [\hat{a}_n^i, \hat{p}_0^j] = 0, \quad [\hat{a}_m^i, \hat{a}_n^j] = 2\alpha' m M^{-1ij}\delta_{m,-n}, \tag{4.37}$$

$$[\hat{p}_0^i, \hat{p}_0^j] = 0, \quad [\hat{x}_0^i, \hat{p}_0^j] = i2\alpha' M^{-1ij}, \quad [\hat{x}_0^i, \hat{x}_0^j] = i2\pi\alpha'(M^{-1}\mathcal{F})^{ij}. \tag{4.38}$$

We now calculate the commutation relations for string coordinates and momenta. We use commutation relations for the modes and note that

$$\sum_{n\neq 0} f(n) = 0, \text{ for } f(n) = -f(-n), \tag{4.39}$$

and that $\sum_{n\neq 0}(\sin nx)/n$ is anti-periodic extension to $\pi - x$, i.e., over the range $[0, 2\pi]$,

$$\sum_{n\neq 0}\frac{1}{n}\sin nx = \begin{cases} 0, & x = 0, 2\pi, \\ \pi - x & x \in (0, 2\pi). \end{cases} \tag{4.40}$$

We obtain

$$[\hat{P}^i(\tau,\sigma), \hat{P}^j(\tau,\sigma')] = 0, \tag{4.41}$$

$$[\hat{X}^k(\tau,\sigma), \hat{X}^l(\tau,\sigma')] = \begin{cases} 2\pi i\alpha'(M^{-1}\mathcal{F})^{kl}, & \sigma = \sigma' = 0, \\ -2\pi i\alpha'(M^{-1}\mathcal{F})^{kl}, & \sigma = \sigma' = \pi, \\ 0 & \text{otherwise}, \end{cases} \tag{4.42}$$

$$[\hat{X}^i(\tau,\sigma), \hat{P}^j(\tau,\sigma')] = i\eta^{ij}\delta(\sigma,\sigma'), \tag{4.43}$$

where $\delta(\sigma,\sigma') = (1+\sum_{n\neq 0}\cos n\sigma\cos n\sigma')/\pi$ is a periodically extended Dirac delta function from $[0,\pi]$.

From the result, we see that the string coordinates become noncommutative at string end points. Since the end points live on D-brane, this means the D-brane becomes noncommutative. This leads to the study of physics on noncommutative spacetime. Normally, in the case of quantum mechanics on commutative spacetime, coordinate space representation can be used. This means that states are represented as wavefunction $\psi(\vec{x})$. The position operator $\hat{x}^i$ acts on the wavefunction by multiplication $\hat{x}^i\psi(\vec{x}) = x^i\psi(\vec{x})$. The momentum operator $\hat{p}^j$ acts on the wavefunction by differentiation $\hat{p}^j\psi(\vec{x}) = -i\partial_j\psi(\vec{x})$. However, in the case of noncommutative spacetime, the operators $\hat{x}^i$ no longer commute among themselves. Therefore the action of position operator on wavefunction can no longer be represented by multiplication.

The situation is even worse in quantum field theory where field operators depend on spacetime which is now noncommutative. The idea to solve this problem is by adapting phase space quantisation method to the noncommutative spacetime. Originally, phase space quantisation was used in order to describe the phase space which is noncommutative in quantum mechanics. Despite the noncommutativity, states and operators are represented by phase space functions. The star-product, which is the product between two functions, is noncommutative but associative, and is used to encode the noncommutativity. For the case of classical field theory on noncommutative spacetime, the adapted phase space quantisation method allows us to view fields as functions on spacetime. However, the (commutative) product between fields becomes star-product. After this step, we can get quantum field theory in noncommutative spacetime by using path integral quantisation.

We will discuss phase space quantisation in details in chapter 5. For the discussion of quantum field theory in noncommutative spacetime see for example [19].

4.2 D-branes and Totally Noncommutative Phase Space

In superstring theory, pp-wave background is given by a plane wave metric supported by RR 5-form field strength

$$ds^2 = -f^2\sum_{i=1}^{8} x^i x^i (dx^+)^2 + 2dx^+dx^- + \sum_{i=1}^{8} dx^i dx^i, \tag{4.44}$$

$$F_5 = f dx^+ \wedge (dx^1 \wedge dx^2 \wedge \ldots \wedge dx^8). \tag{4.45}$$

This background is allowed in superstring theory because it preserves symmetries of the theory [20]. We notice that the Ricci scalar for the metric is zero: $R = 0$. So the first term in the action S_{eff} is zero. As in the case of previous section, we may also try to put

$$H_{\mu\nu\kappa}(X) = 0, \quad \Phi(X) = \text{constant}. \tag{4.46}$$

We can ignore $\Phi(X)$ as before. We can see that by introducing the constant B-field in the pp-wave background, the symmetries of superstring theory are still preserved [17]. If we put a Dp-brane in this background and imposing some gauge choices, the bosonic part and fermionic part can be separated. Furthermore, the RR 5-form field strength will not play the role. So the bosonic action for open string ending on Dp-brane in pp-wave background with constant B-field is

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma[g_{ij}(\eta^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j + m^2 X^i X^j) + \epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j B_{ij}], \tag{4.47}$$

for $i, j = 2, \ldots, p$, and B is turned on only in the directions $2\ldots, p$. Note that we ignored the part that is normal to the D-brane since it does not change from the case of D-brane in flat spacetime. In order to get this action, we used the pp-wave metric in light-cone gauge with $X^+ = \alpha' p^+\tau$, and we choose the gauge $\mathcal{F}_{ij} = B_{ij}$ Furthermore, we have introduced

$$m := \alpha' p^+ f. \tag{4.48}$$

We have also introduced the Euclidean metric g_{ij} which essentially comes from the scaling of X^i. In our normalisation, B_{ij} and m are dimensionless. The equation of motion is

$$(-\partial_\tau^2 + \partial_\sigma^2 - m^2)X^i = 0, \tag{4.49}$$

and the boundary condition at $\sigma = 0, \pi$ is

$$\partial_\sigma X^i + \partial_\tau X^j B_j{}^i = 0, \tag{4.50}$$

where g_{ij} is used to lower indices. The conjugate momentum is given by

$$2\pi\alpha' P^k = \partial_\tau X^k + \partial_\sigma X^j B_j{}^k. \tag{4.51}$$

Without loss of generality, we can take

$$g_{ij} = \lambda\delta_{ij}, \tag{4.52}$$

and

$$B_{ij} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \tag{4.53}$$

and focus on X^2, X^3. Let

$$\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma), \tag{4.54}$$

so the equation of motion becomes

$$2\partial_+\partial_- X^k = -m^2 X^k. \tag{4.55}$$

Assuming X^k is in the form $X^k = X_+^k(\sigma^+)X_-^k(\sigma^-)$ (no summation on k), we have

$$X^k = e^{\pm i\sqrt{c^2+m^2}\tau}e^{\pm ic\sigma}. \tag{4.56}$$

The equation of motion is linear, so we can use superposition principle. Let us consider the solution

$$X^k = e^{i\sqrt{c^2+m^2}\tau}(A_+^k \cos c\sigma + D_+^k \sin c\sigma) + e^{-i\sqrt{c^2+m^2}\tau}(A_-^k \cos c\sigma + D_-^k \sin c\sigma), \tag{4.57}$$

where $A_\pm^k, D_\pm^k$ are constant. Explicitly, the boundary condition at $\sigma = 0, \pi$ is given by

$$\partial_\sigma X^2 - \partial_\tau X^3 \bar{B} = 0, \tag{4.58}$$

$$\partial_\sigma X^3 + \partial_\tau X^2 \bar{B} = 0, \tag{4.59}$$

where $\bar{B} = B/\lambda$. This implies

$$\partial_\sigma^2 X^k + \partial_\tau^2 X^k \bar{B}^2 = 0 \quad \text{at } \sigma = 0, \pi. \tag{4.60}$$

This implies

$$-c^2 - (c^2 + m^2)\bar{B}^2 = 0 \quad \text{or } X^k = 0 \quad \text{at } \sigma = 0, \pi. \tag{4.61}$$

For the first case we have

$$c = \pm \frac{im}{\sqrt{1 + \bar{B}^2}}. \tag{4.62}$$

Applying the full boundary conditions, some constants are eliminated. With some rearrangements, we have

$$X^k_{(0)} = (x_0^k \cos\omega_0\tau + 2\alpha' p_0^k \frac{\sin\omega_0\tau}{\omega_0}) \cosh\omega_0\bar{B}\sigma + (-2\alpha' p_0^j \cos\omega_0\tau + x_0^j \omega_0 \sin\omega_0\tau) B_j{}^k \frac{\sinh\omega_0\bar{B}\sigma}{\omega_0\bar{B}}, \tag{4.63}$$

where x_0^k, p_0^k are constants, and $\omega_0 = m/\sqrt{1+\bar{B}^2} > 0$. Now the condition $X^k = 0$ at $\sigma = 0, \pi$ implies

$$\sin c\pi = 0, \tag{4.64}$$

which in turn implies $c = 0, \pm 1, \pm 2, \ldots$. Applying the full boundary conditions and using superposition principle, the solution for this case is

$$X^k_{(1)} = \sqrt{2\alpha'} \sum_{n \neq 0} e^{-i\omega_n\tau} \left(i\frac{\alpha_n^k}{\omega_n} \cos n\sigma - i\frac{\alpha_n^j}{n} B_j{}^k \sin n\sigma \right), \tag{4.65}$$

where $\omega_n = \text{sign}(n\sqrt{n^2+m^2})$, $n \neq 0$, and α_n^k are constants. By superposition, the full solution is given by

$$X^k = X^k_{(0)} + X^k_{(1)}. \tag{4.66}$$

The momentum is given by

$$P^k = P^k_{(0)} + P^k_{(1)}, \tag{4.67}$$

with

$$\begin{aligned} 2\pi\alpha' P^k_{(0)} &= (-x_0^j \omega_0 \sin\omega_0\tau + 2\alpha' p_0^j \cos\omega_0\tau) M_j{}^k \cosh\omega_0\bar{B}\sigma \\ &\quad + (2\alpha' p_0^j \sin\omega_0\tau + x_0^j \omega_0 \cos\omega_0\tau)(Bg^{-1}M)_j{}^k \frac{\sinh\omega_0\bar{B}\sigma}{\bar{B}}, \end{aligned} \tag{4.68}$$

and

$$2\pi\alpha' P^k_{(1)} = \sqrt{2\alpha'} \sum_{n \neq 0} e^{-i\omega_n\tau} \left(\alpha_n^j M_j{}^k \cos n\sigma + i\frac{m^2}{n\omega_n} \alpha_n^j B_j{}^k \sin n\sigma \right), \tag{4.69}$$

where

$$M_k{}^i = \delta_k{}^i - B_k{}^j B_j{}^i. \tag{4.70}$$

As usual, in order to calculate Poisson bracket, we consider the symplectic form

$$\Omega = \int_0^\pi d\sigma g_{ij} dP^i \wedge dX^j. \tag{4.71}$$

Using equation of motion and boundary conditions, we can check that $d\Omega/d\tau = 0$. Explicitly, by expanding X^k and P^k, finding the inverse of Ω, and quantising, the result is

$$[\hat{x}_0^i, \hat{p}_0^j] = i(M^{-1})^{ij}\frac{\pi\omega_0\bar{B}}{\tanh\pi\omega_0\bar{B}}, \tag{4.72}$$

$$[\hat{x}_0^i, \hat{x}_0^j] = i2\pi\alpha'(g^{-1}BM^{-1})^{ij}. \tag{4.73}$$

$$[\hat{p}_0^i, \hat{p}_0^j] = i\frac{\pi\omega_0^2}{2\alpha'}(g^{-1}BM^{-1})^{ij}, \tag{4.74}$$

$$[\hat{\alpha}_n^i, \hat{\alpha}_s^j] = \omega_n M_n^{ij}\delta_{n,-s}, \tag{4.75}$$

where

$$M_n^{ij} = \left(\frac{1}{g+\frac{\omega_n}{n}B}g\frac{1}{g-\frac{\omega_n}{n}B}\right)^{ij}. \tag{4.76}$$

This implies

$$[\hat{X}^k(\tau,\sigma), \hat{X}^l(\tau,\sigma')] = i2\pi\alpha'(g^{-1}BM^{-1})^{ij} \times \begin{cases} 1, & \sigma=\sigma'=0, \\ -1, & \sigma=\sigma'=\pi, \\ 0 & \text{otherwise,} \end{cases} \tag{4.77}$$

$$[\hat{P}^i(\tau,\sigma), \hat{P}^j(\tau,\sigma')] = \frac{im^2}{2\pi\alpha'}B^{ij} \times \begin{cases} 1, & \sigma=\sigma'=0, \\ -1, & \sigma=\sigma'=\pi, \\ 0 & \text{otherwise,} \end{cases} \tag{4.78}$$

$$[\hat{X}^i(\tau,\sigma), \hat{P}^j(\tau,\sigma')] = ig^{ij}\delta(\sigma,\sigma'). \tag{4.79}$$

We see that on the D-brane the whole phase space becomes noncommutative. In this report we call this kind of noncommutativiy as totally noncommutative phase space. This situation is even more difficult than the case of noncommutative spacetime where we were allowed to view momentum operators as differential operators. Equivalently, we were allowed to use momentum space representation because the momentum space was commutative. However, in totally noncommutative phase space, neither coordinate space representation nor momentum space representations are allowed.

So in order to study physics on totally noncommutative phase space, we may propose to use the generalisation of phase space quantisation. This is fine in the case of quantum mechanics as we will soon see. However, it is difficult to construct a quantum field theory on totally noncommutative phase space. This is because we normally view fields as functions on either spacetime or momentum space. However, we are now forced to use phase space functions. We might need to first construct quantum field theory in usual phase space. Then the next step will by easy: as the fields are already functions of phase space, we may use star-product to give the details of totally noncommutative phase space. Alternatively, we may first try to understand some physics by studying quantum mechanics in totally noncommutative phase space. This might provide some useful insights into constructing quantum field theory in totally noncommutative phase space.

Let us now discuss about phase space quantisation which will be used as a framework for quantum mechanics in totally noncommutative phase space.

Chapter 5

Phase Space Quantisation

Phase space quantisation was developed by several authors for example Weyl, Wigner, Moyal and Groenewold. This method was successful in order to describe quantum mechanics. It was also a success in describing quantum field theory in noncommutative spacetime (see e.g. [19]). Actually, it can also be used to describe more general situations. In this chapter, we will discuss the case of the simplest kind of noncommutative phase space where commutators of phase space coordinates are constant.

5.1 Weyl quantisation and Wigner transformation

Definition 1. *We define $\vec{\xi}$ as a vector in phase space. If the coordinates and conjugate momenta are given by $\{q^i\}, \{p^i\}$, then the vector $\vec{\xi}$ is given by*

$$\vec{\xi} = (q^1, q^2, \ldots, q^d, p^1, p^2, \ldots, p^d), \tag{5.1}$$

where d is the dimension of space. We will call the dimension of phase space as $n = 2d$. We can also define a collection of phase space operators $\hat{\vec{\xi}}$ in the similar way. The general commutation relations is then written as

$$[\hat{\xi}^i, \hat{\xi}^j] = i\Theta^{ij}, \tag{5.2}$$

where Θ^{ij} is constant.

Definition 2 (Weyl Quantisation). *Given a well-behaved function (i.e. the function that derivatives of any order vanish at infinity) $g(\vec{\xi})$, we can get the corresponding operator $\hat{G}$ by (see e.g. [19])*

$$\hat{G} = \hat{W}[g] = \int \frac{d^n\vec{k}}{(2\pi)^n} \tilde{g}(\vec{k}) e^{i\vec{k}\cdot\hat{\vec{\xi}}}, \tag{5.3}$$

where $\tilde{g}(\vec{k})$ is the Fourier transform of $g(\vec{\xi})$. That is

$$\tilde{g}(\vec{k}) = \int d^n\vec{\xi} e^{-i\vec{k}\cdot\vec{\xi}} g(\vec{\xi}). \tag{5.4}$$

We can also write $g(\vec{\xi})$ as a Fourier inverse of $\tilde{g}(\vec{k})$:

$$g(\vec{\xi}) = \int \frac{d^n\vec{k}}{(2\pi)^n} e^{i\vec{k}\cdot\vec{\xi}} \tilde{g}(\vec{k}). \tag{5.5}$$

For convenient, we define a function [21], [19]

$$\hat{\Delta}(\vec{\xi}) = \int \frac{d^n\vec{k}}{(2\pi)^n} e^{i\vec{k}\cdot\hat{\vec{\xi}}} e^{-i\vec{k}\cdot\vec{\xi}}. \tag{5.6}$$

The Weyl operator can now be written as

$$\hat{W}[g] = \int d^n\vec{\xi} g(\vec{\xi})\hat{\Delta}(\vec{\xi}). \tag{5.7}$$

Before we move on, let us remark a useful result from Fourier transform. It is

$$\int \frac{d^n\vec{k}}{(2\pi)^n} e^{i\vec{k}\cdot\vec{\xi}} = \delta^n(\vec{\xi}), \tag{5.8}$$

or similarly,

$$\int d^n\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}} = (2\pi)^n \delta^n(\vec{k}). \tag{5.9}$$

We can verify the result quickly by applying Fourier transform followed by inverse Fourier transform or vice versa.

We have seen that for a given phase space function, the corresponding operator can be written. We now define an inverse of this transformation.

Definition 3 (**Wigner Transformation**). *Given an operator $\hat{G}$, we get, by Wigner transformation, a corresponding function*

$$g(\vec{\xi}) = \mathrm{tr}(\hat{G}\hat{\Delta}(\vec{\xi})), \tag{5.10}$$

where (from [21])

$$\mathrm{tr}\hat{R} = |\det(2\pi\Theta)| \int \frac{d^n\vec{k}}{(2\pi)^n} e^{-i\vec{k}\cdot\hat{\vec{\xi}}} \hat{R} e^{i\vec{k}\cdot\hat{\vec{\xi}}}, \tag{5.11}$$

for any operator $\hat{R}$. Note that for this definition of trace tr *we require Θ to be nonsingular. That is* $\det\Theta \neq 0$.

To get some insight, it will be useful to discuss the properties of trace, tr.

Theorem 1 (Properties of trace). *The trace has the following properties for any operators $\hat{A}, \hat{B}$ and numbers a, b :*

1. $\mathrm{tr}(\hat{A}^\dagger) = (\mathrm{tr}(\hat{A}))^\dagger$.
2. $\mathrm{tr}(\hat{A}^\dagger\hat{A}) \geq 0$ *and* $\mathrm{tr}(\hat{A}^\dagger\hat{A}) = 0$ *iff* $\hat{A} = 0$.
3. $\mathrm{tr}(a\hat{A} + b\hat{B}) = a\mathrm{tr}(\hat{A}) + b\mathrm{tr}(\hat{B})$.
4. $\mathrm{tr}(\hat{\Delta}(\zeta)) = 1$.
5. $\mathrm{tr}(\hat{\Delta}(\xi)\hat{\Delta}(\zeta)) = \delta^n(\vec{\xi} - \vec{\zeta})$.

Proof. The properties 1, 2, and 3 are easy to prove. We use the properties $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$, $\hat{A}^\dagger\hat{A} \geq 0$ and $\hat{A}^\dagger\hat{A} = 0$ iff $\hat{A} = 0$, and we note that the operators $\hat{\vec{\xi}}$ are Hermitian.

We now prove property 4. Let us first consider

$$\begin{aligned}
\mathrm{tr}(e^{i\vec{k'}\cdot\hat{\vec{\xi}}}) &= |\det(2\pi\Theta)| \int \frac{d^n\vec{k}}{(2\pi)^n} e^{-i\vec{k}\cdot\hat{\vec{\xi}}} e^{i\vec{k'}\cdot\hat{\vec{\xi}}} e^{i\vec{k}\cdot\hat{\vec{\xi}}} \\
&= |\det(2\pi\Theta)| \int \frac{d^n\vec{k}}{(2\pi)^n} e^{-i\vec{k}\cdot\hat{\vec{\xi}}} e^{i(\vec{k'}+\vec{k})\cdot\hat{\vec{\xi}}} e^{\frac{i}{2}\Theta^{ij}k'_ik_j} \\
&= |\det(2\pi\Theta)| \int \frac{d^n\vec{k}}{(2\pi)^n} e^{i\vec{k'}\cdot\hat{\vec{\xi}}} e^{i\Theta^{ij}k'_ik_j} \\
&= |\det(2\pi\Theta)| \delta^n(\Theta^{ij}k'_j) e^{i\vec{k'}\cdot\hat{\vec{\xi}}} \\
&= (2\pi)^n \delta^n(\vec{k'}),
\end{aligned}$$

where in the second step we used Baker-Campbell-Hausdorff formula for $[\hat{A}, \hat{B}] = \text{constant}$: $\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A}+\hat{B})\exp\Big([\hat{A},\hat{B}]/2\Big)$. In the last step, we eliminate $e^{i\vec{k'}\cdot\hat{\vec{\xi}}}$ because $\vec{k'}$ will be set to 0 after an integration. We also use the fact that $\delta^n(\Theta^{ij}k'_j) = \delta^n(\vec{k'})/|\det\Theta|$. Now since

$$\hat{\Delta}(\vec{\xi}) = \int \frac{d^n\vec{k'}}{(2\pi)^n} e^{i\vec{k'}\cdot\hat{\vec{\xi}}} e^{-i\vec{k'}\cdot\vec{\xi}}, \tag{5.12}$$

we have

$$\begin{aligned}
\mathrm{tr}(\hat{\Delta}(\vec{\xi})) &= (2\pi)^n \int \frac{d^n\vec{k'}}{(2\pi)^n} \delta^n(\vec{k'}) e^{-i\vec{k'}\cdot\vec{\xi}} \\
&= 1.
\end{aligned}$$

We now prove property 5. First consider

$$\begin{aligned}
\hat{\Delta}(\vec{\xi})\hat{\Delta}(\vec{\zeta}) &= \int \frac{d^n\vec{k'}}{(2\pi)^n} \int \frac{d^n\vec{k''}}{(2\pi)^n} e^{i\vec{k'}\cdot\hat{\vec{\xi}}} e^{i\vec{k''}\cdot\hat{\vec{\xi}}} e^{-i\vec{k'}\cdot\vec{\xi}} e^{-i\vec{k''}\cdot\vec{\zeta}} \\
&= \int \frac{d^n\vec{k'}}{(2\pi)^n} \int \frac{d^n\vec{k''}}{(2\pi)^n} e^{i(\vec{k'}+\vec{k''})\cdot\hat{\vec{\xi}}} e^{\frac{i}{2}\Theta^{ij}k'_ik''_j} e^{-i\vec{k'}\cdot\vec{\xi}} e^{-i\vec{k''}\cdot\vec{\zeta}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{tr}(\hat{\Delta}(\vec{\xi})\hat{\Delta}(\vec{\zeta})) &= \int \frac{d^n\vec{k'}}{(2\pi)^n} \int \frac{d^n\vec{k''}}{(2\pi)^n} (2\pi)^n \delta^n(\vec{k'}+\vec{k''}) e^{\frac{i}{2}\Theta^{ij}k'_ik''_j} e^{-i\vec{k'}\cdot\vec{\xi}} e^{-i\vec{k''}\cdot\vec{\zeta}} \\
&= |\det(2\pi\Theta)| \int \frac{d^n\vec{k}}{(2\pi)^n} \int \frac{d^n\vec{k'}}{(2\pi)^n} \int \frac{d^n\vec{k''}}{(2\pi)^n} e^{-i\vec{k}\cdot\hat{\vec{\xi}}} e^{i\vec{k'}\cdot\hat{\vec{\xi}}} e^{i\vec{k''}\cdot\hat{\vec{\xi}}} e^{i\vec{k}\cdot\hat{\vec{\xi}}} \times \\
&\quad e^{-i\vec{k'}\cdot\vec{\xi}} e^{-i\vec{k''}\cdot\vec{\zeta}} \\
&= \int \frac{d^n\vec{k'}}{(2\pi)^n} e^{-i\vec{k'}\cdot(\vec{\xi}-\vec{\zeta})} \\
&= \delta^n(\vec{\xi}-\vec{\zeta}).
\end{aligned}$$

□

There is an important correspondence between canonical quantisation and phase space quantisation.

Theorem 2 (Weyl-Wigner Correspondence)**.** *Weyl quantisation and Wigner quantisation are inverse of each other. That is*

$$g(\vec{\xi}) = \mathrm{tr}(\hat{W}[g]\hat{\Delta}(\vec{\xi})), \tag{5.13}$$

where

$$\hat{W}[g] = \int d^n\vec{\xi} g(\vec{\xi})\hat{\Delta}(\xi). \tag{5.14}$$

Proof. Using the property 5 of trace we have

$$\begin{aligned}\mathrm{tr}(\hat{W}[g]\Delta(\vec{\xi})) &= \int d^n\vec{\zeta} g(\vec{\zeta})\mathrm{tr}(\hat{\Delta}(\vec{\zeta})\hat{\Delta}(\vec{\xi})) \\ &= \int d^n\vec{\zeta} g(\vec{\zeta})\delta^n(\vec{\zeta}-\vec{\xi}) \\ &= g(\vec{\xi}),\end{aligned}$$

as required. □

Let us denote Weyl-Wigner correspondence by $\xleftrightarrow{\text{ww}}$. Therefore

$$g(\vec{\xi}) \xleftrightarrow{\text{ww}} \hat{W}[g].$$

Before we move on let us make some remarks. From property 4 of trace, we immediately get

$$\mathrm{tr}(\hat{W}[g]) = \int d^n\vec{\xi} g(\vec{\xi}) \tag{5.15}$$

which means that the trace of an operator is the integration of its corresponding function over phase space. Furthermore, in the case of the usual quantum mechanics with commutation relation $[\hat{q}^i, \hat{p}^j] = i\delta^{ij}$, we have

$$\mathrm{tr}\hat{R} = (2\pi)^{n/2}\ \mathrm{Tr}\hat{R}, \tag{5.16}$$

where Tr is the operator trace. Let us prove this for the case of 2 dimensional phase space. Higher dimensional cases can be treated in the same way. Let us first consider

$$\begin{aligned}e^{-i(k_1\hat{q}+k_2\hat{p})} &= e^{-i\frac{k_2}{2}\hat{p}}e^{-ik_1\hat{q}}e^{-i\frac{k_2}{2}\hat{p}} \\ &= \int dxdx' e^{-i\frac{k_2}{2}\hat{p}}|x\rangle\langle x|e^{-ik_1\hat{q}}|x'\rangle\langle x'|e^{-i\frac{k_2}{2}\hat{p}} \\ &= \int dx|x-\frac{k_2}{2}\rangle e^{-ik_1x}\langle x+\frac{k_2}{2}|,\end{aligned} \tag{5.17}$$

where we applied Baker-Campbell-Hausdorff formula twice in the first step, and we used $e^{ia\hat{p}}|x\rangle = |x+a\rangle$, and $\langle x|e^{-ia\hat{p}} = \langle x+a|$, for real number a, in the last step. Now from definition, we have

$$\begin{aligned}\mathrm{tr}\hat{R} &= \int dk_1dk_2 e^{-i(k_1\hat{q}+k_2\hat{p})}\hat{R}e^{i(k_1\hat{q}+k_2\hat{p})} \\ &= \int dk_1dk_2\int dxdx'|x-k_2/2\rangle e^{-ik_1x}\langle x+k_2/2|\hat{R}|x'+k_2/2\rangle e^{ik_1x'}\langle x'-k_2/2| \\ &= 2\pi\int dk_2\int dxdx'|x-k_2/2\rangle\langle x+k_2/2|\hat{R}|x'+k_2/2\rangle\langle x'-k_2/2|\delta(x-x') \\ &= 2\pi\int dxdk_2|x-k_2/2\rangle\langle x+k_2/2|\hat{R}|x+k_2/2\rangle\langle x-k_2/2|.\end{aligned} \tag{5.18}$$

We now make a change of variables $u = x + k_2/2, v = x - k_2/2$ so that $dudv = dxdk_2$. Then we have

$$\begin{aligned}
\mathrm{tr}\hat{R} &= 2\pi \int dudv |v\rangle\langle u|\hat{R}|u\rangle\langle v| \\
&= 2\pi \mathrm{Tr}\left(\hat{R}\int du|u\rangle\langle u|\right) \\
&= 2\pi \mathrm{Tr}(\hat{R}).
\end{aligned} \tag{5.19}$$

5.2 Star Product

In phase space quantisation, star product is very important. It encodes the noncommutative property of phase space. We will also see that star product corresponds to operator product in canonical quantisation formalism.

We motivate this by writing a Poisson bracket in the general form:

$$\{g(\vec{\xi}), h(\vec{\xi})\} = g(\vec{\xi})\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j}h(\vec{\xi}), \tag{5.20}$$

where $\overleftarrow{\partial_i}$ operates on the left by differentiating with respect to ξ^i, and similar for $\overrightarrow{\partial_i}$. We now define a star product.

Definition 4 (Star Product)**.** *The $*$-product between 2 phase space functions $g(\vec{\xi})$ and $h(\vec{\xi})$ is given by*

$$g(\vec{\xi}) * h(\vec{\xi}) = g(\vec{\xi}) \exp\left(\frac{i}{2}\overleftarrow{\partial}_i\Theta^{ij}\overrightarrow{\partial}_j\right) h(\vec{\xi}), \tag{5.21}$$

where $\Theta^{ij} = \hbar\Lambda^{ij}$.

There is another useful form of $*$-product:

Theorem 3. *The $*$-product can also be written as*

$$g(\vec{\xi}) * h(\vec{\xi}) = \int\int \frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}\tilde{g}(\vec{k})\tilde{h}(\vec{k}'-\vec{k})\exp\left(-\frac{i}{2}\Theta^{ij}k_ik'_j\right)e^{i\vec{k}'\cdot\vec{\xi}}. \tag{5.22}$$

Proof.

$$
\begin{aligned}
&\int\int \frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}\tilde{g}(\vec{k})\tilde{h}(\vec{k}'-\vec{k})\exp\left(-\frac{i}{2}\Theta^{ij}k_ik'_j\right)e^{i\vec{k}'\cdot\vec{\xi}}\\
&=\int\underline{\int\frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}}\int d^n\vec{y}\tilde{g}(\vec{k})h(\vec{y})\underline{\exp\left(-\frac{i}{2}\Theta^{ij}k_ik'_j\right)e^{i\vec{k}'\cdot\vec{\xi}}e^{-i\vec{k}'\cdot\vec{y}}}e^{i\vec{k}\cdot\vec{y}}\\
&=\int\frac{d^n\vec{k}}{(2\pi)^n}\int d^n\vec{y}\tilde{g}(\vec{k})h(\vec{y})\delta^n\left(-\frac{1}{2}\Theta^{ij}k_i+\xi^j-y^j\right)e^{i\vec{k}\cdot\vec{y}}\\
&=\int\frac{d^n\vec{k}}{(2\pi)^n}\tilde{g}(\vec{k})h(\xi^i-\frac{1}{2}\Theta^{ij}k_i)e^{i\vec{k}\cdot\vec{\xi}}\\
&=\underline{\int\frac{d^n\vec{k}}{(2\pi)^n}\tilde{g}(\vec{k})e^{i\vec{k}\cdot\vec{\xi}}\exp\left(-\frac{1}{2}\Theta^{ij}k_i\partial_j\right)}h(\vec{\xi})\\
&=g(\xi^i+\frac{i}{2}\Theta^{ij}\partial_j)h(\vec{\xi})\\
&=g(\vec{\xi})\exp\left(\frac{i}{2}\overleftarrow{\partial_i}\Theta^{ij}\overrightarrow{\partial_j}\right)h(\vec{\xi})\\
&=g(\vec{\xi})*h(\vec{\xi}),
\end{aligned}
$$

where the underlines are to emphasise the quantities integrated. □

We now have a useful property.

Theorem 4. $\hat{W}[g]\hat{W}[h]=\hat{W}[g*h]$.

Proof. Consider

$$
\begin{aligned}
\hat{W}[g]\hat{W}[h] &= \int\int\frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}\tilde{g}(\vec{k})\tilde{h}(\vec{k}')e^{i\vec{k}\cdot\hat{\vec{\xi}}}e^{i\vec{k}'\cdot\hat{\vec{\xi}}}\\
&= \int\int\frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}\tilde{g}(\vec{k})\tilde{h}(\vec{k}')\exp\left(-\frac{i}{2}\Theta^{ij}k_ik'_j\right)e^{i(\vec{k}+\vec{k}')\cdot\hat{\vec{\xi}}}\\
&= \int\int\frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}''}{(2\pi)^n}\tilde{g}(\vec{k})\tilde{h}(\vec{k}''-\vec{k})\exp\left(-\frac{i}{2}\Theta^{ij}k_ik''_j\right)e^{i\vec{k}''\cdot\hat{\vec{\xi}}}.
\end{aligned}
$$

In the second step, we used the Baker-Campbell-Hausdorff formula. In the last step, we make the transformation

$$\vec{k}''=\vec{k}'+\vec{k}, \tag{5.23}$$

$$\vec{k}=\vec{k}. \tag{5.24}$$

Since the Jacobi of the transformation is 1, the integration measure remains the same.

Now consider

$$
\begin{aligned}
\hat{W}[g*h] &= \iint\frac{d^n\vec{k}'''}{(2\pi)^n}d^n\vec{\xi}e^{-i\vec{k}'''\cdot\vec{\xi}}g(\vec{\xi})*h(\vec{\xi})e^{i\vec{k}'''\cdot\hat{\vec{\xi}}}\\
&= \underline{\iint\frac{d^n\vec{k}'''}{(2\pi)^n}d^n\vec{\xi}}\iint\frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}\underline{e^{-i\vec{k}'''\cdot\vec{\xi}}}\tilde{g}(\vec{k})\tilde{h}(\vec{k}'-\vec{k})e^{-\frac{i}{2}\Theta^{ij}k_ik'_j}\underline{e^{i\vec{k}'\cdot\vec{\xi}}e^{i\vec{k}'''\cdot\hat{\vec{\xi}}}}\\
&= \iint\frac{d^n\vec{k}}{(2\pi)^n}\frac{d^n\vec{k}'}{(2\pi)^n}\tilde{g}(\vec{k})\tilde{h}(\vec{k}'-\vec{k})e^{-\frac{i}{2}\Theta^{ij}k_ik'_j}e^{i\vec{k}'\cdot\hat{\vec{\xi}}}\\
&= \hat{W}[g]\hat{W}[h],
\end{aligned}
$$

as required. □

From this theorem, we get the Weyl-Wigner correspondence between $*$-product and operator product:

$$g(\vec{\xi}) * h(\vec{\xi}) \stackrel{\text{ww}}{\longleftrightarrow} \hat{W}[g]\hat{W}[h]. \tag{5.25}$$

Since operator product is associative, we can deduce that the $*$-product is also associative.

We now define a Moyal bracket as a quantisation of Poisson bracket:

Definition 5 (Moyal Bracket)**.** *A Moyal bracket between two phase space functions is given by*

$$\{\{g(\vec{\xi}), h(\vec{\xi})\}\} = g(\vec{\xi}) * h(\vec{\xi}) - h(\vec{\xi}) * g(\vec{\xi}). \tag{5.26}$$

We immediately see the Weyl-Wigner correspondence between Moyal bracket and commutator:

$$\{\{g(\vec{\xi}), h(\vec{\xi})\}\} \stackrel{\text{ww}}{\longleftrightarrow} [\hat{W}[g], \hat{W}[h]]. \tag{5.27}$$

5.3 Wigner Function

So far, we have made a correspondence between phase space functions and operators. Actually, states can also be made corresponding to phase space functions called 'Wigner functions'.

Definition 6 (Wigner Function)**.** *Given a density matrix $\hat{\rho}$, we call the corresponding function 'the Wigner function', $f_{\hat{\rho}}(\vec{\xi}) = \mathrm{tr}(\hat{\rho}\Delta(\vec{\xi}))$.*

We can also get a Wigner function from any outer product. So

$$f_{|\psi\rangle\langle\phi|}(\vec{\xi}) \stackrel{\text{ww}}{\longleftrightarrow} |\psi\rangle\langle\phi|. \tag{5.28}$$

Actually, we can treat outer products as operators. Therefore, the results discussed for operators also work for outer products.

In the usual quantum mechanics, we have

$$\begin{aligned} f_{\hat{\rho}}(\vec{\xi}) &= \mathrm{tr}(\hat{\rho}\Delta(\vec{\xi})) \\ &= (2\pi)^{n/2}\mathrm{Tr}(\hat{\rho}\hat{\Delta}(\vec{\xi})) \\ &= (2\pi)^{n/2}\int \frac{d^n\vec{k}}{(2\pi)^n}\mathrm{Tr}(\hat{\rho}e^{i\vec{k}\hat{\vec{\xi}}})e^{-i\vec{k}\cdot\vec{x}}. \end{aligned} \tag{5.29}$$

For the case of pure state with wavefunction $\psi(\vec{q})$, our Wigner function reduces, up to normalisation, to the definition defined in literature [22], [23]. We have

$$f(\vec{\xi}) = \int d^d\vec{y}\psi^*\left(\vec{x} - \frac{1}{2}\vec{y}\right)e^{-i\vec{y}\cdot\vec{p}}\psi\left(\vec{x} + \frac{1}{2}\vec{y}\right). \tag{5.30}$$

In canonical quantisation formalism, the average of an observable $\hat{W}[g]$ in a state with density matrix $\hat{\rho}$ is given by $\mathrm{tr}(\hat{\rho}\hat{W}[g])$. From theorem 4 and equation (5.15), we have

$$\mathrm{tr}(\hat{\rho}\hat{W}[g]) = \int d^n\vec{\xi} f_{\hat{\rho}}(\vec{\xi}) * g(\vec{\xi}). \tag{5.31}$$

Generally, for any well-behaved phase space functions $g(\vec{\xi}), h(\vec{\xi})$, we have

$$\begin{aligned}\int d^n\vec{\xi} g(\vec{\xi}) * h(\vec{\xi}) &= \int d^n\vec{\xi} \int \frac{d^n\vec{k}}{(2\pi)^n} \int \frac{d^n\vec{k'}}{(2\pi)^n} \tilde{g}(\vec{k})\tilde{h}(\vec{k'}-\vec{k}) \exp\left(-\frac{i}{2}\Theta^{ij}k_i k'_j\right) e^{i\vec{k'}\cdot\vec{\xi}} \\ &= \int \frac{d^n\vec{k}}{(2\pi)^n} \tilde{g}(\vec{k})\tilde{h}(-\vec{k}) \\ &= \int d^n\vec{\xi} g(\vec{\xi}) h(\vec{\xi}).\end{aligned} \tag{5.32}$$

So we may as well write equation (5.31) as

$$\mathrm{tr}(\hat{\rho}\hat{W}[g]) = \int d^n\vec{\xi} f_{\hat{\rho}}(\vec{\xi}) g(\vec{\xi}). \tag{5.33}$$

5.4 Eigenvalue Equation

In quantum mechanics it is often useful to study eigenvalue equations of time independent Hamiltonian. It is natural to construct eigenvalue equations for phase space quantisation formalism.

For

$$H(\vec{\xi}) \stackrel{\mathrm{ww}}{\longleftrightarrow} \hat{H}, \tag{5.34}$$

we have

$$H(\vec{\xi}) * f_{|\psi,\alpha\rangle\langle\phi,\beta|}(\vec{\xi}) = E_\psi f_{|\psi,\alpha\rangle\langle\phi,\beta|}(\vec{\xi}) \stackrel{\mathrm{ww}}{\longleftrightarrow} \hat{H}|\psi,\alpha\rangle\langle\phi,\beta| = E_\psi|\psi,\alpha\rangle\langle\phi,\beta|, \tag{5.35}$$

$$f_{|\psi,\alpha\rangle\langle\phi,\beta|}(\vec{\xi}) * H(\vec{\xi}) = E_\phi f_{|\psi,\alpha\rangle\langle\phi,\beta|}(\vec{\xi}) \stackrel{\mathrm{ww}}{\longleftrightarrow} |\psi,\alpha\rangle\langle\phi,\beta|\hat{H} = E_\phi|\psi,\alpha\rangle\langle\phi,\beta|. \tag{5.36}$$

Here α and β are degeneracy indices. From the Weyl-Wigner correspondence we see that $f_{|\psi,\alpha\rangle\langle\phi,\beta|}(\vec{\xi})$ is the left $*$-eigenfunction of $H(\vec{\xi})$ with $*$-eigenenergy E_ϕ, and is the right $*$-eigenfunction of $H(\vec{\xi})$ with $*$-eigenenergy E_ψ.

From Weyl-Wigner correspondence, we obtain the orthogonality condition:

$$f_{|i,\alpha\rangle\langle j,\beta|}(\vec{\xi}) * f_{|k,\gamma\rangle\langle j,\epsilon|}(\vec{\xi}) = f_{|i,\alpha\rangle\langle l,\epsilon|}(\vec{\xi})\delta_{jk}\delta_{\beta\gamma}, \tag{5.37}$$

where the eigenkets and eigenbras are normalised.

5.5 Quantisation Procedure

We have seen that phase space quantisation is not completely new. We can make Weyl-Wigner correspondence to bring phase space quantisation to a more familiar canonical quantisation. However, the phase space quantisation has some advantages. For example, it can be used to visualise observables and states as phase space functions.

We are now going to discuss another advantage of phase space quantisation. Consider phase space observables (phase space functions that correspond to observable operators) that are functions of degree ≤ 2. This type of observables is useful in many important quantum systems.

Theorem 5. *A classical phase space observable $G(\vec{\xi}) = (1/2)G_{ij}\xi^i\xi^j$ remains the same after quantisation.*

Proof. It is natural to quantise a usual function product to symmetrised $*$-product. i.e. we quantise the observable:

$$\begin{aligned} G_{qm}(\vec{\xi}) &= \frac{1}{2}G_{ij}\frac{\xi^i * \xi^j + \xi^j * \xi^i}{2} \\ &= \frac{1}{2}G_{ij}\xi^i\xi^j \\ &= G(\vec{\xi}). \end{aligned}$$

□

From this theorem, we see that the phase space observable of degree ≤ 2 remains the same after quantising.

We now make a note about Moyal bracket.

Theorem 6. *A Moyal bracket between any two phase space observables of degree ≤ 2 reduces, up to constant, to Poisson bracket. i.e.*

$$\{\{g(\vec{\xi}), h(\vec{\xi})\}\} = i\{g(\vec{\xi}), h(\vec{\xi})\}. \tag{5.38}$$

Proof. The proof is straight forward. It follows by using definition of $*$-product and Moyal bracket. □

If a set of phase space observables of degree ≤ 2 forms a group under Poisson bracket, then the set also forms, up to constant i, the same group under Moyal bracket. We say that the algebra is preserved.

Example 2 (Homogeneous Linear Canonical Transformation in Four-Dimensional Phase Space (continued))**.** *This example continues from the Example 1. The analysis in subsection 2.1.3 continue to work in quantum case since the generators are of degree ≤ 2. This example will be useful when studying the main result. Consider a star product between two generators $G_1(\vec{\xi}), G_2(\vec{\xi})$:*

$$G_1(\vec{\xi}) * G_2(\vec{\xi}) = G_1(\vec{\xi})G_2(\vec{\xi}) - \frac{i}{2}(\vec{\xi})^T\mathbf{\Lambda}^{-1}\underline{\underline{G_1 G_2}}\vec{\xi} - \frac{1}{8}\mathrm{Tr}(\underline{\underline{G_1 G_2}}), \tag{5.39}$$

where Tr *is matrix trace. If $[\underline{\underline{G_1}}, \underline{\underline{G_2}}] = \underline{\underline{K}}$, we have*

$$\begin{aligned} G_1(\vec{\xi}) * G_2(\vec{\xi}) - G_2(\vec{\xi}) * G_1(\vec{\xi}) &= -\frac{i}{2}(\vec{\xi})^T\mathbf{\Lambda}^{-1}\underline{\underline{K}}\vec{\xi} \\ &= \frac{1}{2}(\vec{\xi})^T\mathbf{K}\vec{\xi} \\ &= K(\vec{\xi}). \end{aligned} \tag{5.40}$$

So we see that the Moyal bracket for function form of generators is the same as commutator for matrix from of generators. There is another result worth remarking. We have

$$G(\vec{\xi}) * G(\vec{\xi}) = G^2(\vec{\xi}) - \frac{1}{8}\mathrm{Tr}(\underline{\underline{G}}^2). \tag{5.41}$$

So for example, from $J_0^2 - K_2^2 - Q_2^2 = J_2^2$*, and* $\mathrm{Tr}(\underline{\underline{J_0}}^2) = \mathrm{Tr}(\underline{\underline{J_2}}^2) = -\mathrm{Tr}(\underline{\underline{K_2}}^2) = -\mathrm{Tr}(\underline{\underline{Q_2}}^2) = 1$*, we have*

$$J_0 * J_0 - K_2 * K_2 - Q_2 * Q_2 = J_2 * J_2 - \frac{1}{4}. \tag{5.42}$$

So the algebra for the classical function is modified in star product.

In summary, for phase space observables of degree ≤ 2, the classical and quantum version of the observable are represented by the same function. However, the Poisson bracket is quantised to be Moyal bracket.

For general phase space observables we will also make a quantisation so that the algebra is preserved. The form of quantum phase space observables will be differed from that of the classical phase space observables in general case.

Let us remark about vocabulary. We will call the formalism that corresponds to phase space quantisation as phase space formalism. Similarly, for canonical quantisation, we call its corresponding formalism as canonical formalism.

We have finished building the formalism. So let us study some quantum systems using phase space quantisation.

Chapter 6

Topics in Noncommutative Quantum Field Theory

In section 4.1, we have seen that spacetime can become commutative. Let us discuss some physical consequences of noncommutative spacetime in this chapter. We can adapt the phase space quantisation method to study quantum field theory (QFT) in noncommutative spacetime. The idea is that instead of noncommutativity in phase space, we only require the noncommutative spacetime in the formalism. So the star product lives in noncommutative spacetime. The functions in this formalism are now functions of spacetime. So it is convenient to study field theory. We can replace usual products by star products given by

$$* = \exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right). \tag{6.1}$$

We require the matrix with components $\theta^{\mu\nu}$ to be invertible, so let us take the spacetime dimensions to be even.

The features of quantum field theory in noncommutative spacetime are for example, nonlocality, non-planar Feynman diagram, UV/IR mixing, noncommutative Yang-Mills theory, and Morita equivalence. We will go into details in some topics. The discussion in this chapter is based on [24], [19], [9].

6.1 Feynman Diagrams

6.1.1 Feynman Diagrams for QFT

Before we move on to study noncommutative spacetime, let us study about Feynman Diagram which plays an important role when calculating interactions. We consider the case of massive scalar field with ϕ^4 interaction. The action of the field is

$$S = -\int d^Dx\left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{1}{2}m^2\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4\right], \tag{6.2}$$

where λ and m are constants. Note that the first two terms represent the free field. The term with ϕ^4 potential is the interaction term.

Usually, we want to calculate scattering amplitude. This will be useful in order to study the probability of each particle interaction process. This can be done in path integral quantisation using perturbative expansion of coupling parameter λ. Each term

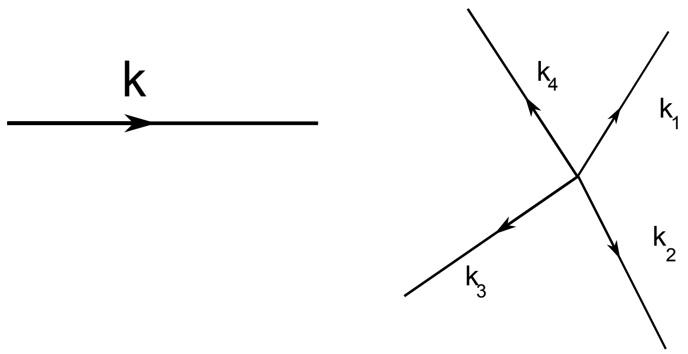

Figure 6.1: Left: Propagator with momentum k. Right: A vertex connecting four propagators.

of the calculation can be represented by Feynman diagrams with associating Feynman rules. So in practice, given a process, we draw all allowed Feynman diagrams up to some order and use Feynman rules to calculate scattering amplitude.

There is a momentum associated at each line in Feynman diagram. The mathematical expression associating to a line is called propagator, and is derived from the free part of the action. For the massive scalar field, the propagator for the line of momentum p^μ is given by

$$G_F(k) = \frac{-i\hbar}{k_\mu k^\mu + m^2 - i\epsilon}, \tag{6.3}$$

where ϵ will be taken to zero after calculation. Momenta of internal lines will be integrated over. Propagators of external lines will be removed. Lines can join at a vertex. For ϕ^4 theory, there are four lines at each vertex. Mathematical expression of vertex can be obtained using Fourier transform of interaction part of the action. We have

$$\begin{aligned} S_I &= -\int d^D x \frac{\lambda}{4!}\phi(x)^4 \\ &= -\int d^D x \frac{\lambda}{4!}\prod_{a=1}^{4}\left(\int \frac{d^D k_a}{(2\pi)^D}\tilde{\phi}(k_a)\right) e^{-i(k_1^\mu + k_2^\mu + k_3^\mu + k_4^\mu)x_\mu} \\ &= -\frac{\lambda}{4!}\prod_{a=1}^{4}\left(\int \frac{d^D k_a}{(2\pi)^D}\tilde{\phi}(k_a)\right)(2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_4). \end{aligned} \tag{6.4}$$

At each vertex, we impose momentum conservation by

$$(2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_4). \tag{6.5}$$

Note that this expression is invariant under any permutation of the propagators at a vertex. We also have to give a factor $-i\lambda/\hbar$ at each vertex.

6.1.2 Feynman Diagrams for Noncommutative QFT

In noncommutative spacetime, we introduce star product into the action. The action now becomes

$$S = -\int d^D x \left[\frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) + \frac{1}{2}m^2\phi(x)^2 + \frac{\lambda}{4!}\phi(x) * \phi(x) * \phi(x) * \phi(x)\right], \tag{6.6}$$

where we used equation (5.32) in the first two terms. We see that the free part of the action remains the same. Therefore the expression for propagator does not change. However, the interaction part is changed. Let us see how this change the momentum conservation on a vertex. Consider Fourier transform of the interaction part:

$$\begin{aligned}
S_I &= -\int d^D x \frac{\lambda}{4!} \phi(x) * \phi(x) * \phi(x) * \phi(x) \\
&= -\int d^D x \frac{\lambda}{4!} \prod_{a=1}^{4} \left(\int \frac{d^D k_a}{(2\pi)^D} \tilde{\phi}(k_a) \right) e^{-ik_1^\mu x_\mu} * e^{-ik_2^\mu x_\mu} * e^{-ik_3^\mu x_\mu} * e^{-ik_4^\mu x_\mu} \\
&= -\int d^D x \frac{\lambda}{4!} \prod_{a=1}^{4} \left(\int \frac{d^D k_a}{(2\pi)^D} \tilde{\phi}(k_a) \right) e^{-\frac{i}{2} k_1 \times k_2} e^{-i(k_1^\mu + k_2^\mu) x_\mu} * e^{-ik_3^\mu x_\mu} * e^{-ik_4^\mu x_\mu} \\
&= -\int d^D x \frac{\lambda}{4!} \prod_{a=1}^{4} \left(\int \frac{d^D k_a}{(2\pi)^D} \tilde{\phi}(k_a) \right) e^{-\frac{i}{2} k_1 \times k_2} e^{-\frac{1}{2}(k_1 + k_2) \times k_3} e^{-i(k_1^\mu + k_2^\mu + k_3^\mu) x_\mu} * e^{-ik_4^\mu x_\mu} \\
&= -\int d^D x \frac{\lambda}{4!} \prod_{a=1}^{4} \left(\int \frac{d^D k_a}{(2\pi)^D} \tilde{\phi}(k_a) \right) V(k_1, k_2, k_3, k_4) e^{-i(k_1^\mu + k_2^\mu + k_3^\mu + k_4^\mu) x_\mu} \\
&= -\frac{\lambda}{4!} \prod_{a=1}^{4} \left(\int \frac{d^D k_a}{(2\pi)^D} \tilde{\phi}(k_a) \right) (2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_4) V(k_1, k_2, k_3, k_4),
\end{aligned} \tag{6.7}$$

where

$$V(k_1, k_2, k_3, k_4) = \prod_{a<b} e^{-\frac{i}{2} k_a \times k_b}, \tag{6.8}$$

and

$$k_a \times k_b \equiv (k_a)_\mu \theta^{\mu\nu} (k_b)_\nu = -k_b \times k_a. \tag{6.9}$$

Therefore the momentum conservation at a vertex becomes

$$(2\pi)^D \delta^D(k_1 + k_2 + k_3 + k_4) V(k_1, k_2, k_3, k_4). \tag{6.10}$$

Note that because of the phase factor this expression is invariant only up to under cyclic permutation of momentum. So Feynman diagrams have to split into two species: planar and non-planar. This means that we can no longer freely permute lines to make all Feynman diagram becomes planar.

Let us first consider planar Feynman diagram. We can keep track of cyclic ordering by using ribbon diagrams. Consider a vertex with cyclically ordered momenta. Introduce l_1, l_2, l_3, l_4 by

$$k_1 = l_1 - l_4, \tag{6.11}$$
$$k_2 = l_2 - l_1, \tag{6.12}$$
$$k_3 = l_3 - l_2, \tag{6.13}$$
$$k_4 = l_4 - l_3. \tag{6.14}$$

The vertex can be drawn as in figure 6.2. In this way, momentum at each vertex is automatically conserved, and the phase factor becomes

$$V = \exp\left(-\frac{i}{2}(l_1 \times l_2 + l_2 \times l_3 + l_3 \times l_4 + l_4 \times l_1) \right). \tag{6.15}$$

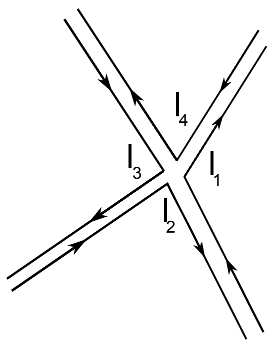

Figure 6.2: Vertex of noncommutative QFT with ϕ^4 potential.

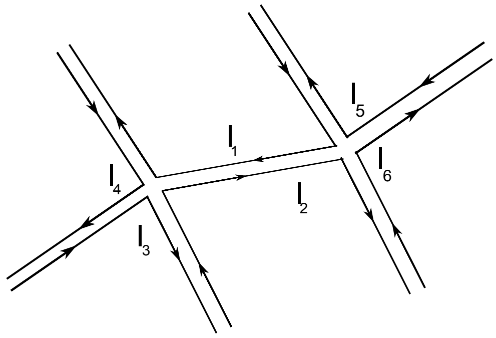

Figure 6.3: Two adjacent vertices in Feynman diagram. The line connecting two vertices make no contribution to overall phase factor V.

Now consider two adjacent vertices in figure 6.3. The overall phase factor of this diagram is

$$V = \exp\left(-\frac{i}{2}(l_2 \times l_3 + l_3 \times l_4 + l_4 \times l_1 + l_1 \times l_5 + l_5 \times l_6 + l_6 \times l_2)\right). \tag{6.16}$$

As far as the overall phase factor concerns, this expression looks as if two vertices join, and the connecting line disappears. i.e. the connecting line makes no contribution to the overall phase factor. Therefore, for a connected Feynman diagram, internal lines make no contribution to the overall phase factor. The resulting phase factor then only depends on external momenta $p_1, p_2, \ldots, p_n$:

$$V_p(p_1, \ldots, p_n) = \prod_{a<b} e^{-\frac{i}{2}p_a \times p_b}. \tag{6.17}$$

Let us now consider non-planar Feynman diagrams. It can be shown that the phase factor is given by

$$V_{np}(p_1, \ldots, p_n) = V_p(p_1, \ldots, p_n) \prod_{a,b} e^{-\frac{i}{2}C_{ab}k_a \times k_b}, \tag{6.18}$$

where C_{ab} is the signed intersection matrix which counts the number of times that the a-th line crosses over the b-th line. Here, the a-th and b-th lines can be either internal or external. If the a-th line cross the b-th line with b-th line goes from right to left then $C_{ab} = 1$.

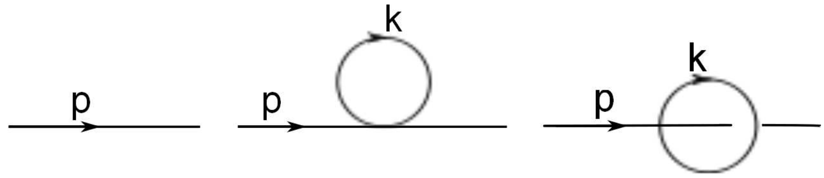

Figure 6.4: All possible $1 \to 1$ scattering amplitudes up to 1-loop order in noncommutative QFT. From left to right: tree-level diagram, 1-loop planar diagram, 1-loop non-planar diagram.

6.2 Divergencies

Consider the $1 \to 1$ scattering process. Feynman diagrams of scattering amplitude have two external lines. Order of $\hbar$ increases with the number of looks. So, to the leading orders, we can consider tree-level diagrams (diagrams with no loop) and 1-loop diagrams. For noncommutative QFT, the diagrams up to 1-loop order are shown in figure 6.4. We apply Wick rotation to zeroth components of momenta in Feynman diagram.

6.2.1 Divergencies in Usual QFT

For the usual QFT, 1-loop planar and non-planar diagrams coincide. Mathematical expression of the tree-level diagram is given by

$$p^2 + m^2, \tag{6.19}$$

where we ignored momentum-conserving delta function and some factors (e.g. $\lambda, -i, \hbar$) because they can be put back in later. In order to obtain this expression, we used Feynman rules. There is one propagator so we have $(p^2 + m^2)^{-1}$. However we have to remove two external propagators. So we have $(p^2 + m^2)^2$. Overall factor then becomes $p^2 + m^2$. For 1-loop diagram, we have

$$\Pi^{(1)}(\vec{k}) = \frac{1}{2} \int \frac{d^D \vec{k}}{(2\pi)^D} \frac{1}{\vec{k}^2 + m^2}, \tag{6.20}$$

where we used Feynman rules given earlier, as well as the rule that we have to integrate over the internal momentum. The factor $1/2$ is coming from symmetry of permutation. This expression is divergence for large $k \equiv |\vec{k}| \to \infty$. This means that if we integrate in some bounded region, then this integral is finite. However, if we extend the integral over all space, the integral becomes infinite. This divergence property is called UV divergence. This usually happens in loop diagrams of massive quantum fields. There is another kind of divergence which is called IR divergence. This divergence happens when momentum approaches zero. We usually encounter IR divergences in theories containing massless quantum fields.

Let us now deal with the UV divergence. In order to do this, we first regularise $\Pi^{(1)}(\vec{k})$ by cutting off the region $k > \Lambda$, and only considering leading divergencies in Λ. Since we are only interested in leading order, we can equivalently regularise $\Pi^{(1)}(\vec{k})$ by using a trick. We first use Schwinger parameterisation

$$\frac{1}{k^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2 + m^2)}. \tag{6.21}$$

We put this in $\Pi^{(1)}(\vec{k})$ and do Gaussian integral for $\vec{k}$. This becomes

$$\Pi^{(1)} = \frac{1}{2}\int_0^\infty d\alpha \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-\alpha k^2}\right)^D e^{-\alpha m^2} = \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{d\alpha}{\alpha^{D/2}} e^{-\alpha m^2}. \tag{6.22}$$

We see that the UV divergence now becomes $\alpha \to 0$ divergence. This can be regularised by introducing a factor $\exp(-1/(\Lambda^2\alpha))$ so that

$$\Pi^{(1)} = \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{d\alpha}{\alpha^{D/2}} e^{-\alpha m^2 - \frac{1}{\Lambda^2\alpha}}. \tag{6.23}$$

This integral can be computed directly (e.g. by using computer programme like Maple). For the case $D = 4$, we have, to leading divergence orders in Λ,

$$\Pi^{(1)} = \frac{1}{2}\frac{1}{(4\pi)^2}\left(\Lambda^2 + m^2 \log\left(\frac{m^2}{\Lambda^2}\right)\right). \tag{6.24}$$

6.2.2 Divergencies in noncommutative QFT

Let us now consider the case of noncommutative QFT. Consider the graphs in figure 6.4. At tree level, the mathematical expression is the same as the usual case. However, at 1-loop level, planar and nonplanar diagrams have to be considered independently. The expression for the planar diagram is given by

$$\Pi_p^{(1)} = \frac{2}{3}\Pi^{(1)}, \tag{6.25}$$

this is because the overall phase factor is 1, and there is less symmetry than the case of usual QFT. So this graph has the same UV divergence property as the usual QFT. The expression for the nonplanar diagram is given by

$$\Pi_{np}^{(1)}(\vec{p}) = \frac{1}{6}\int \frac{d^D\vec{k}}{(2\pi)^D}\frac{e^{i\vec{k}\times\vec{p}}}{\vec{k}^2 + m^2}. \tag{6.26}$$

Note that $2\Pi_{np}^{(1)}(0) = \Pi_p^{(1)}$. Actually, the expression for non-planar graph reduces, up to numerical factor, to the one for planar graph. Therefore, for $p_i\theta^{ij} = 0$, this diagram has UV divergence. Let us now do the regularisation. As before, we apply Schwinger parameterisation. Then

$$\begin{aligned}
\Pi_{np}^{(1)}(\vec{p}) &= \frac{1}{6}\int \frac{d^D\vec{k}}{(2\pi)^D} e^{ik\times p} e^{-\alpha(k^2+m^2)} \\
&= \frac{1}{2}\int_0^\infty d\alpha \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dk' e^{-\alpha k'^2}\right)^D e^{-\frac{p\bullet p}{4\alpha}} e^{-\alpha m^2} \\
&= \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{d\alpha}{\alpha^{D/2}} \exp\left(-\alpha m^2 - \frac{p\bullet p}{4\alpha}\right),
\end{aligned} \tag{6.27}$$

where $p \bullet p = \delta_{lm} p_j \theta^{lj}\theta^{mk} p_k$. Putting in the cut-off and calculating $\Pi_{np}^{(1)}(\vec{p})$ to leading orders, we immediately get, in the case $D = 4$,

$$\Pi_{np}^{(1)} = \frac{1}{6}\frac{1}{(4\pi)^2}\left(\Lambda_{\text{eff}}^2 + m^2 \log\left(\frac{m^2}{\Lambda_{\text{eff}}^2}\right)\right), \tag{6.28}$$

where

$$\Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda^2 + p \bullet p/4}. \tag{6.29}$$

The scattering amplitude up to 1-loop order is given by

$$\Pi(\vec{p}) = p^2 + m^2 + \lambda\Pi_p^{(1)} + \lambda\Pi_{np}^{(1)}(\vec{p}). \tag{6.30}$$

Looking at the tree-level order, we expect that the mass square is given by $\Pi(0)$. So we define a renormalised mass square

$$m_{\text{ren}}^2 = \Pi(0). \tag{6.31}$$

Let us consider two cases:

1. $p \to 0$. In this case, the renormalised mass square reduces to the case of usual QFT. That is

$$m_{\text{ren}}^2 = m^2 + \frac{\lambda\Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2}\log\left(\frac{m^2}{\Lambda^2}\right), \tag{6.32}$$

   which diverges as $\Lambda \to 0$.

2. $\Lambda \to \infty$. The scattering amplitude is given by

$$\Pi(\vec{p}) = p^2 + M^2 + \frac{\lambda}{24\pi^2 p \bullet p} + \frac{\lambda m^2}{96\pi^2}\log\left(\frac{m^2 p \bullet p}{4}\right), \tag{6.33}$$

   where $M^2 = m^2 + \frac{\lambda\Lambda^2}{48\pi^2} + \frac{\lambda m^2}{48\pi^2}\log\left(\frac{m^2}{\Lambda^2}\right)$. Due to the divergence in $p \to 0$, we cannot have a well-defined renormalised mass square $\Pi(0)$.

We see that the two limits $\Lambda \to \infty$ and $p \to 0$ do not commute. This demonstrates the effect called UV/IR mixing. It is surprising that IR divergence appears in massive QFT.

6.3 Other Topics

Let us state some other features of noncommutative QFT. First, due to the infinite spacetime derivative order from star product, the noncommutative QFT is nonlocal. The effect of nonlocality only appears in interacting part of the QFT. It can be seen that nonlocality gives rise to UV/IR mixing. Consider two functions that nonvanish over a very small region. The interaction through star product of the two functions is nonvanished over a much larger region. In momentum space, the small range is equivalent to high momentum. So a virtual particle (a particle in a loop) of high energy produces the effect of small energy (long range). This means that UV cut-off imposes IR cut-off. For more details of this discussion see [24].

In noncommutative spacetime, noncommutative Yang-Mills theory can also be constructed. The construction is more complicated than noncommutative QFT we discussed earlier. We have to define star product for matrix-valued functions.

There is an interesting equivalence in noncommutative Yang-Mills theory. For a Yang-Mills theory in noncommutative torus, there is a dual Yang-Mills theory in a dual noncommutative torus. This equivalence is called Morita equivalence. We may deal with Yang-Mills theory in noncommutative torus by finding a dual theory that is simpler.

Chapter 7

Simple Harmonic Oscillator in Noncommutative Phase Space

We are now ready to study quantum systems using phase space quantisation. We are particularly interested in simple harmonic oscillator (SHO) in totally noncommutative phase space. The simplest situation which noncommutativity comes into play is the case of four dimensional phase space. We first discuss SHO in two dimensional phase space to get the idea of phase space quantisation. We will then turn our attention to SHO in four dimensional phase space. In the case of usual phase space, there are already several results. Based on these results we hope to construct SHO in totally noncommutative four dimensional phase space.

In canonical formalism, we have seen that the energy of SHO is quantised. Each energy state can be constructed from a vacuum $|0\rangle$ by repeatedly applying creation operators on the vacuum. So when we study SHO using phase space formalism, we are particularly interested to see energy levels, energy eigenstates, and ladder operators.

7.1 One Dimensional Simple Harmonic Oscillator

Let us now one dimensional SHO using phase space formalism. There are several techniques to study one dimensional SHO. The most obvious one would be to use Weyl-Wigner correspondence to write the well-known canonical formalism's solutions as quantities in phase space formalism. Alternatively, we may study physics directly in phase space formalism because states and operators are phase space functions, and operations are done by differentiations. So solving some (probably infinite order) partial differential equations might give the required result.

Let us discuss the first technique in more details. In canonical formalism, the Hamiltonian of the SHO, after scaling, is

$$\hat{H} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2), \tag{7.1}$$

and commutation relation is $[\hat{x}, \hat{p}] = i$. We have the ladder operators

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}). \tag{7.2}$$

We now pass to phase space formalism using Weyl-Wigner correspondence. We have the Hamiltonian

$$H = \frac{1}{2}(x^2 + p^2), \tag{7.3}$$

and ladder operators

$$a = \frac{1}{\sqrt{2}}(x + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - ip). \tag{7.4}$$

The operator product is Weyl-Wigner corresponding to the star product. So the number operator is

$$N = a^\dagger * a, \tag{7.5}$$

where the star product is

$$* = \exp \frac{i}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}). \tag{7.6}$$

We can check that

$$H * N = N * H. \tag{7.7}$$

So N shares the $*$-eigenfunctions f_n with H. Consider Wigner functions f_n such that

$$N * f_n = f_n * N = n f_n. \tag{7.8}$$

Using Weyl-Wigner correspondence, we know from canonical formalism that $n = 0, 1, 2, \ldots$, and

$$f_n(x,p) = \int dy \psi_n^* \left(x - \frac{1}{2}y\right) e^{-iyp} \psi_n \left(x + \frac{1}{2}y\right), \tag{7.9}$$

where $\psi_n(x)$ is the wavefunction for the state $|n\rangle$, i.e. $\psi_n(x) = \langle x|n\rangle$. This means that if we know wavefunctions $\psi_n(x)$, we can immediately get the Wigner functions $f_n(x,p)$ which are $*$-eigenfunctions of H with $*$-eigenvalues $n+1/2$. Actually, we only need to know one Wigner function $f_0(x,p)$ which describes ground state. We can then generate $f_n(x,p)$ by repeatedly applying $a^\dagger *$ on the left, and $*a$ on the right. This means that $f_n(x,p)$ is proportional to $(a^\dagger *)^n f_0(x,p)(*a)^n$. Since $\psi_0(x) \propto e^{-x^2/2}$, we have $f_0(x,p) \propto e^{-(x^2+p^2)}$.

For the alternative technique, we begin with the $*$-eigenvalue equation

$$H * f = f * H = E f. \tag{7.10}$$

Explicitly

$$\left[\left(x + \frac{i}{2}\partial_p\right)^2 + \left(p - \frac{i}{2}\partial_x\right)^2 - 2E\right] f(x,p) = 0. \tag{7.11}$$

The imaginary part of this equation implies that f is a function of H. Solving the real part we get [25]

$$f_n(x,p) = f_n(H) \equiv 2(-1)^n e^{-2H} L_n(4H), \tag{7.12}$$

where $n = E - 1/2 = 0, 1, 2, \ldots$, and $L_n(4H)$ are Laguerre's polynomial:

$$L_0(4H) = 1, \quad L_1(4H) = 1 - 4H, \quad L_2(4H) = 8H^2 - 8H + 1, \ldots. \tag{7.13}$$

7.2 Two Dimensional Simple Harmonic Oscillator

Let us now discuss two dimensional SHO. The Hamiltonian is given by $H = \frac{1}{2}(x^2 + y^2 + p_x^2 + p_y^2)$. The star product is

$$* = \exp \frac{i}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_{p_x}} - \overleftarrow{\partial_{p_x}}\overrightarrow{\partial_x} + \overleftarrow{\partial_y}\overrightarrow{\partial_{p_y}} - \overleftarrow{\partial_{p_y}}\overrightarrow{\partial_y}). \tag{7.14}$$

We may use the fact from canonical formalism that for two dimensional SHO, we have two separate sets of ladder operators. Explicitly $\hat{a}_1 = (\hat{x}+i\hat{p}_x)/\sqrt{2}, \hat{a}_2 = (\hat{y}+i\hat{p}_y)/\sqrt{2}$ are the annihilation operators. The number operator associated to each set is then $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$, for $i = 1, 2$. If $|n_1\rangle_1, |n_2\rangle_2$ are eigenstates of $\hat{N}_1$ and $\hat{N}_2$ respectively, then $|n_1\rangle_1 \otimes |n_2\rangle_2$ is the eigenstate of $\hat{H}$ with eigenenergy n_1+n_2+1. The wavefunction $\langle x, y|n_i\rangle_i$ only depends on x_i. So the Wigner function is Weyl-Wigner corresponding to $|n_1\rangle_1\langle n_1|_1 \otimes |n_2\rangle_2\langle n_2|_2$. That is the $*$-eigenfunction of H is

$$\tilde{f}_{n_1,n_2}(x, y, p_x, p_y) = f_{n_1}(H_1)f_{n_2}(H_2), \tag{7.15}$$

where $H_i = N_i + 1/2$. Explicitly

$$\tilde{f}_{n_1,n_2}(x, y, p_x, p_y) = 4(-1)^{n_1+n_2}e^{-2H}L_{n_1}(4H_1)L_{n_2}(4H_2). \tag{7.16}$$

An alternative method can be seen in [26]. In there the analysis is done solely in phase space formalism. The method is Weyl-Wigner corresponding to spectral decomposition method in canonical formalism. We briefly discuss the idea here. First the quantity

$$\mathrm{Exp}(Ht) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(-it)^n(\underbrace{H * \ldots * H}_{n \text{ terms}}) \tag{7.17}$$

is written. It is then expanded around the origin of complex plane of t. For one dimensional SHO, Fourier transform of $\mathrm{Exp}(Ht)$ in this region has Wigner functions $f_n(H)$ as amplitudes and $E = n + 1/2, n = 0, 1, 2, \ldots$ as 'frequency' . For $\mathcal{N}$ dimensional isotropic SHO with Hamiltonian $H = \frac{1}{2}(\vec{p}^2 + \vec{q}^2)$ the quantity $\mathrm{Exp}(Ht)$ can be decomposed as

$$\mathrm{Exp}(Ht) = \mathrm{Exp}(H_1t) * \ldots * \mathrm{Exp}(H_{\mathcal{N}}t) \tag{7.18}$$

where $H_i = \frac{1}{2}(p_i^2 + q_i^2), i = 1, \ldots, \mathcal{N}$. The Fourier amplitude is then obtained by multiplying the $\mathcal{N}$ separate sets of one-dimensional amplitude. In two dimensional SHO, we then get the result in equation (7.16).

Actually, there is a symmetry in two dimensional SHO that allows us to write Wigner funciton in an alternative form. Let us consider an important example which will be a useful starting point for achieving the main result. The analysis is taken from [26]. Consider a two dimensional SHO with Hamiltonian $H = \frac{1}{2}(x^2 + y^2 + p_x^2 + p_y^2)$. We also consider the angular momentum $L = xp_y - yp_x$. Note that $H * L = L * H$. So they shares the same $*$-eigenfunctions. This result is also expected from classical mechanics since the angular momentum is conserved. Let us now solve the eigenvalue problem:

$$H * f = f * H = Ef, \tag{7.19}$$

$$L * f = f * L = Mf, \tag{7.20}$$

$$f * f = f. \tag{7.21}$$

We introduce linear canonical transformation

$$x' = \frac{1}{\sqrt{2}}(x + p_y), \quad p_x' = -\frac{1}{\sqrt{2}}(y - p_x), \tag{7.22}$$

$$y' = \frac{1}{\sqrt{2}}(x - p_y), \quad p_y' = \frac{1}{\sqrt{2}}(y + p_x). \tag{7.23}$$

We now introduce ladder operators and number operators.

$$a_r = \frac{1}{\sqrt{2}}(x' + ip'_x), \quad N_r = a_r^\dagger * a_r, \tag{7.24}$$

$$a_l = \frac{1}{\sqrt{2}}(y' + ip'_y), \quad N_l = a_l^\dagger * a_l. \tag{7.25}$$

Note that $N_r * N_l = N_l * N_r$, and we have

$$N_r + \frac{1}{2} = \frac{H+L}{2}, \tag{7.26}$$

$$N_l + \frac{1}{2} = \frac{H-L}{2}. \tag{7.27}$$

In this way, we have decomposed two dimensional SHO into two sets of separate one dimensional SHO. The solution is then

$$f_{n,m}(H/2, L/2) \equiv 4(-1)^n e^{-2H} L_{(n+m)/2}(2(H+L)) L_{(n-m)/2}(2(H-L)) \tag{7.28}$$

with $E = n+1, M = m$, where $n = 0, 1, 2, \ldots$ and $m = -n, -n+2, \ldots, n-2, n$.

7.3 Two Dimensional SHO in Noncommutative Phase Space

Let us now consider two dimensional SHO in totally noncommutative phase space. After scaling the Hamiltonian is

$$H = \frac{1}{2}(x^2 + y^2 + p_x^2 + p_y^2), \tag{7.29}$$

and the Moyal brackets are

$$\{\{x, y\}\} = i\theta, \quad \{\{x^i, p^j\}\} = i\delta^{ij}, \quad \{\{p_x, p_y\}\} = i\phi. \tag{7.30}$$

We can then define star product accordingly. As in the case of usual phase space, we introduce $L = xp_y - yp_x$. The operators in two dimensional SHO in totally noncommutative phase space are polynomials of degree ≤ 2. So in practice, we may use the tricks discussed in section 5.5, and particularly in example 2.

We first calculate the Moyal bracket between $J_0 = H/2$ and $J_2 = L/2$. We have

$$\{\{J_0, J_2\}\} = i(\phi - \theta)\frac{K_2}{2}, \tag{7.31}$$

where $K_2 = (xp_x + yp_y)/2$. This commutation relation generates a new quantity K_2. We hope that by calculating Moyal bracket within the group of J_0, J_2, K_2, we will get new quantities. Calculating Moyal bracket within a new group would generate other quantities. The process would end when there is no new quantity generated. The result is

$$\{\{J_0, J_2\}\} = -ivK_2, \tag{7.32}$$

$$\{\{J_0, K_2\}\} = -iQ_2 + ivJ_2, \tag{7.33}$$

$$\{\{J_0, Q_2\}\} = iK_2, \tag{7.34}$$

$$\{\{J_2, K_2\}\} = ivJ_0 + iuQ_2, \tag{7.35}$$

$$\{\{J_2, Q_2\}\} = -iuK_2, \tag{7.36}$$

$$\{\{K_2, Q_2\}\} = iJ_0 + iuJ_2, \tag{7.37}$$

$$\tag{7.38}$$

where $Q_2 = -(x^2 - p_x^2 + y^2 - p_y^2)/4, u = (\theta + \phi)/2, v = (\theta - \phi)/2$. Note that the quantities J_0, J_2, K_2 and Q_2 coincide with the ones given in example 1. The commutation relations are messy and difficult to study.

To get some insight, let us study the case of $\theta = \phi = 0$. The commutation relations are then

$$\{\{J_0, J_2\}\} = 0, \tag{7.39}$$
$$\{\{J_0, K_2\}\} = -iQ_2, \tag{7.40}$$
$$\{\{J_0, Q_2\}\} = iK_2, \tag{7.41}$$
$$\{\{J_2, K_2\}\} = 0, \tag{7.42}$$
$$\{\{J_2, Q_2\}\} = 0, \tag{7.43}$$
$$\{\{K_2, Q_2\}\} = iJ_0. \tag{7.44}$$

As expected, these commutation relations reduce to the ones given in example 1. We see that J_0, Q_2, K_2 generates $SO(1,2)$ group. Using the trick in example 2, we have the Casimir operator:

$$J_0 * J_0 - K_2 * K_2 - Q_2 * Q_2 = J_2 * J_2 - \frac{1}{4} = \left(J_2 - \frac{1}{2}\right) * \left(J_2 + \frac{1}{2}\right). \tag{7.45}$$

Let us now analyse the spectrum of two dimensional SHO using $SO(1,2)$ algebra. Let us do this in canonical formalism because it is Weyl-Wigner corresponding to phase space formalism and is more compact in this analysis. Consider the state $|j\ \mu\rangle$ with

$$\hat{J}_2|j\ \mu\rangle = j|j\ \mu\rangle, \tag{7.46}$$
$$\hat{J}_0|j\ \mu\rangle = \mu|j\ \mu\rangle. \tag{7.47}$$

Let $\hat{J}_\pm = i\hat{K}_2 \pm \hat{Q}_2$ so that

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \tag{7.48}$$
$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_0. \tag{7.49}$$

From the commutation relations, we see that

$$\begin{aligned} \hat{J}_0\hat{J}_\pm|j\ \mu\rangle &= (\hat{J}_\pm\hat{J}_0 + [\hat{J}_0, \hat{J}_\pm])|j\ \mu\rangle \\ &= \hat{J}_\pm(\hat{J}_0 \pm 1)|j\ \mu\rangle \\ &= (\mu \pm 1)\hat{J}_\pm|j\ \mu\rangle. \end{aligned} \tag{7.50}$$

That is $\hat{J}_\pm$ raises/lowers the eigenenergy of $\hat{J}_0$. We note that $\hat{J}_+^\dagger = -\hat{J}_-$. So

$$\begin{aligned} 0 \le \left|\hat{J}_\pm|j\ \mu\rangle\right|^2 &= \langle j\ \mu|\hat{J}_\pm^\dagger\hat{J}_\pm|j\ \mu\rangle \\ &= \langle j\ \mu| - \hat{J}_\mp\hat{J}_\pm|j\ \mu\rangle \\ &= -\langle j\ \mu|\hat{J}_2^2 - \frac{1}{4} - \hat{J}_0^2 \mp \hat{J}_0|j\ \mu\rangle \\ &= -(j^2 - \frac{1}{4} - \mu^2 \mp \mu). \end{aligned} \tag{7.51}$$

This implies $-(\mu-1/2) \leq j \leq \mu-1/2$. Actually, the analysis from $SO(1,2)$ is not enough to determine the spectrum. However note that when j take integer or half-integer we get the expected spectrum as in the previous section.

We now have some insight about the commutation relations in the case $\theta = \phi = 0$. Let us come back to the case of general θ and ϕ. We expect to get the algebra as similar as possible to $SO(1,2)$. We do this is steps

STEP 1 We want to see if there is an operator that commutes with J_0. We recognise that $\{\{J_0, J_2\}\}$ and $\{\{J_0, Q_2\}\}$ gives, up to constant, K_2. So we have

$$\{\{J_0, A_1\}\} = 0, \tag{7.52}$$

where $A_1 = J_2 + vQ_2$. As usual, we want a closed algebra of commutation relations. We have, in addition,

$$\{\{K_2, A_1\}\} = -iuA_2, \tag{7.53}$$
$$\{\{J_0, K_2\}\} = -iA_2, \tag{7.54}$$
$$\{\{J_0, A_2\}\} = is^2 K_2, \tag{7.55}$$
$$\{\{K_2, A_2\}\} = is^2 J_0 + iuA_1, \tag{7.56}$$
$$\{\{A_1, A_2\}\} = -ius^2 K_2, \tag{7.57}$$

where $A_2 = Q_2 - vJ_2$, and $s = \sqrt{1+v^2}$.

STEP 2 Looking at $\{\{J_0, K_2\}\}, \{\{J_0, A_2\}\}$, and $\{\{K_2, A_2\}\}$ we are motivated to define

$$A_2'' = \frac{A_2}{s}, \tag{7.58}$$

$$J_0'' = sJ_0 + \frac{uA_1}{s}. \tag{7.59}$$

So now

$$\{\{J_0'', A_1\}\} = 0, \tag{7.60}$$
$$\{\{K_2, A_1\}\} = -iusA_2'', \tag{7.61}$$
$$\{\{J_0'', K_2\}\} = -i(1-\theta\phi)A_2'', \tag{7.62}$$
$$\{\{J_0'', A_2''\}\} = i(1-\theta\phi)K_2, \tag{7.63}$$
$$\{\{K_2, A_2''\}\} = iJ_0'', \tag{7.64}$$
$$\{\{A_1, A_2''\}\} = -iusK_2. \tag{7.65}$$

STEP 3 Let us focus on equations (7.62)-(7.64). We also consider the case $\theta\phi < 1$. Other cases will be discussed later. We are motivated to define

$$J_0' = \frac{J_0''}{1-\theta\phi}, \tag{7.66}$$

$$K_2' = \frac{K_2}{\sqrt{1-\theta\phi}}, \tag{7.67}$$

$$Q_2' = \frac{A_2''}{\sqrt{1-\theta\phi}}. \tag{7.68}$$

So that J_0', K_2', Q_2' generates $SO(1,2)$ group. We want to find the central element J_2' that commutes with J_0', K_2', Q_2'. Let us use the trick in example 2. We realise that $J_0'^2 - K_2'^2 - Q_2'^2 = J_2'^2$, where

$$J_2' = \frac{1}{1-\theta\phi}\left(uJ_0 + A_1\right). \tag{7.69}$$

We can check that this is actually the case, i.e. $J_0' * J_0' - K_2' * K_2' - Q_2' * Q_2' = J_2' * J_2' - 1/4$.

By construction, we see that J_0', K_2', Q_2', J_2' for $\theta\phi < 1$ forms the same set of commutation relations as J_0, K_2, Q_2, J_2 for $\theta = \phi = 0$. So for general θ, ϕ with $\theta\phi < 1$ we can find the spectrum of J_0' in the same way as in the usual phase space. We first note that J_0, J_2, J_0', and J_2' commute with each other. So they share the same eigenstates. Furthermore, we have

$$J_0 = J_0's - uJ_2', \tag{7.70}$$

$$J_2 = J_2's^2 - usJ_0'. \tag{7.71}$$

Let eigenvalues for J_0, J_2, J_0', and J_2' be μ, j, μ', and j', respectively. We carry out the analysis of finding the energy for the state $|j'\ \mu'\rangle$. We set j' being integer or half-integer. As usual, we have $2\mu' = n+1$, for $n = 0, 1, 2, \ldots$, and $2j' = m$, for $m = -n, -n+2, \ldots, n-2, n$. The energy spectrum for $H = 2J_0$ is then

$$E_{n,m}^{(\theta,\phi)} = 2\mu = (n+1)\sqrt{1+\left(\frac{\theta-\phi}{2}\right)^2} - \frac{(\theta+\phi)}{2}m. \tag{7.72}$$

So we see that as opposed to the usual case where $E_{n,m} = n+1$, the energy level for the same value of n is generally split. However, the degeneracy is recovered in the case $\theta+\phi = 0$. For small θ and ϕ, we have $E_{n,m}^{(\theta,\phi)} = E_{n,m} - (\theta+\phi)m/2$. The effect of energy level splitting for two dimensional SHO in totally noncommutative phase space is analogous to Zeeman effect which is the splitting of energy level for Hydrogen atom by weak magnetic field B. The splitting of energy level is proportional to Bm_l, where m_l is the magnetic quantum number.

Let us now discuss what is likely to happen for $\theta\phi \geq 1$. Looking at the equations (7.62)-(7.64), we see that when $\theta\phi = 1$, there are more quantities that commute with Hamiltonian. So the energy should become more degenerate in this case. For the case of $\theta\phi > 1$, the ladder operators $J_\pm = iK_2 \pm Q_2$ switches the role. That is J_+ becomes lowering operator while J_- becomes raising operator. So instead of energy unbounded from above, energy now becomes unbounded from below. We may discuss this in alternative way. Naïvely[1] consider the equation (7.72). When $\theta\phi = 1$, the vacuum energy is degenerate with the value of $(\theta+\phi)/2$ for $m = n$. When $\theta\phi > 1$, the energy is not bounded from below: the vacuum is then unstable. So we may believe that the region $\theta\phi \geq 1$ is unphysical.

There is a better interpretation for the requirement $\theta\phi < 1$. Let us look back at the result of D-brane in pp-wave background with constant B-field. For point particle at $\sigma = 0$, the coordinates and momenta are given by

$$\hat{X}^k = (\hat{x}_0^k \cos\omega_0\tau + 2\alpha'\hat{p}_0^k \frac{\sin\omega_0\tau}{\omega_0}), \tag{7.73}$$

$$2\pi\alpha'\hat{P}^k = (-\hat{x}_0^j\omega_0 \sin\omega_0\tau + 2\alpha'\hat{p}_0^j \cos\omega_0\tau){M_j}^k. \tag{7.74}$$

[1]I would like to thank Prof. Chong-Sun Chu for the discussion leading to this result.

Note that there is no oscillation mode for point particle. The commutation relations for X^k and $\lambda P^k/(2\alpha'(1+\bar{B}^2))$ take the form corresponding to equations (4.72)-(4.74). That is we may take the commutation relations

$$[\hat{x}^j, \hat{p}^k] = i\delta^{jk}\frac{\pi\omega_0\bar{B}}{(1+\bar{B}^2)\tanh\pi\omega_0\bar{B}}, \quad j,k = 2,3, \tag{7.75}$$

$$[\hat{x}^2, \hat{x}^3] = i\frac{2\pi\alpha'\bar{B}}{1+\bar{B}^2}, \tag{7.76}$$

$$[\hat{p}^2, \hat{p}^3] = i\frac{\pi\omega_0^2\bar{B}}{2\alpha'(1+\bar{B}^2)}, \tag{7.77}$$

as a starting point. We want to scale the coordinates and momenta so that the commutation relations take the form of equation (7.30). After scaling, we realise that

$$\theta\phi = \tanh^2\pi\omega_0\bar{B} < 1. \tag{7.78}$$

So we may interpret that string theory only allows the physical region for totally noncommutative phase space.

Previously we have discussed that using the information from $SO(1,2)$ is not enough to determine energy spectrum. However, we may argue that the Wigner functions $f_{n,m}(J_0', J_2')$ are $*$-eigenfunctions of J_0' and J_2'. So they are also $*$-eigenfunctions for Hamiltonian H. In the case where $\theta = \phi = 0$, the energy spectrum is labelled by $n = 0, 1, 2, \ldots,$ and $m = -n, -n+2, \ldots, n-2, n$. So if we continuously turn on θ and ϕ an energy level should not suddenly appear or disappear, i.e. the number of energy levels should remain the same. This is justified because the $*$-eigenfunctions $f_{n,m}(J_0', J_2')$ also have the same set of label n, m. From this argument, we can be more confident to use the result obtained in this section.

Actually, we can justify our result formally. This justification would lead to an interesting consequence. As many readers may have noticed long before that when $\theta = \phi = 0$, the operators J_0, J_2, K_2, Q_2 form a subalgebra of $Sp(4)$. So we may also expect that for general θ, ϕ (probably with $\theta\phi < 1$) the operators J_0', J_2', K_2', Q_2' form a subalgebra of $Sp(4)$. So let us now revisit canonical transformation in four dimensional phase space.

7.4 Canonical Transformation in Four Dimensional Noncommutative Phase Space

Our goal now is to see how the 10 generators of $Sp(4)$ for $\theta = \phi = 0$ deformed and redefined in the case $\theta, \phi \neq 0$ to get an algebra as similar as possible to $Sp(4)$. We may try to use the method discussed in previous section. However, we now have to deal with 10 generators, which means 45 commutation relations have to be studied simultaneously. So this is not a good idea.

7.4.1 Function Form of Generators

Inspired by [1], the 10 generators of $Sp(4)$ can be written in terms of ladder operators. Let us study the case $\theta = \phi = 0$. We want to use ladder operators defined near the end of section 7.2. Since the canonical transformation was used before defining ladder operators, the expressions may not look exactly like the ones in [1]. Working out explicitly, we find that

$$J_0 = \frac{1}{2}(a_r^\dagger * a_r + a_l^\dagger * a_l + 1), \tag{7.79}$$

$$J_1 = \frac{1}{2i}(a_r^\dagger * a_l - a_l^\dagger * a_r), \tag{7.80}$$

$$J_2 = \frac{1}{2}(a_r^\dagger * a_r - a_l^\dagger * a_l), \tag{7.81}$$

$$J_3 = \frac{1}{2}(a_r^\dagger * a_l + a_l^\dagger * a_r), \tag{7.82}$$

$$K_1 = -\frac{1}{4}(a_r^\dagger * a_r^\dagger + a_r * a_r + a_l^\dagger * a_l^\dagger + a_l * a_l), \tag{7.83}$$

$$K_2 = \frac{i}{2}(a_r^\dagger * a_l^\dagger - a_r * a_l), \tag{7.84}$$

$$K_3 = -\frac{i}{4}(a_r^\dagger * a_r^\dagger - a_r * a_r - a_l^\dagger * a_l^\dagger + a_l * a_l), \tag{7.85}$$

$$Q_1 = -\frac{i}{4}(a_r^\dagger * a_r^\dagger - a_r * a_r + a_l^\dagger * a_l^\dagger - a_l * a_l), \tag{7.86}$$

$$Q_2 = -\frac{1}{2}(a_r^\dagger * a_l^\dagger + a_r * a_l), \tag{7.87}$$

$$Q_3 = \frac{1}{4}(a_r^\dagger * a_r^\dagger + a_r * a_r - a_l^\dagger * a_l^\dagger - a_l * a_l). \tag{7.88}$$

So instead of studying 45 commutation relations, we may try to construct 2 sets of ladder operators.

Let us study the case of general θ, ϕ with $\theta\phi < 1$. Recall that we have obtained J_0', J_2', K_2', Q_2'. We expect that

$$J_0' + J_2' - \frac{1}{2} = a_r'^\dagger * a_r', \tag{7.89}$$

$$J_0' - J_2' - \frac{1}{2} = a_l'^\dagger * a_l', \tag{7.90}$$

for new ladder operators $a_r', a_r'^\dagger, a_l', a_l'^\dagger$ to be determined. With direct calculations (see appendix A) we have

$$a_r' = A_r x + B_r p_y + i(-A_r y + B_r p_x), \quad a_r'^\dagger = a_r'^*, \tag{7.91}$$

$$a_l' = A_l x - B_l p_y + i(A_l y + B_l p_x), \quad a_l'^\dagger = a_l'^*, \tag{7.92}$$

where

$$A_r = \sqrt{\frac{(s+u)(s-v)}{4s(1-\theta\phi)}}, \quad B_r = \sqrt{\frac{(s+u)(s+v)}{4s(1-\theta\phi)}}, \tag{7.93}$$

$$A_l = \sqrt{\frac{(s-u)(s+v)}{4s(1-\theta\phi)}}, \quad B_l = \sqrt{\frac{(s-u)(s-v)}{4s(1-\theta\phi)}}. \tag{7.94}$$

These are well-defined because $s > |u|$ for $\theta\phi < 1$. We can check that

$$\{\{a_r', a_r'^\dagger\}\} = \{\{a_l', a_l'^\dagger\}\} = 1, \tag{7.95}$$

$$\{\{a'_r, a'_l\}\} = \{\{a'_r, a'^{\dagger}_l\}\} = \{\{a'^{\dagger}_r, a'^{\dagger}_l\}\} = \{\{a'^{\dagger}_r, a'^{\dagger}_l\}\} = 0. \tag{7.96}$$

So $a'_r, a'^{\dagger}_r, a'_l, a'^{\dagger}_l$ are indeed ladder operators. We use ladder operators to obtain the generators in the similar way to equations (7.79)-(7.88). The result is

$$J'_0 = \frac{1}{1-\theta\phi}[sJ_0 + \frac{u}{s}(J_2 + vQ_2)], \tag{7.97}$$

$$J'_1 = \frac{1}{s\sqrt{1-\theta\phi}}[J_1 + vQ_1], \tag{7.98}$$

$$J'_2 = \frac{1}{1-\theta\phi}[uJ_0 + J_2 + vQ_2], \tag{7.99}$$

$$J'_3 = \frac{1}{s\sqrt{1-\theta\phi}}[J_3 + vQ_3], \tag{7.100}$$

$$K'_1 = \frac{1}{s(1-\theta\phi)}[s^2K_1 + uvJ_3 - uQ_3], \tag{7.101}$$

$$K'_2 = \frac{K_2}{\sqrt{1-\theta\phi}}, \tag{7.102}$$

$$K'_3 = \frac{1}{s(1-\theta\phi)}[s^2K_3 - uvJ_3 + uQ_1], \tag{7.103}$$

$$Q'_1 = \frac{1}{(1-\theta\phi)}[Q_1 - vJ_1 + uK_3], \tag{7.104}$$

$$Q'_2 = \frac{1}{s\sqrt{1-\theta\phi}}[Q_2 - vJ_2], \tag{7.105}$$

$$Q'_3 = \frac{1}{(1-\theta\phi)}[Q_3 - vJ_3 - uK_1]. \tag{7.106}$$

We may check explicitly that the commutation relations for these 10 generators form $Sp(4)$. This is what we expect from the expressions of generators in terms of ladder operators. The commutation relations for ladder operators can then be used to determine the commutation relations for the 10 generators.

Having obtained the two sets of ladder operators, we can completely determine the spectrum of J'_0 in the same way as we did for two dimensional SHO in the usual phase space. Then, using the relationship between J_0, J_2, J'_0, J'_2, we can obtain the result in the equation (7.72). This time we do not need to assume anything.

7.4.2 Matrix Form of Generators

Let us now study the matrix form of generators. We can obtain the matrix form using the equation (2.24). First we have to determine the symplectic matrix $\mathbf{\Lambda}$. Explicitly,

$$\mathbf{\Lambda}(\theta, \phi) = \begin{pmatrix} \theta T & I \\ -I & \phi T \end{pmatrix}, \tag{7.107}$$

where

$$T = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.108}$$

Note that $\det \mathbf{\Lambda}(\theta, \phi) = 1 - \theta\phi$. So this suggests that when $\theta\phi = 1$, our definition of Weyl-Wigner correspondence breaks down. Now we have the matrix form of generators

$$\underline{\underline{J'_0}} = \frac{i}{2s}\begin{pmatrix} vT & I \\ -I & -vT \end{pmatrix}, \tag{7.109}$$

$$\underline{\underline{J'_1}} = \frac{i}{2s\sqrt{1-\theta\phi}}\begin{pmatrix} u\sigma_3 & (s^2+uv)\sigma_1 \\ -(s^2-uv)\sigma_1 & u\sigma_3 \end{pmatrix}, \tag{7.110}$$

$$\underline{\underline{J'_2}} = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{7.111}$$

$$\underline{\underline{J'_3}} = \frac{i}{2s\sqrt{1-\theta\phi}}\begin{pmatrix} -u\sigma_1 & (s^2+uv)\sigma_3 \\ -(s^2-uv)\sigma_3 & -u\sigma_1 \end{pmatrix}, \tag{7.112}$$

$$\underline{\underline{K'_1}} = \frac{i}{2s}\begin{pmatrix} v\sigma_1 & \sigma_3 \\ \sigma_3 & v\sigma_1 \end{pmatrix}, \tag{7.113}$$

$$\underline{\underline{K'_2}} = \frac{i}{2\sqrt{1-\theta\phi}}\begin{pmatrix} I & \theta T \\ \phi T & -I \end{pmatrix}, \tag{7.114}$$

$$\underline{\underline{K'_3}} = -\frac{i}{2s}\begin{pmatrix} v\sigma_3 & -\sigma_1 \\ -\sigma_1 & v\sigma_3 \end{pmatrix}, \tag{7.115}$$

$$\underline{\underline{Q'_1}} = -\frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \tag{7.116}$$

$$\underline{\underline{Q'_2}} = \frac{i}{2s\sqrt{1-\theta\phi}}\begin{pmatrix} -uT & (s^2+uv)I \\ (s^2-uv)I & -uT \end{pmatrix}, \tag{7.117}$$

$$\underline{\underline{Q'_3}} = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{7.118}$$

We may check explicitly that the matrix form of generators satisfy the commutation relations for $Sp(4)$. We may also check explicitly that the transformation matrices generated by these generators indeed preserve Poisson bracket. Surprisingly, these matrices $\underline{\underline{J'_2}}, \underline{\underline{Q'_1}}$ and $\underline{\underline{Q'_3}}$ do not depend on θ and ϕ. That is the matrices remain the same form for any parameter of totally noncommutative phase space. In the future, it might be useful to study why this is the case.

7.5 Discussion

Our main objective in this chapter is to study two dimensional SHO. We mainly use phase space formalism. In quantum case, we have reviewed some methods in obtaining energy spectrum and Wigner functions which solve our problem. We learn that, as expected, energy is degenerate. In the case of totally noncommutative phase space, we tried to find the algebra that is associated to our problem. We have found a complicated algebra for J_0, J_2, K_2, Q_2. This algebra reduces to $SO(1,2)$ in the case of usual phase space. We learn that this algebra produces the expected energy spectrum. Looking back to the case of totally noncommutative phase space, we see that our complicated algebra can be made to $SO(1,2)$ with the shift transformation of J_0, J_2, K_2, Q_2 to J'_0, J'_2, K'_2, Q'_2. From this result, we can also obtain the spectrum of J'_0. This spectrum tells us that the energy is generally nondegenerate.

In the process, we have used the requirement that $\theta\phi < 1$. This is justified from several separated arguments. All of them pointed out that the case $\theta\phi \geq 1$ is unphysical.

The most convincing argument is from string theory. On D-brane in pp-wave background with constant B-field, a point particle has commutation relations that demand $\theta\phi < 1$.

Let us look back to the main discussion. Actually the information from the group $SO(1,2)$ is not enough to determine the spectrum. We noticed that the generators for this group actually belongs to a bigger group which is $Sp(4)$. Studying $Sp(4)$ might help us justify the result. Instead of using the method discussed previously, we use a more elegant method. We can do this by writing the 10 generators in term of ladder operators. In the case of totally noncommutative phase space we have already obtained 4 (from $SO(1,2)$) generators out of 10. So it is possible to determine the ladder operators. After getting the ladder operators and checking that they give correctly the generators of $Sp(4)$, we can use these ladder operators to construct the spectrum of J_0' in the same way as we did for the two dimensional SHO in usual phase space. So this justifies our result.

Reading this chapter, the readers may be motivated that we have used some kind of changing the basis. Essentially, we realise that a new basis can be defined by

$$\tilde{x} = \sqrt{2}(A_r x + B_r p_y), \quad \tilde{y} = \sqrt{2}(A_l x + B_l p_y), \tag{7.119}$$

$$\tilde{p}_x = \sqrt{2}(-A_r y + B_r p_x), \quad \tilde{p}_y = \sqrt{2}(A_l y + B_l p_x). \tag{7.120}$$

This is motivated from the equations (7.91)-(7.92). We can check that the symplectic matrix is transformed as

$$\mathbf{\Lambda}(\theta,\phi) \to \widetilde{\mathbf{\Lambda}}(\theta,\phi) = \mathbf{\Lambda}(0,0). \tag{7.121}$$

That is the symplectic matrix take the form of the one in usual phase space. Explicitly, we have the transformation matrix

$$S = \sqrt{2}\begin{pmatrix} A_r & 0 & 0 & B_r \\ A_l & 0 & 0 & -B_l \\ 0 & -A_r & B_r & 0 \\ 0 & A_l & B_l & 0 \end{pmatrix}. \tag{7.122}$$

So that

$$S\mathbf{\Lambda}(\theta,\phi)S^T = \mathbf{\Lambda}(0,0). \tag{7.123}$$

The generators $\underline{\underline{G'}}$ of $\mathbf{\Lambda}(\theta,\phi)$ are related to the generators $\underline{\underline{G}}$ of $\widetilde{\mathbf{\Lambda}}(\theta,\phi)$ by

$$S\underline{\underline{G'}}S^{-1} = \underline{\underline{G}}. \tag{7.124}$$

For example $S\underline{\underline{J_0}}'S^{-1} = \underline{\underline{J_0}}$, which can be verified explicitly. More elegantly, let us consider canonical transformation matrices $\underline{\underline{M'}} = \exp(-i\alpha\underline{\underline{G'}})$, $\underline{\underline{M}} = \exp(-i\alpha\underline{\underline{G}})$ for some real parameter α. Using equations (7.123) and (7.124), we can verify that if $\underline{\underline{M}}$ and $\mathbf{\Lambda}(0,0)$ satisfy the equation (2.19) (this is true; see example 1), then $\underline{\underline{M'}}$ and $\mathbf{\Lambda}(\theta,\phi)$ also satisfy the equation (2.19).

Chapter 8

Conclusions and Discussions

In this report, we have seen how noncommutative geometry arises in string theory. We have also discussed physical consequences. In particular, we studied simple harmonic oscillator (SHO) in totally noncommutative phase space. The idea is that in the case of noncommutative spacetime, quantum field theory can be constructed. However, it is difficult to directly construct quantum field theory in totally noncommutative phase space. So studying SHO in totally noncommutative phase space might give us some insights.

We have started this report by presenting essential basics in Part I. We started by discussing theories of point particles. In particular we discussed canonical transformation which serves as a basic for the main results. We then discussed string theory. When quantised, string creates particles and fields. In low energy limit, massless background that strings can live in must satisfies the condition of vanishing $\beta-$functions.

We then discuss some advanced topics in Part II. We have seen that when quantising an open string attached to D-brane in constant B-field, the D-brane worldvolume becomes noncommutative. Quantum field theory in noncommutative spacetime has been discussed in literatures. We also see that in the case of D-brane in pp-wave background with constant B-field, the phase space of D-brane becomes totally noncommutative.

Quantum field theory in noncommutative spacetime has been studied in literatures. We have reviewed some features in chapter 6.

Actually, we see that the phase space can become totally noncommutative. So the ultimate goal is to construct quantum field theory in totally noncommutative phase space. However, it is difficult to do so because neither coordinate space representation nor momentum space representation are allowed. We consider phase space quantisation which allows us to view phase space as a whole. The noncommutative property is encoded in star product. So in the future this formalism can be useful to construct quantum field theory in totally noncommutative phase space.

We have constructed phase space quantisation formalism as a direct generalisation to the one discussed in literatures. In this generalisation we can study quantisation in totally noncommutative phase space. We can view observables and states as phase space functions. There is an important property called Weyl-Wigner correspondence which relates phase space formalism to canonical formalism. So if we have a theory in canonical formalism we can immediately get the theory in phase space formalism, or vice versa. The calculation in phase space formalism is simplified if an observable is a polynomial of degree ≤ 2 : the quantum observable take the same function as the classical observable.

After we discussed phase space quantisation, we studied two dimensional SHO in totally noncommutative phase space. This is the simplest system where noncommutativity comes in. We studied a closed algebra of observables and realised that this algebra can be transformed to be the algebra of $SO(1,2)$. This allows us to construct energy spectrum, and we found that the energy is generally nondegenerate. In the process we have restricted ourselves to the case $\theta\phi < 1$. This is justified by comparing with the result from string theory.

Actually the analysis up to this point is still not complete. The information from $SO(1,2)$ is not enough to determine energy spectrum. So we look at a bigger group $Sp(4)$, which is a homogeneous linear canonical transformation group that preserves Poisson bracket in four dimensional phase space. We realise that in order to study this group, we only need to find ladder operators. When we get the ladder operators, we can use them to construct energy spectrum. This justifies our result of energy spectrum discussed earlier. Additionally, when we study generators of $Sp(4)$ in matrix form, we see that $\underline{\underline{J'_2}}, \underline{\underline{Q'_2}}$, and $\underline{\underline{Q'_3}}$ do not depend on θ and ϕ. This might be related to some symmetries. Further investigations may be required.

Essentially, what we have done is the change of basis. We transform the phase space so that it looks like a usual phase space. The symplectic group $Sp(4)$ can then be constructed naturally. This group can then be transformed back into the totally noncommutative phase space to describe physics there.

Actually, we have only discussed the two dimensional isotropic SHO. The SHO with Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2)$$

will need to be studied in the future. After this, we still have to deal with arbitrary dimensional SHO before we can construct quantum field theory in totally noncommutative phase space. This is quite a long way to go along this path. However, we believe that for each system of SHO, there is a change of basis that allows us to construct separate ladder operators. Once we get some ideas about the general form of separate ladder operators we might be able to at least construct free quantum fields in totally noncommutative phase space.

The problem now is how to make a change of basis. Notice that the change of basis (7.119)-(7.120) reduces, when $\theta = \phi = 0$, to the canonical transformation discussed in [26] or near the end of subsection 7.2. Why does [26] define such a transformation when studying two dimensional SHO? Their motivation might be to simultaneously diagonalise Hamiltonian and a conserved quantity which in this case is angular momentum. So when we study a general SHO, we may first try to simultaneously diagonalise Hamiltonian and conserved quantities in usual phase space. This might lead us to canonical transformation. When we turn on noncommutativity, this canonical transformation might be modified to become the change of basis that transforms the totally noncommutative phase space into a usual phase space. Ladder operators can then be constructed, and physics can then be described.

We have discussed the possible future works in the direction of our ultimate goal. We may as well study some side tasks. In this report, we have constructed phase space quantisation for the case of totally noncommutative phase space. So we may be able to use it to describe other quantum systems for example hydrogen atom in totally noncommutative phase space. Additionally, we have discussed about the deformation of symplectic group $Sp(4)$. This group is not limited to two dimensional SHO. We may also use this group to study quantum optics in totally noncommutative phase space.

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Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

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Appendix A

Calculations of Ladder Operators in Section 7.4

Recall that in section 7.4, we need ladder operators to construct the 10 generators of $Sp(4)$ for general θ, ϕ. Let us see the calculations.

From section 7.3, we obtained

$$J_0' = \frac{1}{1-\theta\phi}\left[sJ_0 + \frac{u}{s}(J_2 + vQ_2)\right], \tag{A.1}$$

$$J_2' = \frac{1}{1-\theta\phi}\left[uJ_0 + J_2 + vQ_2\right]. \tag{A.2}$$

$$\tag{A.3}$$

Our goal is to solve equations (7.89)-(7.90). Here let us show the calculation for a_r'. The solution for a_l' can be obtained similarly.

Let us first observe that in the case $\theta = \phi = 0$, the annihilation operator is given by

$$a_r = \frac{1}{2}(x - iy + ip_x + p_y). \tag{A.4}$$

We require that when $\theta = \phi = 0$, a_r' should reduce to a_r, so we try $a_r' = z_1 x + z_2 y + z_3 p_x + z_4 p_y$ for some complex numbers z_1, z_2, z_3, z_4. Now consider

$$\begin{aligned} J_0' + J_2' - \frac{1}{2} &= \frac{s+u}{s(1-\theta\phi)}[sJ_0 + J_2 + vQ_2] - \frac{1}{2} \\ &= \frac{s+u}{s(1-\theta\phi)}\left[\frac{s-v}{4}(x^2+y^2) + \frac{s+v}{4}(p_x^2+p_y^2) + \frac{1}{2}(xp_y - yp_x)\right] - \frac{1}{2}. \end{aligned} \tag{A.5}$$

Also consider

$$a_r'^\dagger * a_r' = a_r'^\dagger a_r' + \frac{i}{2}\{a_r'^\dagger, a_r'\}. \tag{A.6}$$

The second term on the right-hand-side is just a number. So we require

$$\frac{i}{2}\{a_r'^\dagger, a_r'\} = -\frac{1}{2}. \tag{A.7}$$

We now write the equation (7.89) in terms of x, y, p_x, p_y and compare the coefficients.

Consider the coefficiencts of x^2 and y^2. We have

$$|z_1|^2 + |z_2|^2 = \frac{s+u}{s(1-\theta\phi)}\frac{s-v}{4} \equiv A_r^2, \tag{A.8}$$

where A_r is positive. So we have

$$z_1 = A_r e^{i\alpha_1}, \quad z_2 = A_r e^{i\alpha_2}, \tag{A.9}$$

for some real numbers α_1 and α_2. Consider the coefficients of xy :

$$\begin{aligned} 0 &= z_1^* z_2 + z_1 z_2^* \\ &= 2A_r^2 \cos(\alpha_1 - \alpha_2). \end{aligned} \tag{A.10}$$

So by requiring $\alpha_1 \to 0$ and $\alpha_2 \to -\pi/2$ when $\theta = \phi = 0$, we pick the solution

$$\alpha_1 - \alpha_2 = \frac{\pi}{2}. \tag{A.11}$$

Similarly, consider the coefficiencts of p_x^2 and p_y^2. We have

$$|z_3|^2 + |z_4|^2 = \frac{s+u}{s(1-\theta\phi)} \frac{s+v}{4} \equiv B_r^2, \tag{A.12}$$

where B_r is positive. So we have

$$z_3 = B_r e^{i\alpha_3}, \quad z_4 = B_r e^{i\alpha_4}, \tag{A.13}$$

for some real numbers α_3 and α_4. Consider the coefficients of $p_x p_y$:

$$\begin{aligned} 0 &= z_3^* z_4 + z_3 z_4^* \\ &= 2B_r^2 \cos(\alpha_3 - \alpha_4). \end{aligned} \tag{A.14}$$

By requiring $\alpha_3 \to \pi/2$ and $\alpha_4 \to 0$ when $\theta = \phi = 0$, we pick the solution

$$\alpha_3 - \alpha_4 = \frac{\pi}{2}. \tag{A.15}$$

Now consider the coefficients of xp_y :

$$\begin{aligned} \frac{1}{2}\frac{s+u}{s(1-\theta\phi)} &= z_1^* z_4 + z_1 z_4^* \\ &= 2A_r B_r \cos(\alpha_1 - \alpha_4) \\ &= \frac{1}{2}\frac{s+u}{s(1-\theta\phi)} \cos(\alpha_1 - \alpha_4). \end{aligned} \tag{A.16}$$

So

$$\alpha_1 - \alpha_4 = 0. \tag{A.17}$$

Consider the coefficients of yp_x :

$$\begin{aligned} -\frac{1}{2}\frac{s+u}{s(1-\theta\phi)} &= z_2^* z_3 + z_2 z_3^* \\ &= 2A_r B_r \cos(\alpha_2 - \alpha_3) \\ &= \frac{1}{2}\frac{s+u}{s(1-\theta\phi)} \cos(\alpha_2 - \alpha_3). \end{aligned} \tag{A.18}$$

We require that the solution must be consistent with equations (A.11), (A.15), and (A.17). So

$$\alpha_2 - \alpha_3 = -\pi. \tag{A.19}$$

So we have

$$a'_r = e^{i\alpha_1}(A_r x - iA_r y + iB_r p_x + B_r p_y). \tag{A.20}$$

Finally we have to check that the equation (A.7) is satisfied. This requirement does not restrict the choice of α_1. So we simply choose $\alpha_1 = 0$ to be consistent with the case $\theta = \phi = 0$. Now the equation (A.7) implies

$$A_r^2\theta + B_r^2\phi - 2A_r B_r = -\frac{1}{2}. \tag{A.21}$$

By noting that $\theta = u + v, \phi = u - v$, and $s^2 = 1 + v^2$, we can check that the equation (A.7) is indeed satisfied.

So we have finished deriving the ladder operators $a'_r, a'^{\dagger}_r$, and in the process, the commutation relation

$$\{\{a'_r, a'^{\dagger}_r\}\} = 1 \tag{A.22}$$

is proved.