M-Theory Solutions in Multiple Signatures from E_{11}

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(based on work[1] with Peter West)

M-Theory and E_{11}

M-theory is the logical culmination of the attempt to unify gravity with quantum theory using strings. M-theory itself is a set of dualities linking the five ten-dimensional supersymmetric string theories and its low energy limit is eleven-dimensional supergravity. Attempts to find non-perturbative results from string theory have relied heavily on maximal supergravity theories, which possess coset space symmetries determining the way scalars appear in these theories. Dimensional reduction on a torus to three, four and five dimensions, leads to scalars belonging to cosets of the exceptional groups E_8, E_7 and E_6[2]. The origin of these symmetry groups at lower dimensions appears mysterious, and motivates, in retrospect, the conjecture that a much larger symmetry exists hidden within eleven-dimensional supergravity that gives rise to the coset symmetries upon dimensional reduction. This large symmetry group is E_{11} and possesses an infinite-dimensional Kac-Moody algebra, whose action leaves the 11-dimensional theory invariant[3]. In other words, **E_{11} is a conjectured symmetry algebra of M-theory**.

Kac-Moody Algebras

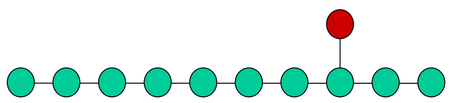
A Kac-Moody algebra is constructed from its Cartan matrix, A_{ab} and the Chevalley generators. A Cartan matrix A_{ab} is given in terms of the simple roots, α_a, of an algebra by

$$A_{ab} = 2\frac{(\alpha_a,\alpha_b)}{(\alpha_a,\alpha_a)}$$

Where (α_a, α_b) is the generalised Killing form acting on the corresponding elements of the Cartan sub-algebra. The diagonal elements of A_{ab} are all 2 and the off-diagonal elements are negative integers, with the zeroes being symmetrically distributed such that $A_{ab} = 0 \Leftrightarrow A_{ba} = 0$. The generators associated with positive simple roots are labelled E_a and those associated to negative simple roots F_a. A Kac-Moody algebra is then uniquely constructed from the commutators of E_a , F_a and the Cartan sub-algebra generators, H_a, subject to the following Serre relations:

$$[H_a, H_b]=0$$
$$[H_a, E_b]= A_{ab}E_b, \qquad [H_a, F_b]= -A_{ab} F_b$$
$$[E_a, F_b]= \delta_{ab}H_b$$
$$[E_a, \ldots [E_a, E_b]]=0, \qquad [F_a, \ldots [F_a, F_b]]=0$$

In the final line there are $(1-A_{ab})$ E_a's in the first equation and the same number of F_a's in the second equation. E_{11} is a Kac-Moody algebra whose Dynkin diagram is formed by adding three nodes to the E_8 Dynkin diagram giving,



The Dynkin diagram encodes the Cartan matrix. Each node corresponds to a simple root of the algebra and the number of lines joining node "a" to node "b" is A_{ab}. In the diagram above the line of green nodes, A_{10}, is called the gravity line. The generators of its Cartan sub-algebra are[4]:

$$H_a=K^a{}_a - K^{a+1}{}_{a+1} \text{ for } a = 1\ldots10,$$
$$H_{11}= -{}^1/_3(K^1{}_1+\ldots K^8{}_8) + {}^2/_3(K^9{}_9+\ldots K^{11}{}_{11})$$

The generators of the positive roots are:

$$E_a=K^a{}_{a+1} \text{ for } a = 1\ldots10 \text{ and } E_{11}=R^{91011}$$

Together these generators form the Borel sub-algebra.

The l_1 Representation of E_{11}

E_{11} may be studied by considering representations of the A_{10} sub-algebra (the gravity line). These are found[5] by first adding a node, which we indicate by a 0, to the Dynkin diagram of E_{11} attached to the very-extended node, the first node on the left of the gravity line, and then analysing the possible weights that remain upon deletion of both this additional node and the node off the gravity line of E_{11} (coloured red above), which we label by an 11. Because this decomposition involves the deletion of an additional node attached to the first root of the gravity line we say it gives rise to l_1 representations. These are representations of A_{10} which are categorised by the coefficients (n_0,n_{11}) of the deleted roots for a generic root, β:

$$\beta = \sum_{i=0}^{11} n_i\alpha_i$$

Due to the fact that roots of a Kac-Moody algebra constructed from a simply-laced Lie algebra must have length squared of two or less, and in addition must be even valued in this simply-laced case, there are only a finite number of possibilities for the remaining representations of A_{10} for each fixed level (n_0,n_{11})[5]. Up to n_{11}=3, the representations match the bosonic sector of the eleven-dimensional supergravity fields.

Low level A_{10} Representations of E_{11}

(n_0,n_{11})	β	β^2	Generator	Gauge Field
(0,1)	(0,0,0,0,0,0,0,0,0,0,1)	2	R^{abc}	A_{abc}
(0,2)	(0,0,0,0,0,1,2,3,2,1,2)	2	R^{abcdef}	A_{abcdef}
(0,3)	(0,0,0,1,2,3,4,5,3,1,3)	2	$R^{abcdefgh,j}$	$A_{abcdefgh,j}$
(0,3)	(0,0,1,2,3,4,5,6,4,2,3)	0	$R^{abcdefghi}$	$A_{abcdefghi}$
(1,0)	(0,0,0,0,0,0,0,0,0,0,0)	2	$K^a{}_b$	$h_a{}^b$

Algebraic Structure at Low Levels

Analysis of the low level root structure and the Serre relations give rise to the following commutators for the low level algebra[3]:

$$[K^a{}_b, K^c{}_d]= \delta^c{}_bK^a{}_d-\delta^c{}_bK^a{}_d$$
$$[K^a{}_b, R^{cde}]=\delta^c{}_bR^{ade}+\ldots$$
$$[K^a{}_b, R^{cdefgh}]=\delta^c{}_bR^{adefgh}+\ldots$$
$$[K^a{}_b, R^{cdefghij,k}]=(\delta^c{}_b R^{adefghij,k}+\ldots)+ \delta^k{}_b R^{cdefghij,a}$$
$$[R^{abc}, R^{def}]=2R^{abcdef}$$
$$[R^{abcdef}, R^{ghi}]=3R^{abcdef[ghi]}$$

Solution Generating Group Element

Representations of A_{10}, or SU(11), may be made directly applicable to eleven dimensional gravity theories. By first restricting to the real form of SU(11) we obtain SL(11,R), and then by including the eleventh generator from E_{11} (the one from the node that sticks out on the Dynkin diagram) this algebra is enlarged to GL(11,R). For the usual (1,10) signature of low energy M-theory or supergravity, the vielbein is an element of a coset of this group, namely of GL(11,R)/SO(1,10). Furthermore, the usual half-BPS solutions of supergravity (the M2-brane, M5-brane and the *pp*-wave) are encoded in a generic group element, g[4,6],

$$g = \exp(-\frac{1}{(\beta,\beta)}\ln N\beta\cdot H)\exp((1-N)E_\beta)$$

Where for any β arising in the decomposition to A_{10} representations above, E_β is its corresponding generator, and N is a harmonic function, whose exterior derivative is a field strength in a generic gravity action of the form. Specifically, R^{abc}, R^{abcdef} and $K^a{}_b$, give rise to the M2-brane, the M5-brane and the *pp*-wave solutions respectively[4]. $R^{abcdefgh,j}$ is the imperfectly understood dual to the pure gravity solution. The generator $R^{abcdefghi}$ has an associated root which is null and gives a singular group element, g, so is not treated by this approach.

As an example of a known solution encoded in the group element we consider the M2-brane. We take therefore β=(0,0,0,0,0,0,0,0,0,0,1) and $E_\beta=R^{91011}$, with associated gauge field A_{91011} and for historic reasons we consider the coordinate x^1 to be the single temporal coordinate. Such an approach motivates considering the lowest weight of the representation, which we achieve by taking a series of commutators with the appropriate $K^a{}_{a+1}=E_a$ the generators of the positive roots. We now work with β=(1,2,3,3,3,3,3,3,2,1,1) and $E_\beta=R^{123}$ and gauge field A_{123} giving,

$$(\beta, \beta)=2,$$
$$\beta.H=H_1+2H_2+3(H_3+\ldots H_8)+2H_9+H_{10}+H_{11}$$
$$= {}^2/_3\,(K^1{}_1+\ldots K^3{}_3)-{}^1/_3(K^4{}_4+\ldots K^{11}{}_{11})$$

The group element now has the form,

$$g = exp\left(-\frac{1}{2}lnN_2(\frac{2}{3}(K^1{}_1+K^2{}_2+K^3{}_3)-\frac{1}{3}(K^4{}_4+\ldots K^{11}{}_{11}))\right)exp\left((1-N_2)R^{123}\right)$$

The veilbein components may be read directly from the group element to give the line element of the M2-brane[7],

$$ds^2 = N_2^{-\frac{2}{3}}(-(dx_1)^2+(dx_2)^2+(dx_3)^2)+N_2^{\frac{1}{3}}((dx_4)^2+\ldots+(dx_{11})^2)$$

Which can be readily verified to be a solution to the equations of motion of a truncated form of the eleven-dimensional gravity action,

$$A = \frac{1}{16\pi G_{11}}\int d^{11}x\sqrt{-g}(R-\frac{1}{2.4!}F_{a_1\ldots a_n}F^{a_1\ldots a_n})$$

Where for the M2-brane n=4 and the field strength is given in terms of N_2 by,

$$F_{123i} = \pm\frac{\partial}{\partial x^i}N_2^{-1},\ i = 4,\ldots,11.$$

Weyl Reflections

The Weyl reflections of a Lie group are reflection symmetries of the associated root lattice. A Weyl reflection, S_α, in a root, α, reflects the root lattice in a hyperplane perpendicular to the root α, so that the Weyl reflection of an arbitrary root, β, is:

$$S_\alpha\beta \equiv \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$$

The Local Sub-algebra and Signature

The group element, g, was constructed using a nonlinear realisation of E_{11} as a coset symmetry E_{11}/H_{11}. Choosing the local denominator sub-algebra H_{11} to be the sub-algebra formed by generators invariant under the Cartan involution, Ω, leads to a theory with a Euclidean signature. Recall that $\Omega(E_\alpha)= -F_\alpha$ and $\Omega(F_\alpha)= -E_\alpha$, so that a Cartan involution invariant set of generators of the form $(E_\alpha - F_\alpha)$ may be chosen as a basis of H_{11}. To consider theories with a non-Euclidean local signature one may add a parameter ε_i which may be either +/–1 to the Cartan involution to form a "temporal involution" $\hat{\Omega}$, whose action on the local generators is,

$$\hat{\Omega}(K^i{}_{i+1}) = -\epsilon_i K^{i+1}{}_i$$

The new involution picks out a different local sub-algebra and ε_i encodes its signature. For example, $\varepsilon_i = +1$ is identical to the Cartan involution and gives a Euclidean signature, whereas taking $\varepsilon_1 = -1$ and the remaining $\varepsilon_i = +1$ gives a (1,10) theory where x^1 is the single temporal coordinate. However, it was observed[9] that the temporal involution operator does not commute with the Weyl reflections, with the consequence that a Weyl reflection picks out a new local sub-algebra, as is indicated by considering,

$$S_i\hat{\Omega}(K^j{}_{j+1}) = S_i(\epsilon_j K^j{}_{j+1}) = \epsilon_j\rho_j K^{j+1}{}_j \equiv \rho_j\hat{\Omega}'(K^j{}_{j+1})$$

Where $\rho_j = +/-1$ is a factor arising because $K^j{}_{j+1}$ are only representations of the Weyl group up to a sign. In short, the Weyl reflections may lead to a different choice of local sub-algebra and, potentially, a new signature. Signature diagrams[10] are a useful way to keep track of the signature changes induced by Weyl reflections. The information encoded in ε_i is transformed into a vector *f* in the weight space whose components, written $f(\alpha_i)$ are 1 or 0, and may be written in the form,

$$f = \sum_i n_i\lambda_i$$

Where λ_i are the fundamental weights and n_i may take the values 0 or 1, so that $f(\alpha_i)=(f, \alpha_i)$. If $f(\alpha_i) = 0$, the coordinates x^i and x^{i+1} are both of the same type (i.e. both spatial or both temporal) and if $f(\alpha_i) = 1$ one of the coordinates is spatial and the other temporal. If we write the values of this vector on top of their corresponding roots in the shape of the Dynkin diagram, we obtain the signature diagram. For example a (1,10) signature diagram may look like,

$$\begin{array}{c} 0 \\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \\ s\ s\ s\ s\ t\ s\ s\ s\ s\ s\ s \end{array}$$

The nature of all eleven space-time coordinates is decided from the ten values of *f* along the gravity line. The value of $f(\alpha_{11})$ gives the sign of the kinetic term in the gravity action, if 0 we have the usual $-F^2$ term and if it equals 1 we have a theory with a $+F^2$ term[9]. The effect of a Weyl reflection, S_i, on *f* is,

$$S_i(f) = \sum_j n_j\lambda_j - \sum_j n_j(\alpha_i,\lambda_j)\alpha_i$$
$$\equiv \sum_j m_j\lambda_j$$

Where, upon taking the inner product with α_j we find the relation between m_i and n_i to be given by,

$$m_k = n_k - n_iA_{ki}$$

In terms of the signature diagram, a Weyl reflection in the root α_i adds the value of $f(\alpha_i)$ to all the nodes it is connected to, modulo two[10].

Solutions in Multiple Signatures

The group element, g, relates a root of the decomposed A_{10} sub-algebra to a supergravity solution. It is possible to apply a series of E_{11} Weyl reflections that change the signature of the local sub-algebra but leave the root unaltered. One might suspect that the Weyl reflected group element also encodes a supergravity solution but in the new signature, and this is often the case. Consequently there exists an ambiguity as to which signature a solution exists in[1]. Let us briefly consider the example of the M2-brane. The only Weyl reflections that change the signature as well as the signature diagram are those in α_{11}. As it is always possible to transform the root, β, into a root that is invariant under S_{11} using a series of gravity line Weyl reflections which we call *U*, one may apply a signature-changing Weyl reflection to the root but without altering it: $\beta=U^{-1}S_{11}U\beta$. Discovering all the signatures that hold related solutions is best achieved using a computer program, and yields the following results[1] for the lowest weight of the R^{abc} generator (associated to the M2-brane):

Signature (temporal x^1)	Longitudinal Signature	Transverse Signature	Number of reflections in α_{11}	Trivially Related Signatures	κ
$(1,10,+)$	$(1,2)$	$(0,8)$	0	1	-1
$(10,1,+)$	$(2,1)$	$(8,0)$	0	2	$+1$
$(9,2,+)$	$(3,0)$	$(6,2)$	1	15	-1
$(9,2,-)$	$(3,0)$	$(6,2)$	1	13	$+1$
$(6,5,+)$	$(2,1)$	$(4,4)$	4	70	$+1$
$(6,5,-)$	$(2,1)$	$(4,4)$	4	70	-1
$(5,6,+)$	$(1,2)$	$(4,4)$	4	35	-1
$(5,6,-)$	$(1,2)$	$(4,4)$	4	35	$+1$
$(5,6,+)$	$(3,0)$	$(2,6)$	126	13	-1
$(5,6,-)$	$(3,0)$	$(2,6)$	126	15	$+1$
$(2,9,-)$	$(2,1)$	$(0,8)$	261	2	-1
$(9,2,-)$	$(1,2)$	$(8,0)$	261	1	$+1$
				272	

When $\kappa = -1$ these are solutions to the appropriate Einstein equations[1]. Interestingly, the solutions related by Weyl reflections are those of the M, M* and M'-theories[11].

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