# Classification of $c < 3$ Unitary Minimal $N = 2$ Superconformal Field Theories

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August 8, 2005

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## Abstract

The (bosonic) minimal unitary conformal field theories (CFTs) are known to form a discrete sequence [1, 7] and have been shown to be in one to one correspondence with the ADE sequences of simple algebras [2]. My aim is to investigate to what extent such a classification exists for the $N = (2, 2)$ supersymmetric minimal conformal field theories. Gannon [5] has constructed all the possible modular invariant partition functions for such theories. These far outnumber the ADE algebras but it is not yet clear whether each of the partition functions can be realised in a CFT; the 3-point functions have not yet been constructed. However, it is expected that each partition function does indeed give rise to a CFT, so the question arises of whether there is a classification in terms of certain singularities as there is in the non-super case.


## What is Conformal Field Theory?

### The Virasoro Algebra

The conformal transformations of the puntured complex plane are exactly the holomorphic and the anti-holomorphic functions. The infinitesimal conformal transformations of $\mathbb{C}^*$ form a Lie algebra which can be extended to the Witt algebra, $W$, given by the generators $\{l_n, \bar{l}_n \mid n \in \mathbb{Z}\}$ with the relations

$$\begin{aligned}[l_m, l_n] &= (n-m)l_{m+n}\\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n}\\ [l_m, \bar{l}_n] &= 0.\end{aligned}$$

Note that the Witt algebra can be decomposed into 'left' and 'right' handed parts: $W = W_l \oplus W_r$. We centrally extend the left Witt algebra $W_l$ to obtain the Virasoro algebra at central charge $c$, $Vir_c$, generated by $\{L_n \mid n \in \mathbb{Z}\}$ with the relations

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}\delta_{m+n,0}n(n^2-1).$$

The same can be done for the right Witt algebra to obtain an isomorphic algebra with central charge $\bar{c}$ generated by $\{\overline{L}_n \mid n \in \mathbb{Z}\}$. The full Virasoro algebra is the direct sum of the two commuting parts.

### Definition of a CFT

We present the main points in the definition of a CFT: A (2-dimensional, Euclidean) unitary CFT at central charges $c$, $\bar{c}$ is a pre-Hilbert space $\mathbb{H}$ with a real structure along with a system of n-point functions $\langle ... \rangle$ on $\mathbb{H}$ such that

1. $\mathbb{H}$ is a unitary representation of $Vir_c \oplus Vir_{\bar{c}}$;
2. $\mathbb{H} = \bigoplus \mathbb{H}_{h,\bar{h}}$ where $\mathbb{H}_{h,\bar{h}}$ are $L_0$,$\overline{L}_0$ eigenspaces, $h - \bar{h} \in \mathbb{Z}$, dim $\mathbb{H}_{h,\bar{h}} < \infty$ and $\mathbb{H}_{0,0} \cong \mathbb{C}$;
3. $\langle ... \rangle$ are Poincaré covariant and are a representation of an operator product expansion;
4. $\mathbb{H}$ has a modular invariant partition function;

We say a CFT is minimal if the direct sum in (2) is finite.

### n-point functions

An n-point function $\langle ... \rangle$ on a pre-Hilbert space $\mathbb{H}$ maps n states $\phi_i \in \mathbb{H}$ to a function in $z_1, ..., z_n$ which is real analytic except at points for which $z_i = z_j$ for some $i, j$. We say an n-point function is Poincaré covariant if for $\phi_i \in \mathbb{H}_{h_i,\bar{h}_i}$ and $f \in PSL_2(\mathbb{Z})$ we have

$$\langle \phi_1(f(z_1))...\phi_n(f(z_n))\rangle = \prod_{i=1}^{n}\left(\frac{\partial f}{\partial z}(z_i)\right)^{-h_i}\left(\frac{\partial \bar{f}}{\partial \bar{z}}(z_i)\right)^{-\bar{h}_i}\langle \phi_1(z_1)...\phi_n(z_n)\rangle$$

### Operator Product Expansions

The operator product expansion (OPE) is a map $\mathbb{H} \otimes \mathbb{H} \to \mathbb{H}\{z\}$ such that for $\phi_1, \phi_2 \in \mathbb{H}$

$$OPE(\phi_1 \otimes \phi_2) = \sum_{r,m} C_{r,m} z^{r+m}\bar{z}^r$$

where $C_{r,m} \in \mathbb{H}$ satisfy certain conditions. Once the 2- and 3-point functions are known, the OPE can be used to find all the remaining n-point functions of a theory.

### Partition Functions

We can assume that $\mathbb{H}_{h,\bar{h}} = \mathbb{H}_h \otimes \mathbb{H}_{\bar{h}}$ where $\mathbb{H}_h$,$\mathbb{H}_{\bar{h}}$ are $L_0, \overline{L}_0$ simultaneous eigenspaces. $\mathbb{H}_h$ has a character

$$\chi_h(\tau) = Tr_{\mathbb{H}}\left(q^{L_0 - \frac{c}{24}}\right), \, q = e^{2\pi i \tau}$$

and $\tau$ is in the upper half plane. For a minimal CFT these give rise to a partition function of the form

$$Z(\tau) = \sum_{h,\bar{h}} M_{h,\bar{h}}\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau})$$

where $M_{h,\bar{h}}$ is the multiplicity of $\mathbb{H}_{h,\bar{h}}$ in $\mathbb{H}$. We require these partition functions to be *modular invariant* which means that they are invariant under the action of the group $PSL_2(\mathbb{Z})$, generated by $S(\tau) = -\frac{1}{\tau}, T(\tau) = \tau + 1$.

## The Classification of $c < 1$ Minimal Unitary CFTs

It has been shown [4] that for a minimal unitary CFT with $c < 1$, $c$ and $h$ can only take values from a discrete list:

$$\begin{aligned} c &= 1 - \tfrac{6}{m(m+1)}, \qquad m = 2, 3, ...\\ h &= \tfrac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \; 1 \leqslant p \leqslant m-1, 1 \leqslant q \leqslant p.\end{aligned}$$

Moreover, each of these possibilities has been realised [7] via a coset construction, which we will now explain.

### The Coset Construction

Suppose we are given a simple, finite Lie algebra $\mathfrak{g}$ with generators $\{t^a\}$ and commutators

$$\left[t^a, t^b\right] = f^{abc}t^c.$$

We define the Kac-Moody (or affine) algebra, $\hat{\mathfrak{g}}_k$ associated to $\mathfrak{g}$ at integer level $k$ by the commutators

$$\left[T^a_n, T^b_m\right] = f^{abc}T^c_{n+m} + km\delta^{ab}\delta_{m+n,0}$$

for $n, m \in \mathbb{Z}$. It was shown in [9, 8] that the elements

$$\mathfrak{L}_n := \sum_{m\in\mathbb{Z}} \frac{1}{2k+g} \vdots \sum_{a=1}^{dim\mathfrak{g}} T^a_{m-n}T^a_{-m} \vdots, \quad n \in \mathbb{Z}$$

(where $g$ is the dual Coxeter number of $\mathfrak{g}$ and $\vdots ... \vdots$ represent normal ordering) form a representation of the Virasoro algebra at central charge

$$c_{\mathfrak{g}} = \frac{kdim(\mathfrak{g})}{k+g}.$$

This result shows that every simple, finite Lie algebra gives rise to a Virasoro algebra at some central charge $c \geqslant 1$. The result can be extended to semi-simple algebras.

In order to construct representations with $c < 1$, we consider a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. It can be shown that the operators $\mathfrak{L}_{\mathfrak{g}/\mathfrak{h}} := \mathfrak{L}_{\mathfrak{g}} - \mathfrak{L}_{\mathfrak{h}}$ generate a Virasoro algebra at central charge $c_{\mathfrak{g}} - c_{\mathfrak{h}}$.

If we consider the diagonal coset construction for

$$\frac{\widehat{su(2)}_k \oplus \widehat{su(2)}_1}{\widehat{su(2)}_{k+1}}$$

we find representations of the Virasoro algebra at central charges

$$c_k = 1 - \frac{6}{(k+2)(k+3)}, \quad k \geqslant 1;$$

i.e. (when $k = m + 2$) exactly the values of the central charges of the unitary minimal models. It is then possible to use known representation theory of $\widehat{su(2)}_k$ to deduce the existence of unitary minimal models for all of the values allowed by the Kac determinant formula.

The classification of the unitary minimal models was thus completed by the classification of the algebras $\widehat{su(2)}_k$ [2]. A surprising outcome was that the allowed partition functions are in a natural 1-1 correspondence with the ADE pattern of simply-laced simple algebras.

## The $N = 2$ Superconformal Algebra

The Virasoro algebra can be augmented to

$$\begin{aligned}\left[L_m, L_n\right] &= (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}\\ \left[L_m, G^i_n\right] &= \left(\frac{-m}{2} - n\right)G^i_{m+n}\\ \left[L_m, T_n\right] &= -nT_{n-m}\\ \left[T_m, T_n\right] &= \frac{c}{3}m\delta_{m+n,0}\\ \left[T_m, G^i_n\right] &= i\epsilon^{ij}G^j_{m+n}\\ \left\{G^i_m, G^j_n\right\} &= 2\delta^{ij}L_{m+n} + i\epsilon^{ij}(m-n)T_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta^{ij}\delta_{m+n,0}\end{aligned}$$

where $i = 1, 2$ and $\epsilon$ is totally anti-symmetric. This is the N=2 superconformal algebra (SCA). Analogous to the N=0 case, only a discrete set of $c < 3$ are allowed in a unitary minimal representation of the SCA [3]:

$$c = 3 - \frac{6}{m}, \, m \geqslant 2$$

and these have all been constructed in terms of the coset

$$\frac{\widehat{su(2)}_k \oplus \widehat{u(1)}_4}{\widehat{u(1)}_{2k+4}}.$$

Gannon [6] has shown how to reduce the problem of finding modular invariant partition functions for cosets to the problem of finding those for semi-simple algebras, in this case $\widehat{su(2)} \oplus u\widehat{(1) \oplus u}(1)$. Gannon then went on to classify all modular invariant partition functions of the form $\widehat{su(2)} \oplus \widehat{u(1)}^{\oplus m}$ [5] and hence completed the classification. But as Gannon writes, "the A-D-E classification of these ... turns out to be a very coarse-grained one: e.g. associated with the names $E_6$, $E_7$, $E_8$, respectively, are precisely 20, 30, 24 different partition functions."

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