HOMOTOPY ALGEBRAS AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. We provide a summary of some results and methodology of our recent paper [4]. In particular we advocate working with a geometrical definition of an infinity-algebra. We mention how it is possible to treat the associated cohomology theories in this framework and describe a remarkable result linking symplectic and nonsymplectic C_∞-structures. Applications to string topology are subsequently discussed.

Homotopy algebras, or more accurately *strong* homotopy algebras, were originally introduced by Stasheff in [9] where they were used to study group like structures on a topological space which were associative only up to homotopy. Here he introduced the term 'A_∞-algebra' to describe such structures satisfying an infinite sequence of higher homotopy associativity conditions; hence an A_∞-algebra may be regarded as the homotopy invariant notion of an associative algebra. The homotopy invariant notions of a commutative and Lie algebra, called C_∞ and L_∞-algebras respectively, subsequently appeared and were used extensively in rational homotopy theory, cf. [10]. Relatively recently they have found applications in mathematical physics; a prime example of this is Kontsevich's interpretation of mirror symmetry as an equivalence between A_∞-categories [7].

We wish to advocate working with a more geometrical definition of an infinity-algebra. In this context an infinity-structure is described by a homological vector field on a certain formal supermanifold. This approach is not new, cf. [1] and [8], but we feel that it has been rather undersubscribed. In certain situations this notion is indispensable.

Definition. Given a vector space V, an A_∞-structure on V is a *continuous* derivation

$$m : \widehat{T}\Sigma V^* \to \widehat{T}\Sigma V^*$$

of degree one and vanishing at zero, such that $m^2 = 0$. Here Σ denotes the suspension, $*$ denotes the dual and $\widehat{T}$ denotes the completed tensor algebra. There are similar definitions of a C_∞ and L_∞-structure where $\widehat{T}\Sigma V^*$ is replaced with $\widehat{L}\Sigma V^*$ and $\widehat{S}\Sigma V^*$.

In [6] Kontsevich introduced the notion of an 'infinity-algebra *with an invariant inner product*'. In the C_∞ case this is a higher homotopy generalisation of a commutative Frobenius algebra. He showed that infinity-algebras with invariant inner products have a close relationship with graph homology and therefore with the intersection theory on the moduli spaces of complex algebraic curves and invariants of differentiable manifolds. The advantage of the above definition is that it is possible to describe such an infinity-structure as a *symplectic* vector field on a certain formal *symplectic* supermanifold.

There is a noncommutative version of differential geometry developed by Connes [3] with further crucial input by Kontsevich [5], which allows one to define a complex $DR^{\bullet}(A)$ for any associative algebra A. This complex is the noncommutative analogue of the de Rham complex. A symplectic form is then defined to be a nondegenerate closed 2-form $\omega \in DR^2(A)$. We can now give the definition of a symplectic infinity-algebra:

Definition. Given a vector space V, a *symplectic* A_∞-structure on V is a pair (m, ω) such that:

(1) $\omega \in DR^2(\widehat{T}\Sigma V^*)$ is a symplectic form.
(2) $m : \widehat{T}\Sigma V^* \to \widehat{T}\Sigma V^*$ is an A_∞-structure.
(3) m is a *symplectic* vector field, i.e. $L_m\omega = 0$.

It turns out that the notion of a symplectic infinity-algebra is equivalent to the notion of an infinity-algebra with an invariant inner product. A proof of this fact is contained in [4] although it seems to have previously been known to a few experts in the field, e.g. Kontsevich.

Another advantage of working with the above definitions of an infinity-algebra is that we can engage the apparatus of noncommutative differential geometry in order to define their associated cohomology theories. These cohomology theories can be interpreted as spaces of formal noncommutative differential forms where the differential is provided by the Lie derivative of the homological vector field defining the infinity-structure. This definition is very practical, for instance we were able to construct the Hodge decomposition of Hochschild cohomology for an arbitrary C_∞-algebra; our geometrical approach allowed us to considerably simplify the combinatorics of previous authors.

In [4] we used our description of these cohomology theories to develop an obstruction theory for C_∞-structures. Using our main tool, the Hodge decomposition, this enabled us to prove a remarkable result describing the similarity between C_∞-structures and *symplectic* C_∞-structures:

Theorem. *Any unital C_∞-algebra whose homology algebra can be given the structure of a Frobenius algebra has a minimal model which has the structure of a* symplectic *C_∞-algebra.*

This theorem is very powerful and as an application we were able to construct 'string topology' operations on the ordinary and equivariant homology of the loop space of a formal Poincaré duality space X in a homotopy invariant way by interpreting these structures as the natural Gerstenhaber algebra structures on the Hochschild cohomology of the singular cochain algebra of X; such operations were originally introduced by Chas and Sullivan in their influential paper [2].

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