OSCILLATION OF EIGENFUNCTIONS OF FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

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Durham, August 2005.
1- We begin by quoting:

[Ince, 1926] (referring to Sturm’s oscillation theorem)
“It is important from the point of view of physical applications, and not without theoretical interest, to determine the number of zeros which the solution has in the interval \((a,b)\).”
2- Consider the regular Sturm-Liouville problem:

spectral parameter

\[-(py')' + (q - E)y = 0\]

\[\sin(\alpha)y(0) - p(0)\cos(\alpha)y'(0) = 0\]

\[\sin(\beta)y(1) - p(1)\cos(\beta)y'(1) = 0\]

\[p', q \text{ continuous (or } L^\infty), p > 0, q \text{ real-valued.}\]

**Sturm’s Theorem: [1826]** (*) has an infinite set of eigenvalues all real, simple and accumulating only at \(E = +\infty\). If we arrange the eigenvalues in increasing order \(E_0 < E_1 < \ldots\) and \(\{y_n\}\) are the corresponding eigenfunctions, then \(y_n\) has exactly \(n\) zeros in \((0, 1)\).
3- The fourth-order regular Sturm-Liouville e.v.p.

Let the eigenvalue problem:

\[(EQ) \quad (py''')'' + (q - E)y = 0,\]

where \(p'', q\) are continuous, \(p > 0\) and \(q\) is real-valued.

Let the separated b.c. given in general form:

\[(BC)_{MN} \quad M\overline{y}_0 = 0 = N\overline{y}_1,\]

where \(M, N \in \mathbb{R}^{2 \times 4}, \quad \overline{y}_a := [(py'')', py'', y', y]^t |_{x=a} .\)

Below we always assume that \(M\) and \(N\) are of maximal range and are given in reduced row form.
\[(EQ) \quad (py'')'' + (q - E)y = 0\]
\[(BC)'_{MN} \quad M\overline{y}_0 = 0 = N\overline{y}_1\]

Can we recover oscillation results in the spirit of Sturm’s Theorem when conditions on the coefficients of \(M\) and \(N\) ensure self-adjointness?

[Atkinson, 1964]: conjugate points...

But what about the counting of zeros?
4- A very simple example. The equation \( y^{(4)} - E y = 0 \) with

\[
M = N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

In this case \( E_n = (n \pi)^4 \) and \( y_n(x) = \sin(n \pi x) \). Notice that if \( \mathcal{L}y := -y'' \) with Dirichlet b.c., then the above e.v.p. is the one associated to \( \mathcal{L}^2 \).
5- Background.

[S. Janczewsky, 1928]: Sturm’s Theorem is completely recovered for a special class of $M$ and $N$, and sufficiently large eigenvalues. This class includes

$$M = N = D := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$


Universal estimates for general self-adjoint b.c.?
6- Weak formulation, self-adjointness.

\[ a(u, v) := \int_0^1 pu''v'' \, dx, \quad c(u, v) := \int_0^1 quv \, dx, \]

and

\[ b(u, v) := [(pu'')'v - pu''v'] \bigg|_{x=0}^{x=1}. \]

Denote by \( \tilde{M}, \tilde{N} \) the reduced boundary conditions. Let

\[ \tilde{\mathcal{D}} := \{ y \in H^2(0, 1) : \tilde{M} \mathbf{y}_0 = 0 = \tilde{N} \mathbf{y}_1 \} \quad \text{and} \quad \mathcal{D} := \{ y \in H^4(0, 1) : \mathcal{M} \mathbf{y}_0 = 0 = \mathcal{N} \mathbf{y}_1 \}. \]

\[
\begin{bmatrix}
  r_{11} & r_{14} \\
  r_{21} & r_{24}
\end{bmatrix} \begin{bmatrix}
  r_{12} & r_{13} \\
  r_{22} & r_{23}
\end{bmatrix} = \quad \text{for } [r_{ij}] = M \text{ and } N.
\]

The form \( a + c + b \) is symmetric, bounded below and closable in the domain \( \mathcal{D} \subset L^2(0, 1) \). Its closure has domain \( \tilde{\mathcal{D}} \).

Notice that (s) is necessary and sufficient for the first statement to hold true.
Interlacing properties of the eigenvalues.

\((EQ) - (BC)_{MN}\) has an infinite set of eigenvalues, they are of multiplicity no greater than two and satisfying

\[ E_n = \alpha_p n^4 + O(n^3) \quad \text{as} \quad n \to \infty, \quad \alpha_p := \left[ \frac{2\pi}{\int_0^1 p^{-1/4} \, dx} \right]^4 \]

Let \(\mu_n\) be the eigenvalues of the “Dirichlet” \((EQ) - (BC)_{DD}\).

**Lemma 1.** Let \(j_{(MN)}\) be the total of vanishing rows in the matrices \(\tilde{M}\) and \(\tilde{N}\) \((= 0, 1, 2, 3, 4)\). Then

\[ \mu_n - j_{(MN)} \leq E_n \leq \mu_n. \]

\[ \uparrow \quad \uparrow \]

\[ n \geq 4 \quad n \geq 0 \]
Proof. $E_n \leq \mu_n$.

$$E_n = \min_S \max_{u \in S, \int |u|^2 = 1} (a + c + \tilde{b})(u, u),$$

min over $S \subset \tilde{D}_{MN}$ of dim $= n$.

$$\mu_n = \min_{S_d} \max_{u \in S_d, \int |u|^2 = 1} (a + c)(u, u)$$

min over $S_d \subset \tilde{D}_{DD}$ of dim $= n$.

Notice $\tilde{D}_{DD} \subseteq \tilde{D}_{MN}$. 
Proof. $\mu_{n-j} \leq E_n$.

$j =$ total of rows vanishing in $(\tilde{M}, \tilde{N})^t$.

$$E_n = \max_S \min_{u \in S, \|u\|^2 = 1} (a + c + \tilde{b})(u, u),$$

max over $S \subset \tilde{D}_{MN}$ of $\dim(\tilde{D}_{MN}/S) \leq n$.

$$\mu_{n-j} = \max_{S_d} \min_{u \in S_d, \|u\|^2 = 1} (a + c)(u, u)$$

max over $\dim(\tilde{D}_{DD}/S_d) \leq n - j$.

Since $\dim(\tilde{D}_{MN}/\tilde{D}_{DD}) = j$, then $\dim(\tilde{D}_{MN}/S_d) \leq n$. 
8- Upper bound on the number of zeros.

\[#[f] := \text{number of vanishing pts. of } f \text{ in } (0,1) \text{ count. mult.}\]

**Theorem 2.** Let \( n \geq 4 \). If \( E_n \) is such that \( E_n > q(x) \) for \( 0 \leq x \leq 1 \), then \( #[y_n] \leq n + c_{(MN)} \). Where \( c_{(MN)} \leq 12 \) is a constant.

Sharp? Probably not: if \( p, q \) are constant, \( c_{(MN)} \leq 3 \).

Lemma 1 implies that there is at most 4 eigenvalues below the minimum of \( q(x) \).
Sketch of the proof:

- If $(w_n, \mu_n) \in \mathcal{D} \times \mathbb{R}$ is eigenpair of $(EQ) - (BC)_{DD}$, 
  $\#[w_n] = n$.

- Any other solution of $(EQ)$ with $E = \mu_n$ has at most 
  $\#[w_n] + 3$ zeros in $(0, 1)$.

- Lemma 1.

- Comparison result of [Leighton, Nehari]: the solutions 
  $u, v$ of $(ru'')''' - Q_1u = 0$ and $(rv'')''' - Q_2v = 0$ for 
  $0 < Q_1(x) \leq Q_2(x)$ subject to $u(0) = v(0) = u(1) = 
  v(1) = 0$ satisfy $\#[u] \leq \#[v] + 3$. 
9- Lower bounds?

There is no lower bound for large positive e.v.!

The problem $y^{(4)} - Ey = 0$ with b.c. given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & b_2 & 1 \end{pmatrix},$$

$$b_1 = \frac{u'(1) - u'''(1)}{u(1)}, \quad b_2 = -\frac{u''(1) + u(1)}{u'(1)},$$

$$u(x) = \cosh(\beta\pi x/2) - \alpha \cos(\beta\pi x/2) \quad \text{for} \quad \alpha = \cosh(2k\pi).$$

$E = (\beta\pi/2)^4$ is e.v. with e.f. $u(x)$ and $\#[u] = 2k + 1$.

Fix $k = 0, 1, 2, \ldots$, choose $\beta$ arbitrarily large and recall Lemma 1.
10- Negative eigenvalues.

$(EQ) - (BC)_{MN}$ can have negative eigenvalues with arbitrarily large modulus. There is at most 4 of them. As $|E|$ increases, the oscillation count increases.

We speculate that the correct asymptotic is

$$
\#[y] = (\alpha_p)^{-1/4}|E|^{1/4} + o(|E|^{1/4}), \quad E \to \infty.
$$

for the eigensolution $(y, E)$.

This is certainly the case for constant coefficients.
11- Explanation?

\[ E_0 \leq \ldots \leq E_3 \leq E_4 \leq \ldots \]

Spectral theorem?