Interpolation on Semigroupoid Algebras

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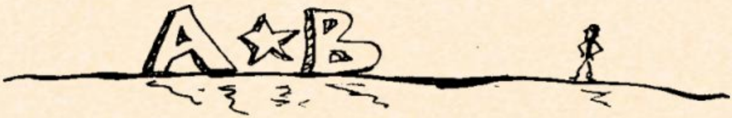

The classical Nevanlinna-Pick interpolation problem

Problem: Given n points $z_1, \ldots z_n$ in $\mathbb{D}$, and n complex values $w_1, \ldots, w_n$, does there exist a analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_k) = w_k$, $k = 1, \ldots, n$?

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$H^\infty(\mathbb{D})$ is the *multiplier algebra* for $H^2(\mathbb{D})$, the analytic functions on the disk with square summable power series. That is, for all $f \in H^2(\mathbb{D})$ and for any $\varphi \in H^\infty(\mathbb{D})$, the function with values $M_\varphi f(z) = \varphi(z)f(z)$ is in $H^2(\mathbb{D})$.

Solution to the classical Nevanlinna-Pick interpolation problem

A solution exists $\Longleftrightarrow$ the matrix

$$\left(\frac{1-w_j w_k^*}{1-z_j z_k^*}\right)_{j,k=1,\ldots n}$$

is positive (ie, positive semidefinite).

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We call

$$\mathbf{k}(w, z) = (1 - zw^*)^{-1}$$

the Szegő kernel. Write $k : \mathbb{D} \to \mathbb{C}$ as $k(z)(\cdot) = \mathbf{k}(\cdot, z)$.

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A straightforward calculation shows $M_\varphi^* k(z) = \varphi(z)^* k(z)$,
(so in particular, if $\varphi(z_k) = w_k$ then $M_\varphi^* k(z) = w_k^* k(z)$).

Solution to the classical Nevanlinna-Pick interpolation problem, cont.

So the N-P problem has a solution iff

$I - M_\varphi M_\varphi^* \geq 0$ on $\mathcal{M} = \operatorname{span}\{k(z_1), \ldots, k(z_n)\}$,

in which case φ can be extended to all of $H^2(\mathbb{D})$ without increasing the norm.

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Recall that if A, B are matrices, the *Schur product* of A and B (write $A \star B$) is the entrywise product.

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We can rewrite the condition for the existence of a solution of the Pick problem as

$$([1] - \varphi\varphi^*) \star \mathbf{k} \geq 0,$$

where $[1]$ is the matrix which has all entries equal to 1.

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Let $\delta(z) = 1$, $Z(z) = z$. Then the positivity condition can be rewritten as

$$\begin{aligned}
([1] - \varphi\varphi^*) \star kk^* = \gamma\gamma^* \quad &\iff \\
\delta\delta^* - \varphi\varphi^* = \gamma\gamma^* \star ([1] - ZZ^*) \quad &\iff \\
Z\gamma\gamma^* Z^* + \delta\delta^* = \gamma\gamma^* + \varphi\varphi^* &
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Then there is an isometry V such that

$$V \begin{pmatrix} Z(z)\gamma(z) \\ \delta(z) \end{pmatrix} = \begin{pmatrix} \gamma(z) \\ \varphi(z) \end{pmatrix}$$

Solution to the classical Nevanlinna-Pick interpolation problem, cont.

Let $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then we have

$$Az\gamma(z) + B = \gamma(z)$$
$$Cz\gamma(z) + D = \varphi(z)$$

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Solve for γ in the first equation:

$$\gamma(z) = (1 - Az)^{-1}B.$$

Plug into the second equation:

$$\varphi(z) = D + zC(1 - Az)^{-1}B.$$

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We refer to this as a *transfer function representation* for φ.

Solution to the classical Nevanlinna-Pick interpolation problem, cont.

The fact that M_φ^* can be extended from $\mathcal{M}$ to the whole space without increasing the norm is referred to as the *Pick property* (for $H^2(\mathbb{D})$). Spaces with this property are rather special.

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For example, there is no Hilbert space of functions with the Pick property having $H^\infty(\mathbb{D}^2)$ as the space of multipliers. So how do we solve interpolation problems in the function algebra $H^\infty(\mathbb{D}^2)$?

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The approach we use, due to Jim Agler, is to look at all of the spaces (and their kernels) having $H^\infty(\mathbb{D}^2)$ as the multiplier algebra, and require that for all such kernels, $([1] - \varphi\varphi^*) \star \mathbf{k} \geq 0$.

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Semigroupoids—Definition and basic properties

Let G be a set with a function from $X \to G$, where $X \subset G \times G$, called a *partial multiplication* and written xy for $(x, y) \in X$.

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The following laws are assumed to hold:

1. (associative law) If either $(ab)c$ or $a(bc)$ is defined, then so is the other and they are equal. If ab, bc are defined, then so is $(ab)c$.
2. (existence of idempotents) For all $a \in G$, there exist $e, f \in G$ with $ea = a = af$. Also, if $e^2 = e$, then e is idempotent.
3. (nonexistence of inverses) If $a, b \in G$ and $ab = e$ where e is idempotent, then $a = b = e$.
4. (strong artinian law) For all $a \in G$, the set $\{z, b, w : zbw = a\}$ is finite, $+ \ldots$.

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No commutativity or cancellation required!

Semigroupoids—Order

Define a partial order on G as follows:

$b \leq a$ if there exist $z, w \in G$ such that $a = zbw$

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A set $F \subset G$ is *lower* if $a \in F$ and $b \leq a$ then $b \in F$.

Convolution products

The formal "power series" on a lower set F (ie, functions $f, g : F \to \mathbb{C}$) form a complex vector space $\mathcal{P}(F)$ indexed by F with pointwise addition.

Since G is artinian there is a well-defined product given by

$$(f \star g)(a) = \sum_{rs=a} f(r)g(s) \in \mathcal{P}(F).$$

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Multiplicative unit:

$$\delta(x) = \begin{cases} 1 & x \in F_e, \\ 0 & \text{otherwise.} \end{cases}$$

A function f is invertible if and only if $f(x)$ is invertible for all $x \in F_e$.

Convolution products, cont.

We also introduce the reverse product

$$(f \mathbin{\hat{\star}} g)(a) = \sum_{rs=a} f(s)g(r).$$

The multiplicative unit remains δ and the invertibility condition is the same.

It is unimportant that the functions map into $\mathbb{C}$.

A bivariate $\star$ product—definition

Michael Jury defines a generalization of the Schur product which is a useful tool for interpolation problems. The equivalent in our setting is the following, which can be viewed as a bivariate version of the convolution product.

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The assumption that the entries of A and B are in $\mathbb{C}$ is not important, and we will at times use the $\star$ and $\hat{\star}$ product when the entries are in other algebras.

The bivariate $\star$ product—properties

- Associative: $C \star (A \star B) = (C \star A) \star B$.
- *Not necessarily commutative!*
- $A, B \geq 0 \Longrightarrow A \star B \geq 0$.
- $[1] = \delta\delta^*$ (that is, $[1]_{a,b}$ is 1 for a, b both elements of F_e and zero otherwise) is the multiplicative unit.
- A invertible $\Longleftrightarrow A_{ab}$ is invertible for all $a, b \in F_e$. The inverse is unique.
- *The inverse of a positive matrix need not be positive!*
- Inverses of selfadjoint elements are selfadjoint.
- $(A \star B)^* = A^* \star B^*$.
- Equivalent statements apply to $A \hat{\star} B$.

Toeplitz representations

Let φ be a function on a (finite) lower set F. Define the associated Toeplitz representation $\mathfrak{T}$ by

$$(\mathfrak{T}(\varphi))_{a,b} = \begin{cases} \sum_c \varphi(c), & cb = a; \\ 0 & \text{otherwise.} \end{cases}$$

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In the case that G is the semigroupoid $\mathbb{N}$, $\mathfrak{T}(\varphi)$ is precisely the Toeplitz matrix associated with the sequence $\{\varphi(j)\}$.

At the other extreme, when $G = G_e$, $\mathfrak{T}(\varphi)$ is simply the diagonal matrix with diagonal entries $\varphi(a)$ for $a \in G$ which, despite our terminology, is very un-Toeplitz like!

Generalized Szegő kernels

Theorem

Let $A \in M(F)$ *be positive, and suppose* $\|A \star 1\| < 1$*. Then* $[1] - A$ *is invertible (with respect to the* $\star$ *product) and* $([1] - A)^{-1} \geq 0$*.*

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Set $A_{a,b} = \varphi(a)\varphi(b)^*$. Then $\|A \star 1\| < 1$. The result then follows from the last theorem.

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Example

Take $G = G_e = \mathbb{D}$, $\varphi(z) = z$, then $([1] - \varphi\varphi^*)^{-1}$ is the Szegő kernel.

Interpolation problem

Let G be semigroupoid, $\mathcal{A}$ a normed algebra of functions on G.

Let F be a finite lower set, $\xi : F \to \mathbb{C}$ given.

Does there exists a $\varphi \in \mathcal{A}$ with $\|\varphi\|_{\mathcal{A}} \leq 1$ and $\varphi|F = \xi$?

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Ideally, we want to not only characterize when a solution exists, but also explicitly give the solution.

Examples

- If $G = G_e = \mathbb{D}$, F a finite subset, $\mathcal{A} = H^\infty(\mathbb{D})$, this is the classical Nevanlinna-Pick interpolation problem.
- More generally, we could take $G = G_e = R \subset \mathbb{C}^n$, again F a finite subset, $\mathcal{A} = H^\infty(R)$. The case R a polydisk was done by Agler. Other generalised Cartan domains by Ambrozie, Ball, Timotin and others.
- We don't need R simply connected. For example $R \subset \mathbb{C}$ an annulus was considered by Abrahamse.
- Let $G = \mathbb{N}$, $G_e = \{0\}$, the $\star$ product given by addition. Let $F = \{0, \ldots, n\}$, a lower set. In this case we view $\xi(k)$ as the k^{th} Taylor coefficient of a function expanded about 0. We then have the Carathéodory-Fejér interpolation problem.
- G is a free semigroup on d letters, G_e contains only the empty word, the $\star$ product is concatenation. We can take G to be commutative or noncommutative. The latter case is the sort of generalization of Carathéodory-Fejér interpolation considered by Popescu and others.

More Examples

- More generally, it is possible to consider mixtures of problems from the last slide.
- There are also lots of exotic examples!
- In the above, the semigroupoids were rather tame. For these, if a is not an idempotent and $eaf = a$, then $f = e$. Also, there is cancellation, which is not necessary.

Reproducing kernel Hilbert spaces

We say that a function $\mathbf{k} : G \times G \to \mathbb{C}$ is a *positive kernel* on G if for any finite subset A of G, the matrix $(\mathbf{k}(a, b))_{a,b \in A}$ is positive semidefinite.

Define $k : G \to \mathbb{C}$ as $k(b) = \mathbf{k}(\cdot, b)$, $b \in G$.

In the usual way we form a sesquilinear form $\langle \cdot, \cdot \rangle$ with $\langle k(b), k(a) \rangle = \mathbf{k}(a, b)$, mod out by the kernel, complete to a Hilbert space $\mathcal{H}(\mathbf{k})$.

On $\mathcal{H}(\mathbf{k})$ addition is defined termwise.

Reproducing kernels—the multiplier algebra for a single kernel

Define the *multiplier algebra* $H^\infty(\mathbf{k})$ as the collection of operators $\mathfrak{T}(\varphi) : f \mapsto \varphi \star f$ for functions $\varphi : G \to \mathbb{C}$ satisfying $\varphi \star f \in \mathcal{H}(\mathbf{k})$ for each $f \in \mathcal{H}(\mathbf{k})$.

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The closed graph theorem implies that the elements of $H^\infty(\mathbf{k})$ are bounded.

For $f \in \mathcal{H}(\mathbf{k})$,

$$\langle \mathfrak{T}(\varphi) f, k(a) \rangle = \left\langle f, \sum_{bc=a} \varphi(b)^* k(c) \right\rangle.$$

So $\mathfrak{T}(\varphi)^* k(a) = \sum_{bc=a} \varphi(b)^* k(c)$;
ie, $\quad \mathfrak{T}(\varphi)^* k(a) = (\varphi^* \star k)(a)$.

The multiplier algebra, cont.

For a lower set F, if we set $\mathcal{M}_F$ to the closed linear span of kernel functions $k(a)$, $a \in F$, then the usual sort of argument gives $\mathcal{M}_F$ invariant for adjoints of multipliers $\mathfrak{T}(\varphi)^*$.

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The $\star$-product is useful in characterising multipliers.

$\|\mathfrak{T}(\varphi)^*|\mathcal{M}_F\| \leq 1 \iff$

$$
\begin{aligned}
&(\langle(1-\mathfrak{T}(\varphi)\mathfrak{T}(\varphi)^*)k(a),k(b)\rangle)\\
&=\left(\sum_{pq=a}\sum_{sr=b}([1]_{pr}-\varphi(p)\varphi(r)^*)\mathbf{k}(q,s)\right)\\
&=([1]-\varphi\varphi^*)\star\mathbf{k}\geq 0
\end{aligned}
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Test functions and families of reproducing kernels

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The *family of reproducing kernels associated to* Ψ is $\mathcal{K}_\Psi = \{\mathbf{k}\}$, where

$$([1] - \psi\psi^*) \star \mathbf{k} \geq 0$$

for all $\psi \in \Psi$ and $\mathbf{k} \in \mathcal{K}_\Psi$. Our defintion of a semigroupoid

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Define the *multiplier algebra* $H^\infty(\mathcal{K})$ as the intersection of all $\bigcap_{\mathbf{k}\in\mathcal{K}} H^\infty(\mathbf{k})$, with norm of an element the infimum of its norm over all $H^\infty(\mathbf{k})$.

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If $G = G_e = \mathbb{D}$, $\Psi = \{z\}$, then the family of kernels consists of kernels of the form $\gamma \star k \,\hat{\star}\, \gamma^*$, where $\mathbf{k}(x, y) = (1 - xy^*)^{-1}$ (the Szegő kernel).

The evaluation map

Let $C(\Psi)$ be the continuous functions on Ψ, the collection of test functions.

Define $E \in B(G, C(\Psi))$ by

$$E(x)(\psi) = \psi(x), \qquad \psi \in \Psi,$$

and

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- $E(x)$ is the evaluation map on Ψ.
- $\|E(x)\| < 1$ for each $x \in G_e$ and $\|E(x)\| \leq 1$ otherwise.
- The collection $\{E(x) : x \in G\}$ separates points, so the smallest unital C^*-algebra containing all the $E(x)$ is $C(\Psi)$.

Colligations and transfer functions

Let E be an evaluation map,

$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ unitary on $\mathcal{E} \oplus \mathbb{C}$, $\mathcal{E}$ a Hilbert space

$\rho : \mathfrak{B} \to B(\mathcal{E})$ a unital $*$-representation.

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Write $\Sigma = (U, \mathcal{E}, \rho)$ (called a *colligation*).

Define the *transfer function* by

$$W_\Sigma(x) = \big(D\delta + C\rho(E) \star (\delta - A\rho(E))^{-1} \star (B\delta)\big)\,(x).$$

The Main Result

Theorem (**Realization**)

Suppose Ψ is a collection of test functions over a semigroupoid G, with associated family of kernels $\mathcal{K}$. Further, assume $\|T_E\| < 1$.
The following are equivalent,

(i) $\varphi \in H^\infty(\mathcal{K})$ *and* $\|\varphi\|_{H^\infty(\mathcal{K})} \leq 1$;

(iiF) *for each finite lower set $F \subset G$ there exists a positive kernel $\Gamma : F \times F \to (C(\Psi))^*$ so that for all $x, y \in F$*

$$([1] - \varphi\varphi^*)(x,y) = (\Gamma \hat{\star} ([1] - EE^*))(x,y);$$

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(iii) *there is a colligation Σ so that $\varphi = W_\Sigma$.*

How the main result is proved

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(i) $\Longrightarrow$ (ii): Hahn-Banach separation argument.

Agler-Ambrozie-Jury interpolation

Let F be a finite lower set, $\xi : F \to \mathbb{C}$ given.

Then there exists a $\varphi \in H^{\infty}(\mathcal{K})$ with $\|\varphi\|_{H^{\infty}(\mathcal{K})} \leq 1$ and $\varphi|F = \xi$ $\iff$ for each $k \in \mathcal{K}_{\Psi}$, the kernel

$$F \times F \ni (x, y) \mapsto (([1] - \phi\phi^*) \star k)(x, y)$$

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Moreover, in this case there is a transfer function representation for the solution.

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There is a similar result corresponding to left/right tangential interpolation (eg, solving $(\varphi \star z)(a) = w(a)$ for all a in a finite lower set F).

More on the proof of the realization theorem

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By contradiction:

Define the cone

$$\mathcal{C}_F = \{\big(\Gamma \hat{\star}([1] - EE^*)\big)_{x,y \in F} : \Gamma \in M(F, \mathfrak{B}^*)^+\},$$

and assume that

$$M_\varphi = \big(([1] - \varphi\varphi^*)(x,y)\big)_{x,y \in F} \notin \mathcal{C}_F.$$

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$\|\mu(\psi)\| \leq 1$ for test functions. A cyclic representation with this property comes from a reproducing kernel on F which extends to a reproducing kernel in $k \in \mathcal{K}_\Psi$.

Since $\|\mu(\varphi)\| > 1$, $([1] - \varphi\varphi^*) \,\hat{\star}\, k \not\geq 0$.

Applications

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`http://front.math.ucdavis.edu/math.FA/0507083`

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