The spectral function and the remainder in local Weyl’s law:

View from below

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\( X^n, \ n \geq 2 \) — compact manifold

\( g_{ij} \) — Riemannian metric

\( \Delta \) — Laplace operator

\( \Delta \phi_i = \lambda_i \phi_i, \ \{ \phi_i \} \) — orthonormal basis of eigenfunctions

\( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) — spectrum

**Spectral function:**

\[
N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)
\]

If \( x \neq y \), \( N_{x,y}(\lambda) = O(\lambda^{n-1}) \)
If \( x = y \), set \( N_{x,y}(\lambda) := N_x(\lambda) \)

**Weyl’s law:**

\[
N(\lambda) = C \text{Vol}(X) \lambda^n + R(\lambda),
\]

\[
R(\lambda) = O(\lambda^{n-1})
\]

**Local Weyl’s law:**

\[
N_x(\lambda) = C \lambda^n + R_x(\lambda), \quad R_x(\lambda) = O(\lambda^{n-1})
\]

Remainder estimates are sharp (attained on a round sphere)
MAIN RESULTS: lower bounds for $N_{x,y}(\lambda)$ and $R_x(\lambda)$

Notation: $f_1(\lambda) = \Omega(f_2(\lambda))$, $f_2 > 0$ iff

$$\limsup_{\lambda \to \infty} \frac{|f_1(\lambda)|}{f_2(\lambda)} > 0$$

**Theorem 1** If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} \right)$$
On-diagonal version:

Set \( u_j(x, x) \) — \( j \)-th local heat invariant

For example, \( u_1(x, x) = \frac{\tau(x)}{6} \), where \( \tau \) is scalar curvature

Denote \( \kappa_x = \min\{j \geq 1 | u_j(x, x) \neq 0\} \).

If \( u_j(x, x) = 0 \) for all \( j \geq 1 \), set \( \kappa_x = \infty \).

**Theorem 2** If \( n - 2\kappa_x - 1 > 0 \) then

\[
R_x(\lambda) = \Omega(\lambda^{n-2\kappa_x-1}).
\]

If \( n - 4\kappa_x - 1 < 0 \), and \( X \) has no conjugate points, then

\[
R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})
\]
Example: flat square 2-torus

\[ \lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbb{Z} \]

\[ \phi_j(x) = e^{2\pi i(n_1x_1 + n_2x_2)}, \quad x = (x_1, x_2) \]

\[ |\phi_j(x)| = 1 \Rightarrow N(\lambda) \equiv N_x(\lambda) \]

Gauss’s circle problem: estimate \( R(\lambda) \)

Theorem 2 \( \Rightarrow \) \( R(\lambda) = \Omega(\sqrt{\lambda}) \)

This is classical Hardy–Landau bound.

Theorem 2 \( \Rightarrow \) Hardy–Landau bound for the local remainder on any surface without conjugate points.
Manifolds of negative curvature

Suppose sectional curvatures satisfy

\[-K_1^2 \leq K(\xi, \eta) \leq -K_2^2\]

**Theorem** (Berard '77) \( R_x(\lambda) = O\left(\frac{\lambda^{n-1}}{\log \lambda}\right) \)

**Conjecture** (Randol' 81) On a surface of constant negative curvature

\[ R(\lambda) = O\left(\lambda^{\frac{1}{2}} \epsilon\right) \]

**Conjecture** (attributed to ?) On a *generic* negatively curved surface

\[ R(\lambda) = O(\lambda^{\epsilon}) \text{ for any } \epsilon > 0. \]
**Theorem** On a negatively curved surface

\[ R_x(\lambda) = \Omega(\sqrt{\lambda}). \]

This result was proved in an unpublished Ph.D. thesis of A. Karnaukh (Princeton, 1996) under the supervision of P. Sarnak.

It served as a starting point and a motivation for our work.
Thermodynamic formalism

$G^t$ — geodesic flow on a unit tangent bundle $SX$. **Topological pressure** of $f : SX \to \mathbb{R}$:

$$P(f) = \sup_{\mu} \left( h_{\mu} + \int f \, d\mu \right),$$

$\mu$ is $G^t$-invariant, $h_{\mu}$ — measure–theoretic entropy.

Variational principle: $P(0) = h$,

$h$ — **topological entropy** of $G^t$. 
On negatively curved manifolds geodesic flows are Anosov.

\( U(\xi) \) — unstable subspace of \( T_\xi SX \)

**Sinai-Ruelle-Bowen potential**

\[
\mathcal{H}(\xi) = \left. \frac{d}{dt} \right|_{t=0} \ln \det dG^t|_{U(\xi)}
\]

\( P(-\mathcal{H}) = 0 \) and the *equilibrium measure* (attaining the supremum) for \( \mathcal{H} \) is the Liouville measure \( \mu_L \) on \( SX \). Thus

\[
h_{\mu_L} = \int_{SX} \mathcal{H} d\mu_L
\]
Off-diagonal:

**Theorem 3.** If $X$ is negatively curved then for any $\delta > 0$ and $x \neq y$

\[ N_{x,y}(\lambda) = \Omega \left( \lambda \frac{n-1}{2} \left( \log \lambda \frac{P(-\mathcal{H}/2)}{h} \right) - \delta \right) \]

Power of the logarithm is positive

\[ \frac{P(-\mathcal{H}/2)}{h} \geq \frac{K_2}{2K_1} \]

and equals $\frac{1}{2}$ if curvature is constant.
On-diagonal:

**Theorem 4.** $X$ — negatively curved. If $n \leq 5$ then for any $\delta > 0$

$$R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-H/2)}{h}} - \delta \right)$$

If $n \geq 6$ then

$$R_x(\lambda) = \Omega(\lambda^{n-3})$$

Note different asymptotics for small and large $n$. 
Sketch of Proofs

Wave kernel on $X$:

$$e(t, x, y) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t) \phi_i(x) \phi_i(y)$$

Let $\psi \in C^\infty_0([-1, 1])$, even, monotone decreasing on $[0,1]$, $\psi \geq 0$, $\psi(0) = 1$. Fix $\lambda, T \gg 0$, consider the function

$$\frac{1}{T} \psi\left(\frac{t}{T}\right) \cos(\lambda t)$$

For $x, y \in M$, let

$$k_{\lambda, T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) e(t, x, y) dt$$
Off-diagonal case: $x \neq y$.

The following lemma is used in the proofs:

**Lemma 5** If $N_{x,y}(\lambda) = o(\lambda^a), \ a > 0$ then

$$k_{\lambda,T}(x, y) = o(\lambda^a).$$

If $N_{x,y}(\lambda) = O(\lambda^a(\log \lambda)^b), \ a, b > 0$ then

$$k_{\lambda,T}(x, y) = O(\lambda^a(\log \lambda)^b).$$

Let us start with **Theorem 3**:

$X$ — negatively curved.
Pretrace formula. Let $E(t, x, y)$ be the wave kernel on the universal cover $M$.

Given $x, y \in X$, we have

$$e(t, x, y) = \sum_{\omega \in \Gamma = \pi_1(X)} E(t, x, \omega y)$$

Given $x, y \in M$, define $K_{\lambda, T}(x, y)$ by

$$K_{\lambda, T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) E(t, x, y) dt$$

Then for $x, y \in X$

$$k_{\lambda, T}(x, y) = \sum_{\omega \in \Gamma} K_{\lambda, T}(x, \omega y)$$
Hadamard parametrix

Let $x, y \in M$, $r = d(x, y)$.

\[
E(t, x, y) = \frac{1}{\pi^{n-1}} |t| \sum_{j=0}^{\infty} u_j(x, y) \frac{(r^2 - t^2)^{j-n/2-3/2}}{4^j \Gamma(j - \frac{n-3}{2} - 1)}
\]

modulo a smooth function.

Here $u_j(x, y)$ solve transport equations along the geodesic joining $x$ and $y$. 
Leading term asymptotics

**Proposition 6** Let $x \neq y \in M$, $r = d(x, y)$. Then $K_{\lambda,T}(x, y)$ satisfies as $\lambda \to \infty$:

$$K_{\lambda,T}(x, y) = \frac{Q \lambda^{n-1} \psi(r/T)}{T \sqrt{g(x, y)} r^{n-1}} \sin(\lambda r + \phi_n) + O(\lambda^{n-3/2}).$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates centered at $x$,

$$\phi_n = \frac{\pi}{4}(3 - (n \mod 4))$$

and $Q \neq 0$. 
Proof of Theorem 3. Assume for contradiction that for some $\delta > 0$,

$$N_{x,y}(\lambda) = O\left(\lambda^{\frac{n-1}{2}}(\log \lambda)\frac{P(-\mathcal{H}/2)}{n}-\delta\right).$$

Lemma 5 implies a similar bound for $k_{\lambda,T}(x, y)$.

Proposition 6 implies

$$k_{\lambda,T}(x, y) = \sum_{r_\omega < T} \frac{\lambda^{\frac{n-1}{2}} A \psi\left(\frac{r_\omega}{T}\right)}{T \sqrt{g(x, \omega y)} r_\omega^{n-1}} \sin(\lambda r_\omega + \phi_n)$$

$$+ O(\lambda^{\frac{n-3}{2}}) \exp(O(T)),$$

for some $A \neq 0$. 
Consider the sum

\[
S_{x,y}(T) = \sum_{r\omega \leq T} \frac{1}{\sqrt{g(x,\omega y)} r^{n-1}_\omega}
\]

It follows from results of Parry and Pollicott that

**Theorem 7** As \( T \to \infty \),

\[
S_{x,y}(T) \geq C_0 e^{P\left(-\frac{H}{2}\right) \cdot T}
\]

Here \( P\left(-\frac{H}{2}\right) \geq \frac{(n-1)K_2}{2} \).
Case $n \not\equiv 3 \pmod{4} \Rightarrow \phi_n \not\equiv 0 \pmod{\pi}$.

Dirichlet box principle $\Rightarrow$ can choose $\lambda$ large so that

$$|e^{i\lambda r \omega} - 1| < \epsilon, \; \epsilon \text{ small},$$

for all $r \omega \leq T$. Then

$$|\sin(\lambda r \omega + \phi_n)| \approx |\sin \phi_n| > 0.$$

For Dirichlet principle need

$$T \approx \frac{1}{h} \log \log \lambda$$

Thus, exponential bound in Theorem 7 yields log–improvement in Theorem 3.
**Case** \( n = 3 \text{ (mod 4) } \Rightarrow \phi_n = 0 \text{ (mod } \pi \text{).} \)

Need a separate argument to establish

\[
\sin(\lambda r_\omega) > \frac{\nu}{T}, \quad \forall \omega : \frac{T}{\alpha} \leq r_\omega \leq T,
\]

\( \alpha > 0 \) some constant.

Combined with Theorem 7, this contradicts Lemma 5 and proves Theorem 3 in all dimensions.
Proof of Theorem 1 Assume

\[ N_{x,y}(\lambda) = o(\lambda^{\frac{n-1}{2}}). \]

Lemma 5 ⇒ \[ k_{\lambda,T}(x,y) = o(\lambda^{\frac{n-1}{2}}). \]

Work directly on \( X \) and adapt parametrix construction.

Let \( x, y \in X \) not conjugate along any shortest geodesic ⇒ finitely many shortest geodesics of length \( r = d(x,y) \).

Also, there are no geodesics from \( x \) to \( y \) of length \( l \in ]r, r + \epsilon[ \) for some \( \epsilon > 0 \).
Let $T = r + \frac{\epsilon}{2}$. Sum the parametrices along shortest geodesics and get

$$k_{\lambda,T}(x,y) = \beta \lambda^{\frac{n-1}{2}} \sin (\lambda r + \phi_n) + O(\lambda^{\frac{n-3}{2}}),$$

where $\beta$ is a non-zero constant.

Choose a sequence $\lambda_k \to \infty$ such that

$$|\sin(\lambda_k r + \phi_n)| > \nu > 0$$

Contradiction with $k_{\lambda,T}(x,y) = o(\lambda^{\frac{n-1}{2}})$. 
On-diagonal case, \( x = y \). Theorems 2 and 4 are proved similarly to Theorems 1 and 3 using the on-diagonal counterparts of Lemma 5 and Proposition 6.

The 0-th term of the parametrix on the diagonal cancels out with the main term in the Weyl’s law.

Consider Theorem 4 in more detail.
First on-diagonal term of the parametrix: \( c \lambda^{n-3} \) (for \( n > 3 \)).

Sum of the 0-th off-diagonal terms (by Theorem 3):

\[
O \left( \lambda^{\frac{n-1}{2}} \left( \log \lambda \right) \frac{P(-\mathcal{H}/2)}{h} - \delta \right)
\]

**Dimension** \( n \leq 4 \):

\[ n - 3 < \frac{n - 1}{2}, \]

so “diagonal<off-diagonal.”
**Dimension** \( n = 5 \):

\[ n - 3 = \frac{n - 1}{2}, \]

but “diagonal < off-diagonal” due to the power of log.

**Dimension** \( n \geq 6 \):

\[ n - 3 > \frac{n - 1}{2}, \]

so “diagonal > off-diagonal.”

Hence different bounds in Theorem 4 for \( n \geq 6 \) and \( n \leq 5 \).
Concluding remarks

• $R_x(\lambda) = \Omega(\sqrt{\lambda})$ in dimension 2. Together with the prediction $R(\lambda) = O(\lambda^\epsilon)$ on negatively curved surfaces this looks intriguing!

• Can one apply our method to estimate $R(\lambda)$ from below? We believe YES (in progress).