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Invariant subspaces of dissipative operators in Krein spaces

Let $H$ be a separable Hilbert space and $J = P_+ - P_-$ be a canonical symmetry ($J^2 = P_+ + P_- = 1$).

$K = \{H, J\}$ equiped with indefinite inner product

$[x, y] = (Jx, y), \quad x, y \in H$

is called Krein space (or Pontrjagin space $\Pi_x = \{H, J\}$ if rank $P_+ = x < \infty$).

Def. A subspace $L$ is nonnegative in $K$ if $[x, x] \geq 0 \forall x \in L$.

It is maximal nonnegative if there are no proper extensions of $L$. 
Def. An operator $A$ is dissipative in $H$ if
\[ \text{Im} \ (Ax, x) \geq 0 \quad \forall x \in D(A). \]
It is max. dissipative if there are no proper extensions of $A$ ($\iff \mathcal{C}^- \subset \rho(A)$, where $\mathcal{C}^-$ is open lower-half plane).

Def. $A$ is dissipative in Krein space $K = \{H, J\}$ if $JA$ is dissipative in $H$. $A$ is $m$-dissipative in $K$ if $JA$ is $m$-dissipative in $H$.

Symmetric and selfadjoint operators in $K$ are defined analogously.

Let $H = H_+ \oplus H_-$, $H_\pm = P_\pm(H)$, $D_\pm = D(A) \cap H_\pm$.

Assumption: $D(A) = D_+ \oplus D_-$ (it is sufficient to assume that $D_+ \oplus D_-$ is a core of $A$) $\iff A$ admits matrix representation
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} P_+ A P_+ & P_+ A P_- \\ P_- A P_+ & P_- A P_- \end{pmatrix},
\]
where $x = x_+ + x_-$ are identified with columns $x = (x_+)$.
Background:

Th. (Sobolev, 1941, 1962). A selfadjoint operator in $\Pi$ has at least one eigenvector corresponding to an eigenvalue $\lambda \in \mathbb{C}^+$.

Th. (Pontrjagin, 1944). Let $A$ be self-adjoint in $\Pi_\alpha$, $\alpha < \infty$. Then

(a) $\exists$ max. nonnegative subspace $L^+$ invariant with respect to $A$;
(b) among these subspaces $\exists L^+$ such that $\overline{\sigma(A^+)} \subseteq \mathbb{C}^+$, $A^+ = A / L^+$.

Th. (Langer, 1961). Let $A$ be selfadjoint in $K$ and

(i) $D(A) = H^+$ ($\iff A_{11}$ and $A_{21}$ are bounded);
(ii) $A_{12}$ is compact.

Then (a) & (b) hold.

Th. (Krein, 1948, 1964). Analogues of Pontrjagin and Langer theorems are true for unitary operators in $\Pi_\alpha$ and $K$, respectively.

M. Krein proposed a shorter elegant approach to prove (a) by means of Schauder-Tikhonov fixed point theorem.
Th. (Krein and Langer, 1971; Azizov 1972). Let \( A \) be \( m \)-dissipative in \( H \). Then (a) and (b) hold.

Th. (Azizov, Khoroshavin 1981). Let \( A \) be a contraction in Krein space and \( A_{12} \) be compact. Then (a)\&(b) hold if \( C^- \) is replaced by the open unit disk.

Th. (Azizov, 1985). Analogue of the previous result holds for \( m \)-dissipative operators in \( K \) provided that \( D(A) = H_+ \) and \( A_{12} \) is \( A_{22} \)-compact.

Th. (Shkalikov, 2004). Let

(i) \( A \) be dissipative in \( K \);
(ii) \( A_{22} \) be \( m \)-dissipative in \( H_- \);
\( \Leftrightarrow \exists (A_{22}-\mu)^{-1} \) for some \( \mu \in C^- \);
(iii) \( F(\mu) := (A_{22}-\mu)^{-1}A_{21} \) be bounded;
(iv) \( G(\mu) := A_{12}(A_{22}-\mu)^{-1} \) be compact;
(v) \( S(\mu) := A_{11} - A_{12}(A_{22}-\mu)^{-1}A_{21} \) be bounded.

Then (a) and (b) hold.
The main result in the paper is that firstly the larger condition $\mathcal{D}(A) \supset H^+$ was dropped out. In particular, for a model matrix operator

$$A = \begin{pmatrix} u(x) & \frac{d}{dx} \\ \frac{d}{dx} & \frac{d^2}{dx^2} \end{pmatrix}, \quad g = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

which is self-adjoint in $\mathcal{K} = \{H, I\}$, $H = L^2(0,1) \times L^2(0,1)$, provided that the domain of $A$ is chosen properly, one can guarantee the validity of properties (a) and (b).

The main goal of this talk is to prove (a) and (b) provided that only assumptions (i) - (iv) are valid.

It turns out that we need no assumptions for the transfer function $S(\mu)$.

New problems arise if we start working with unbounded entries and reject larger condition $\mathcal{D}(A) \supset H^+$. In this case, if we
succeed to prove (a) & (b), we come to the following interesting problems
(c) does the operator $A^+ = A / x^+$ generate a $C_0$-semigroup, or holomorphic semigroup?
We shall provide some sufficient conditions for positive answer to this question.
A subspace $L$ is $A$-invariant in classical sense if $L \subset D(A)$ and $A(L) \subset L$. We accept the following
\begin{definition}
$L$ is $A$-invariant if
\[ D(A) \cap L \text{ is dense in } L \text{ and } Ax \in L \text{ for all } x \in D(A) \cap L. \]
\end{definition}
Let us formulate the main results.
\begin{theorem}
A. Conditions (i)-(iv) imply (a).
\end{theorem}
\begin{theorem}
B. Property (b) holds if and only if assumption (i) is replaced by
(i') $A$ is $m$-dissipative in $K$.
\end{theorem}
For convenience we accept

**Def.** $B$ is a generator of $H_0$-semigroup if $A_{>0}$ $B - \varepsilon$ generates a holomorphic semigroup.

**Theorem C.** $iA^+$ generates a $C_0$-semigroup of exponential type $0$ is one of the following conditions holds

1. $A_{12}$ is compact
2. $-iA_{22}$ generates an $H_0$-semigroup.

**Theorem D.** $iA^+$ generates an exponentially stable semigroup if either (1) or (2) holds and $A$ is uniformly dissipative in $I_K$.

**Theorem E.** There is $\mu \in C^+$ such that $iS(\mu)$ generates an $H_0$-semigroup. Then $iA^+$ generates $H_0$-semigroup.
The main steps of the proof of the first two theorems.

Assumptions (ii)–(iv) allow to use Frobenious–Shur factorization
\[ A - \mu = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S - \mu & 0 \\ 0 & A_{22} - \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \]
where \( A = A(\mu) \), \( F = F(\mu) \) and \( S = S(\mu) \) is the transfer function defined on the domain \( \mathcal{D}(S) = \mathcal{D}_+ \).

**Lemma 1**
\[ JA + \mu = J \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S + \mu & 0 \\ 0 & A_{22} - \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \]
Proof by direct verification.

**Lemma 2** \( \forall \mu \in \mathbb{C}^+ \) and \( \forall x \in \mathcal{D}_+ \)
we have
\[ (Sx_+, x_+) = (JA (x_+, -Fx_+), (x_+)) + \]
\[ + \mu (Fx_+, Fx_+). \]
Proof by direct verification.
Corollary (important). \( S = S(\mu) \) with domain \( D(S) = D_+ \) is dissipative in \( H_+ \) provided that assumption \((i)\) holds. Also, \( S \) is closable. The closure of \( S \) is \( m \)-dissipative in \( H_+ \). \( \iff \) \( A \) is \( m \)-dissipative in \( K \).

**Lemma 3** (important). Let a subspace \( \mathcal{L} \) have a representation of the form
\[
\mathcal{L} = \{ x : x = (x_+^T), x_+ \in H_+ \}
\]
where \( K : H_+ \to H_- \) is a bounded operator. Then \( \mathcal{L} \) is \( \mathcal{A} \)-invariant \( \iff \)
\[
(1 - KG) (A_{22} - \mu) (F + K) = K (S - \mu)
\]
(the so-called Riccati equation for \( K \)).

**Proof.** For \( x_+ \in D_+ \)
\[
(A - \mu) \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix} = \begin{pmatrix} (S - \mu)x_+ + G (A_{22} - \mu) (F + K) x_+ \\ (A_{22} - \mu) (F + K) x_+ \end{pmatrix}.
\]
Assuming that \( \mathcal{L} \) is \( \mathcal{A} \)-invariant we find \( y_+ \in H_+ \) such that...
\begin{align*}
    \left[(S - \mu) + G \left(A_{22} - \mu\right) (F + k)\right] x^+ = y^+, \\
    (A_{22} - \mu) (F + k) x^+ = Ky^+.
\end{align*}

Substituting the first equality in the second one we come to Riccati equation for \( k \).

Conversely, Riccati equation for \( k \) implies the last two equations with some \( y^+ \), therefore the graph subspace \( S \) is \( A \)-invariant. \( \square \)

\underline{Remark.} Pontrjagin used:
\[ S \text{ is } A \text{-invariant} \iff \]
\[ A_{21} + A_{22} K - K A_{11} - K A_{12} K = 0 \]

However this form of Riccati equation is inconvenient while working with unbounded entries \( A_{ij} \).

\underline{Lemma 4.} Assume that \( G(\mu) \) is compact for some \( \mu \in \mathbb{C}^+ \). Then it is compact for all \( \mu \in \mathbb{C}^+ \) and \( \|G(\mu)\| \to 0 \) as \( \mu \to \infty \) and \( \mu \in \Lambda_e^+ \).

\underline{Proof} is simple.
Lemma 5. A subspace $L$ is max nonnegative $\iff$ $L$ has graph representation

$L = \{ x = (x_+, x_-) \mid x_+ \in H_+ \}$

with the angle operator $K$, $\|K\| \leq 1$.

Corollary. Take $\mu \in \mathbb{C}^+$ such that $\|A(\mu)\| < 1/2$. Then (a) holds $\iff \exists$ a contraction $K$ s. th.

$$K + K = (A_{22} - \mu)^{-1}(1 - KA)^{-1}K(S - \mu).$$

Lemma 6. Denote $H_S = \mathcal{D}(\overline{S}) \cap H_+$ where $\overline{S}$ is the closure of $S$ and the norm in $H_S$ is defined by

$$\|x_+\|_{H_S} = \sqrt{\|\overline{S}x_+\|^2 + \|x_+\|^2}.$$ 

Then $\exists$ a complete orthogonal system $\{y_k\}_{k=1}^\infty$ in $H_+$ such that $\{y_k\}_{k=1}^\infty$ is a Riesz basis in $H_S$.

Proof. If $H_S$ is compactly embedded in $H_+$ we take $\{y_k\}_{k=1}^\infty$ consisting of eigenvectors of $S^*S$. In general case
Proof of Theorem A.

Let \( P_n \) be orthogonal projectors onto \( \text{Lin} \{ \Phi_k \}_k^n \) in \( H_+ \). Then \( P_n \rightarrow 1 \) in \( H_+ \) and \( P_n \rightarrow 1 \) in \( H_- \).

Consider

\[
A_n = \begin{pmatrix}
P_n A_{11} P_n & P_n A_{12} \\
A_{21} P_n & A_{22}
\end{pmatrix}
\quad \text{in} \quad H_n^+ \oplus H_n^-,
\]

\( H_n^+ = P_n (H^+) \).

Then \( A_n \) is \( m \)-dissipative in Pontrjagin space \( \Pi_n \) and due to Krein-Bang-Adzizov theorem (a) holds. This implies (Lemma 3) that

\[
(*) \quad F_n + K_n = (A_{22} - \mu)^{-1} (1 - K_n G)^{-1} K_n (S_n - \mu).
\]

Choose \( K_n \rightarrow K \). Since \( \|K_n\| \leq 1 \), we have \( \|K\| \leq 1 \). Then

\[
F_n = F P_n \rightarrow 1
\]

\( K_n G \Rightarrow KG \) and \( (1 - K_n G)^{-1} \Rightarrow (1 - KG) \)

(we essentially use here that \( G \) is compact!!)
Further,
\[ K_n S_n = K_n S P_n , \]
\[ \overline{S} P_n x \to \overline{S} x \quad \forall x \in \mathcal{D}(\overline{S}) , \]
Hence, \[ K_n S P_n x \to KS x . \]
Therefore we can pass to the weak limit in the equation (*) and obtain
\[ F+K = (A_{22}-\mu)^{-1}(F-KL)^{-1}K(S-\mu) \]
and by virtue of Lemma 3 property (a) holds.
\[ \square \]
Let \( A^+ = \overline{A}/\xi^+ \). How to prove
\[ (b): \exists \xi^+ \text{ such that } \mathcal{D}(A^+) \subset \mathcal{D}^+ ? \]
We have
\[ (\overline{A}-\mu) (Kx^+) = (\overline{S}-\mu + A L) x^+ , \]
where \( L = (A_{22}-\mu)(F+K), \mathcal{D}(L) = \mathcal{D}(\overline{S}). \)
Consider
\[ Q: \xi \to H_+ \text{ defined by } Q(Kx^+) = x^+ . \]
\( Q \) is bounded and boundedly invertible \( \| Q^{-1} \| \leq 2 . \)
We have
\[
\bar{A}/\mathcal{L}_+ = Q^{-2}(S + QA)Q = 
\]
\[
= Q^{-2}[1 + G(1-KA)^{-2}K(S-\mu)]Q,
\]
hence
\[
(\ast)(\bar{A} - \lambda)/\mathcal{L}_+ = Q^{-1}[1 + T(\lambda)](S(\mu) - \lambda)Q,
\]
where
\[
T(\lambda) = G(1-KA)^{-2}K(S-\mu)(\bar{S} - \lambda)^{-1}
\]
is a holomorphic operator function whose values are compact operators. Here we assumed that \((\bar{S} - \lambda)^{-1}\) exists \(\iff\) \(\bar{S}\) is m-dissipative in \(H_+\) \(\iff\) \(\bar{A}\) is m-dissipative in \(K\).

It can be shown that \(\|T(\lambda)\| \to 0\) as \(\lambda \to \infty\) along negative imaginary axis, therefore \(1 + T(\lambda)\) has only discrete spectrum in \(\mathcal{C}^-\). We use the following

\textbf{Lemma 7.} \(\text{Im} [Ax_0, x_0] = \text{Im}d_0 [x_0, x_0]\)
if \(Ax_0 = d_0 x_0\).
Therefore, all eigenvectors of \( A \) corresponding to the eigenvalues from \( \mathbb{C}^- \) are of negative type provided that \( A \) is strictly dissipative in \( \mathbb{K} \).

This proves Theorem B if we assume in addition that \( A \) is strictly dissipative in \( \mathbb{K} \).

If not, we consider

\[
A_\varepsilon = A + i\varepsilon P^+,
\]

\( \varepsilon > 0 \).

Assertion (a) is valid for \( A_\varepsilon \) and it does not have spectrum in \( \mathbb{C}^- \). Since

\[
\text{Im} \left[ (A + i\varepsilon P^+) \left( (Kx_+) \right)^t \right] \geq \varepsilon (x_+, x_+).
\]

Write Riccati equation for \( A_\varepsilon \):

\[
F + K_\varepsilon = (A_{22} - \mu)^{-1} (1 - K_\varepsilon G)^{-1} K_\varepsilon (S+i\varepsilon - \mu).
\]

Take \( \varepsilon_n \to 0 \) and \( K_{\varepsilon_n} = :K_n \to K \).

We have
$$A_t^e = Q^{-1} \left[ 1 + T_e(x) \right] (S + ie - \lambda) Q$$

and

$$T_e(x) = C (1 - K_e Q)^{-1} K_e (S + ie - \mu)(S + ie - \lambda)^{-1} \Rightarrow T^e(x).$$

Since $1 + T_e(x)$ is a holomorphic operator function of Fredholm type in $\mathbb{C}^-$, boundedly invertible $\forall \lambda \in \mathbb{C}^-$, so is $1 + T^e(x)$. $\square$

Theorems C-E are proved by analyzing representation $(**)$.