On possible spectral structure of linear continuous operators

Gallia est omnis divisa in partes tres

Ceasar,
Bellum Gallicum

$X$ - a topological vector space (over $\mathbb{C}$)

$T : X \to X$ - a continuous linear operator

$\sigma(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible} \}$;

$\sigma_p(T) = \{ \lambda \in \mathbb{C} : \ker(\lambda I - T) = (\lambda I - T)^2(0) \neq \{0\} \}$;

$\sigma_c(T) = \{ \lambda \in \sigma(T) \setminus \sigma_p(T) : (\lambda I - T)(X) \text{ is dense in } X \}$;

$\sigma_e(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T))$;

$\sigma^{(n)}(T) = \{ \lambda \in \mathbb{C} : \dim \ker(\lambda I - T) = n \}$;

$\sigma_{p,n}(T) = \{ \lambda \in \mathbb{C} : \dim \ker(\lambda I - T) = n \}$.

Question 1. For a given class $X$ of topological vector spaces, which triples of subsets of $\mathbb{C}$ are point, continuous and residual spectra of continuous linear operators acting on a space from $X$. 
1. Necessary conditions

**Theorem 1.** Let \( X \) be a separable Fréchet space and \( T: X \to X \) be a closed densely defined linear operator. Then the sets \( \delta^n_p(T) \) are Souslin for \( n = 1, \ldots, \infty \), \( \delta_c(T), \delta_r(T) \) and \( \delta_{p,n}(T) \) are co-Souslin.

**Morozevich**\( \exists \) an \( F_\sigma \)-set \( D = \bigcap_{n \in \mathbb{N}} \delta_p(T) \cup \delta_r(T) = \delta_p(T) \cup D \)

**Remark.** If \( \delta_{p,\infty}(T) \) is a Borel measurable set, then all above sets are Borel measurable.

**Theorem 2.** Let \( X \) be a reflexive separable Banach space and \( T \) be a closed densely defined linear operator acting on \( X \). Then the set \( \delta_c(T) \) is \( G_\sigma \)
\( \delta^n_p(T) \) is \( F_\sigma \) for any \( n = 1, \ldots, \infty \).

**Remark.** Reflexivity can be replaced by quasireflexivity.

2. Spectral synthesis

**Theorem 3.** Let \( A_1, A_2, \ldots \) be a decreasing sequence of \( F_\sigma \)-sets (subsets of \( C \)), \( A_0 \) be a \( G_\delta \)-set, and \( A \subseteq C \) be such that

\( A_{-1}, A_0, A_1 \) are disjoint

\( A_{-1} \cup A_0 \cup A_1 \) is a non-empty compact set. Then there exists a continuous linear operator \( T \) on \( l_2 \) such that \( \delta_r(T) = A_{-1}, \delta_c(T) = A_0 \) and \( \delta^n_p(T) = A_n \) for \( n = 1, \ldots \).
Theorem 4. Let $K \subseteq C$ be a non-empty compact set, being a disjoint union of $A, B$ and $C$, where $A$ is Souslin, $B$ is co-Souslin and there exists an $F_\sigma$-set $D$ for which $A \cup D = A \cup C$. Then there exists a separable Banach space $X$ and a continuous linear operator $T : X \to X$ for which $6^p (T) = A$, $6^c (T) = B$ and $6^r (T) = C$.

Theorem 5. Let $A_1, A_2, \ldots$ be a decreasing sequence of Borel sets, $A_0, A_1$ be Borel sets such that $A_0, A_0, A_1$ are disjoint $A_0 \cup A_0 \cup A_1$ is a non-empty compact set and there exists an $F_\sigma$-set $D$ for which $A_0 \cup A_\infty = A_0 \cup D$. Then there exists a separable Banach space $X$ and a continuous linear operator $T : X \to X$ for which $6^p (T) = A_0$, $6^c (T) = A_0$ and $6^r (T) = A_0$, $n \in \mathbb{N}$.

Theorem 6. Let $X$ be a separable Fréchet space and $T$ be a closed densely defined linear operator acting on $X$. Then $6(T)$ is a $G_{\delta , c}$-set. Conversely, for any $G_{\delta , c}$-set $A \subseteq C$, there exists a separable Fréchet space $X$ and a continuous linear operator $T : X \to X$ for which $6^c (T) = A$. 
of subsets of \( C \) there exists \( T \) acting on \( X \in C \) for which
\[
\varepsilon (T) = A_{\infty}, \quad \varepsilon_c (T) = A_0, \quad \varepsilon^\infty (T) = A_n, \quad n \neq 1
\]

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**History:**

1) G. Kalisch [1972]: for any nonempty compact set \( K \in \mathbb{R} \) there exists a \( L^0 \)
\( T \) on a separable Hilbert space for which
\[
\varepsilon (T) = \varepsilon_p (T) = \varepsilon_{p,1} (T) = K
\]

2) L. Nikol'skiaia [1974]: the point spectrum of a closed densely defined linear operator acting on a separable reflexive Banach space is an \( F_\sigma \) -set. Any \( F_\sigma \) -set is the point spectrum of such an operator acting on a separable Hilbert space.

3) R. Kaufman [1981, 1985]: the point spectrum of a continuous linear operator acting on a separable Banach space is a Souslin set. Any bounded Souslin set is the point spectrum of such an operator.

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