Dirichlet spaces with no reference measure

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Weak solutions

\[ \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} \partial_{\alpha} (a_{\alpha\beta} \partial_{\beta} u) = F \text{ (a measure)} \]

\[ \mathcal{E}[u, \phi] = \sum_{|\alpha|,|\beta| \leq m} \int a_{\alpha\beta} (\partial_{\beta} u)(\partial_{\alpha} \phi) \, dx = F(\phi), \quad \forall \phi \in \mathcal{D} - \text{test functions} \]

Green function \( G : \mathcal{E}[G(\cdot, x), \phi] = \phi(x), \quad \forall \phi \in \mathcal{D} \)

Super-harmonic \( u : \mathcal{E}[u, \phi] \geq 0, \quad \forall \phi \in \mathcal{D}^+ \)
The Fukushima construction

$m$ - a full support measure

$(\mathcal{E}, \mathcal{D})$ - a closable Markov form in $L^2(m)$, associates Markov SG $P_t$ on $L^p(m)$, $p \in [1, \infty]$

$P_t$ is transient

$\Leftrightarrow \forall f \in L^1_+(m) : Gf = \int_0^\infty P_t f \, dt < \infty \ m$-a.e.

$m(\phi f) - m(\phi P_T f) \quad \left( \rightarrow m(\phi f) \right)

= \int_0^T \mathcal{E}[P_t f, \phi] \, dt \quad \left( \rightarrow \mathcal{E} \left[ \int_0^\infty P_t f \, dt, \phi \right] \right)

\exists g \in L^1(m), \ g > 0 \ m$-a.e.: $\sqrt{\mathcal{E}[\phi]} \geq \int |\phi| \, dm, \ \forall \phi \in \mathcal{D}$

Fukushima: transience (recurrence) depends on measure $m$. 
Examples

\( \Omega \subset \mathbb{R}^N \) smooth bdd connected

\[ E(u) := \int_{\Omega} |\nabla u|^2 \, dx, \ D_0 = H^1_0(\Omega), \ D_1 = H^1(\Omega). \]

\( \lambda \) be the \( N \)-dim Lebesgue measure on \( \Omega \)

\[ \Delta := \sum_{q \in Q^N \cap \Omega} c_q \delta_q. \]

\( H^1 \) is recurrent wr to any reference measure it is closable.

\( H^1_0 \) is transient wr to \( m = \lambda \).

\( H^1_0 \) is recurrent wr to \( m = \lambda + \Delta \).

\( H^1 \) and \( H^1_0 \) are not closable wr to \( m = \Delta \), \( N \geq 3 \).
Philosophy: measure as a clocking device

Let $m_0 \leftrightarrow \frac{du}{dt} = Au$.

Then $dm := \rho dm_0 \leftrightarrow \frac{du}{d\tau} = \frac{1}{\rho} Au$, i.e., $t = \frac{\tau}{\rho}$.

Fukushima: for $m$ not charging sets of zero capacity,

$t = T_\tau(\omega)$:

$$\frac{1}{\tau} \int_0^\tau f(X_\tau)d\tau \to m(f), \quad \tau \to 0$$

$X_t(\omega)$:

$\mathbb{E}_x f(X_t) = P_t f(x)$, $P_t \leftrightarrow (\mathcal{E}, \mathcal{D})$ on $L^2(m_0)$. 
**Transient Dirichlet space** $(\mathcal{H}, [\cdot, \cdot])$

*Given:* state space $\Omega$, $\mathcal{B}$ - Borel $\sigma$-algebra on $\Omega$, $\mathcal{B}(\Omega)$ - $\mathcal{B}$-measurable functions of $\Omega$

1. $\mathcal{H}$ is a separable Hilbert space.

2. $\mathcal{H}$ is an ordered vector space
   $\mathcal{H}^+$ closed, $\mathcal{H}^+ \cap (-\mathcal{H}^+) = \{0\}$.

3. $\mathcal{H}$ is a *Stone lattice* i.e. a vector lattice with an order-convex sub-lattice $\mathcal{H}^\wedge \subset \mathcal{H}^+$ of "positive elements not exceeding the unit". $\mathcal{H}^\wedge$ is closed.

4. $\mathcal{H} \overset{\text{dense}}{\hookrightarrow} \mathcal{D} \subset \mathcal{B}(\Omega)$, a Stone sub-lattice in the pointwise order, generating $\mathcal{B}$.

5. For all $\in \mathcal{H}$: $\|(u^+)^\wedge\|_\mathcal{H} \leq \|u\|_\mathcal{H}$. 
Stone lattice $\mathcal{V}$

- vector lattice ($\equiv$ ordered vector space with $\land, \lor$ operations);

- countable type ($\equiv$ a majorized family of disjoint elements is at most countable);

- $\exists$ order-convex sub-lattice $\mathcal{V}^\land \subset \mathcal{V}^+$ such that:
  $0 = \min \mathcal{V}^\land$;
  $\forall u \in \mathcal{H}^+ : \exists u^\land := \sup\{v \in \mathcal{H}^\land, \ v \leq u\}$;
  $\forall u \in \mathcal{H}^+ : (\forall \alpha \in \mathbb{R}^+ : \alpha u \in \mathcal{H}^\land) \Rightarrow u = 0$. 
Daniell Stone integral

A Stone lattice allows for an abstract version of the Lebesgue (Daniell-Stone) integral:

- order completion $\hat{\mathcal{V}} (\hat{\mathcal{V}}^+ = \text{limits of increasing positive sequences})$ is an analog of the measurable functions space;

- $\sigma(\mathcal{V}) := \left\{ \sup_n [(nu)^\wedge] | u \in \mathcal{V}^+ \right\} \subset \hat{\mathcal{V}}$ is a (Boolean) $\sigma$-algebra of "(indicators of) supports of elements of $\mathcal{V}$"

- Daniell-Stone theorem: an order continuous positive linear functional on $\mathcal{V}$ is a positive measure on $\sigma(\mathcal{V})$. 
Properties of a transient Dirichlet space

1. $\sigma(\mathcal{H}) \supset B$.

2. $S^+ := \left(\mathcal{H}^*\right)^+$ separates points on $\mathcal{H}$. They are positive measures on $\sigma(\mathcal{H})$ satisfying $\mu(u) \leq c\|u\|_\mathcal{H}$, $u \in \mathcal{D}^+$

3. $\exists m \in S^+$ of a full support. $\left([,], \mathcal{H} \cap L^2(m)\right)$ is a transient Dirichlet form in $L^2(m)$ in the Fukushima sense.

4. The Green operator $G$ is the Riesz isometry $\mathcal{H}^* \rightarrow \mathcal{H}$ restricted to (signed) measures on $\sigma(\mathcal{H})$. 
Construction

$\mathcal{D} \subset C_c(\Omega)$

- Stone lattice with the pointwise order;

- dense in $C_c(\Omega)$;

- $\forall v \in \mathcal{D}^+ \exists u \in \mathcal{D}^\wedge$ such that $\forall \epsilon > 0$ 
  $(u + \epsilon v)^\wedge = u$ ("$u = 1$ on supp $v$);

- $\|(u^+)^\wedge\|_\mathcal{H} \leq \|u\|_\mathcal{H}$;

- for any $\|u_n\|_\mathcal{H} \to 0$, $\sup_n \|v_n\|_\mathcal{H} < \infty$: 
  $0 \leq v_n \leq u_n \Rightarrow v_n \to 0$ (weakly) in $\mathcal{H}$. 