Commutative algebras
of Toeplitz operators
and Berezin quantization

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\( \mathbb{D} \) is the unit disk in \( \mathbb{C} \),
\( L_2(\mathbb{D}) \) with the Lebesgue plane measure \( d\mu(z) = dx dy \),
Bergman space \( \mathcal{A}^2(\mathbb{D}) \) consists of analytic functions in \( \mathbb{D} \),
Bergman orthogonal projection \( B_\mathbb{D} \) of \( L_2(\mathbb{D}) \) onto \( \mathcal{A}^2(\mathbb{D}) \):

\[
(B_\mathbb{D} \varphi)(z) = \frac{1}{\pi} \int_\mathbb{D} \frac{\varphi(\zeta) d\mu(\zeta)}{(1 - z\zeta)^2},
\]

Toeplitz operator \( T_a \) with symbol \( a = a(z) \):

\[
T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \mapsto B_\mathbb{D} a \varphi \in \mathcal{A}^2(\mathbb{D}).
\]
Unit Disk as a Hyperbolic Plane

Consider the unit disk $\mathbb{D}$ endowed with the hyperbolic metric

$$g = ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

A geodesic, or a hyperbolic straight line, in $\mathbb{D}$ is (a part of) an Euclidian circle or a straight line orthogonal to the boundary $S^1 = \partial \mathbb{D}$.

Each pair of geodesics, say $L_1$ and $L_2$, lie in a geometrically defined object, one-parameter family $\mathcal{P}$ of geodesics, which is called the pencil determined by $L_1$ and $L_2$. Each pencil has an associated family $\mathcal{C}$ of lines, called cycles, the orthogonal trajectories to geodesics forming the pencil.
The pencil $\mathcal{P}$ determined by $L_1$ and $L_2$ is called

*parabolic* if $L_1$ and $L_2$ are parallel, in this case $\mathcal{P}$ is a set of all geodesics parallel to both $L_1$ and $L_2$, and cycles are called *horocycles*;

*elliptic* if $L_1$ and $L_2$ are intersecting, in this case $\mathcal{P}$ is a set of all geodesics passing through the common point of $L_1$ and $L_2$;

*hyperbolic* if $L_1$ and $L_2$ are disjoint, in this case $\mathcal{P}$ is a set of all geodesics orthogonal to the common orthogonal of $L_1$ and $L_2$, and cycles are called *hypercycles*. 
Each Möbius transformation $g \in \mathrm{Möb}(\mathbb{D})$ is a movement of the hyperbolic plane, determines a certain pencil of geodesics $\mathcal{P}$, and its action is as follows:

* each geodesic $L$ from the pencil $\mathcal{P}$, determined by $g$, moves along the cycles in $\mathcal{C}$ to the geodesic $g(L) \in \mathcal{P}$, while each cycle in $\mathcal{C}$ is invariant under the action of $g$. 
Theorem 1  Given a pencil $\mathcal{P}$ of geodesics, consider the set of symbols which are constant on corresponding cycles. The $C^*$-algebra generated by Toeplitz operators with such symbols is commutative.

That is, each pencil of geodesics generates a commutative $C^*$-algebra of Toeplitz operators.

Theorem 2  Given a Möbius transformation $g \in \text{Möb}(\mathbb{D})$, consider the set of symbols which are invariant with respect to the one-parameter group generated by $g$. The $C^*$-algebra generated by Toeplitz operators with such symbols is commutative.

That is, each one-parameter group of Möbius transformations ($\equiv$ maximal commutative subgroup of $\text{Möb}(\mathbb{D})$) generates a commutative $C^*$-algebra of Toeplitz operators.
Model cases:
Parabolic case

Consider the upper half-plane $\Pi$ in $\mathbb{C}$. Introduce the unitary operators

$$U_1 = F \otimes I : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}+),$$

where $F : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ is the Fourier transform

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iu\xi} f(\xi) \, d\xi,$$

and

$$U_2 : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}+)$$

which is defined by the rule

$$U_2 : \varphi(u, v) \longmapsto \frac{1}{\sqrt{2|x|}} \varphi(x, \frac{y}{2|x|}).$$

Letting $\ell_0(y) = e^{-y/2}$, we have $\ell_0(y) \in L_2(\mathbb{R}+)$ and $\|\ell_0(y)\| = 1$. Denote by $L_0$ the one-dimensional subspace of $L_2(\mathbb{R}+)$ generated by $\ell_0(y)$.

**Theorem 3** The unitary operator $U = U_2U_1$ is an isometric isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}+)$ under which the Bergman space $A^2(\Pi)$ is mapped onto $L_2(\mathbb{R}+) \otimes L_0$,

$$U : A^2(\Pi) \longrightarrow L_2(\mathbb{R}+) \otimes L_0.$$
Introduce the isometric imbedding
\[ R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \]
by the rule \( (R_0f)(x, y) = \chi_+(x) f(x) \ell_0(y), \) where \( \chi_+(x) \) is the characteristic function of \( \mathbb{R}_+. \)

Now the operator \( R = R_0^* U \) maps the space \( L_2(\Pi) \) onto \( L_2(\mathbb{R}_+) \), and the restriction
\[ R|_{A^2(\Pi)} : A^2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \]
is an isometric isomorphism. The adjoint operator
\[ R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow A^2(\Pi) \subset L_2(\Pi) \]
is an isometric isomorphism of \( L_2(\mathbb{R}_+) \) onto the subspace \( A^2(\Pi) \) of the space \( L_2(\Pi) \). Moreover,
\[ RR^* = I : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \]
\[ R^* R = B_\Pi : L_2(\Pi) \longrightarrow A^2(\Pi). \]

**Theorem 4** Let \( a = a(v) \) be a measurable function on \( \mathbb{R}_+ \). Then the Toeplitz operator \( T_a \) acting on \( A^2(\Pi) \) is unitary equivalent to the multiplication operator \( \gamma_a I = RT_a R^* \), acting on \( L_2(\mathbb{R}_+) \). The function \( \gamma_a(x) \) is given by
\[ \gamma_a(x) = \int_{\mathbb{R}_+} a\left(\frac{y}{2x}\right) e^{-y} \, dy, \quad x \in \mathbb{R}_+. \]
Berezin quantization on the hyperbolic plane

We consider the pair \((\mathbb{D}, \omega)\), where \(\mathbb{D}\) is the unit disk and

\[
\omega = \frac{1}{\pi} \frac{dx \wedge dy}{(1 - (x^2 + y^2)^2)} = \frac{1}{2\pi i} \frac{dz \wedge dz}{(1 - |z|^2)^2}.
\]

Poisson brackets:

\[
\{a, b\} = \pi (1 - (x^2 + y^2))^2 \left( \frac{\partial a}{\partial y} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} \right)
= 2\pi i (1 - z\overline{z})^2 \left( \frac{\partial a}{\partial \overline{z}} \frac{\partial b}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial b}{\partial \overline{z}} \right).
\]

Laplace-Beltrami operator:

\[
\Delta = \pi (1 - (x^2 + y^2))^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
= 4\pi (1 - z\overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}.
\]

Introduce weighted Bergman spaces \(A_h^2(\mathbb{D})\) with the scalar product

\[
(\varphi, \psi) = \left( \frac{1}{h} - 1 \right) \int_{\mathbb{D}} \varphi(z) \overline{\psi(z)} (1 - z\overline{z})^{\frac{1}{h}} \omega(z).
\]

The weighted Bergman projection has the form

\[
(B_{\mathbb{D}, h} \varphi)(z) = \left( \frac{1}{h} - 1 \right) \int_{\mathbb{D}} \varphi(\zeta) \left( \frac{1 - \zeta\overline{z}}{1 - z\overline{\zeta}} \right)^{\frac{1}{h}} \omega(\zeta).
\]
Let \( E = (0, \frac{1}{2\pi}) \), for each \( \hbar = \frac{h}{2\pi} \in E \), and consequently \( h \in (0, 1) \), introduce the Hilbert space \( H_{\hbar} \) as the weighted Bergman space \( A_{\hbar}^{2}(\mathbb{D}) \).

For each function \( a = a(z) \in C^{\infty}(\mathbb{D}) \) consider the family of Toeplitz operators \( T_{a}^{(h)} \) with (anti-Wick) symbol \( a \) acting on \( A_{\hbar}^{2}(\mathbb{D}) \), for \( h \in (0, 1) \), and denote by \( \mathcal{T}_{h} \) the *-algebra generated by Toeplitz operators \( T_{a}^{(h)} \) with symbols \( a \in C^{\infty}(\mathbb{D}) \).

The Wick symbols of the Toeplitz operator \( T_{a}^{(h)} \) has the form

\[
\tilde{a}_{h}(z, \zeta) = (\frac{1}{h} - 1) \int_{\mathbb{D}} a(\zeta) \left( \frac{(1 - |z|^{2})(1 - |\zeta|^{2})}{(1 - z\zeta)(1 - \zeta\zeta)} \right)^{\frac{1}{h}} \omega(\zeta).
\]

For each \( h \in (0, 1) \) define the function algebra

\[
\tilde{A}_{h} = \{ \tilde{a}_{h}(z, \zeta) : a \in C^{\infty}(\mathbb{D}) \}
\]

with point wise linear operations, and with the multiplication law defined by the product of Toeplitz operators:

\[
\tilde{a}_{h} \ast \tilde{b}_{h} = (\frac{1}{h} - 1) \int_{\mathbb{D}} \tilde{a}_{h}(z, \zeta) \tilde{b}_{h}(\zeta, \zeta) \left( \frac{(1 - |z|^{2})(1 - |\zeta|^{2})}{(1 - z\zeta)(1 - \zeta\zeta)} \right)^{\frac{1}{h}} \omega.
\]
The correspondence principle is given by
\[
\tilde{a}_h(z, \bar{z}) = a(z, \bar{z}) + O(\hbar),
\]
\[
(\tilde{a}_h \ast \tilde{b}_h - \tilde{b}_h \ast \tilde{a}_h)(z, \bar{z}) = i\hbar \{a, b\} + O(\hbar^2).
\]

**Three term asymptotic expansion:**
\[
(\tilde{a}_h \ast \tilde{b}_h - \tilde{b}_h \ast \tilde{a}_h)(z, \bar{z})
\]
\[
= i\hbar \{a, b\}
\]
\[
+ \frac{i\hbar^2}{4} \{\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\} + 8\pi \{a, b\}\}
\]
\[
+ i\frac{\hbar^3}{24} \left[ \{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2 \{a, b\}\right]
\]
\[
+ \Delta \{a, \Delta b\} + \Delta \{\Delta a, b\}
\]
\[
+ 28\pi (\Delta \{a, b\} + \{a, \Delta b\} + \{\Delta a, b\})
\]
\[
+ 96\pi^2 \{a, b\}\right] + o(\hbar^3)
\]

**Corollary 5** Let \( \mathcal{A}(\mathbb{D}) \) be a subspace of \( C^\infty(\mathbb{D}) \) such that for each \( h \in (0, 1) \) the Toeplitz operator algebra \( \mathcal{T}_h(\mathcal{A}(\mathbb{D})) \) is commutative.
Then for all \( a, b \in \mathcal{A}(\mathbb{D}) \) we have
\[
\{a, b\} = 0,
\]
\[
\{a, \Delta b\} + \{\Delta a, b\} = 0,
\]
\[
\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0.
\]
Let $A(\mathbb{D})$ be a linear space of smooth functions which generates for each $h \in (0, 1)$ the commutative $C^*$-algebra $T_h(A(\mathbb{D}))$ of Toeplitz operators.

**First term:** $\{a, b\} = 0$:

**Lemma 6** All functions in $A(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.

**Second term:** $\{a, \Delta b\} + \{\Delta a, b\} = 0$:

**Theorem 7** The space $A(\mathbb{D})$ consists of functions whose common gradient lines are geodesics in the hyperbolic geometry of the unit disk $\mathbb{D}$.

**Third term:** $\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0$:

**Theorem 8** The space $A(\mathbb{D})$ consists of functions whose common level lines are cycles.
Dynamics of spectra of Toeplitz operators

Let $D$ be either the unit disk $\mathbb{D}$, or the upper half-plane $\Pi$ in $\mathbb{C}$.

For a symbol $a = a(z)$, $z \in D$, the Toeplitz operator $T_a^{(\lambda)}$ acts on $A^2_\lambda(D)$ as follows

$$T_a^{(\lambda)} \varphi = B_D^{(\lambda)} a \varphi, \quad \varphi \in A^2_\lambda(D).$$

**Theorem 9** Given any model pencil and a symbol $a \in L_\infty(D)$, constant on corresponding cycles, the Toeplitz operator $T_a^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a, \lambda} I$, where

in the parabolic case: $a = a(y)$, $y \in \mathbb{R}_+$,

$\gamma_{a, \lambda} I : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$,

$$\gamma_{a, \lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda + 1)} \int_0^\infty a(y/2) y^\lambda e^{-xy} dy;$$

in the elliptic case: $a = a(r)$, $r \in [0, 1)$, $\gamma_{a, \lambda} I : l_2 \to l_2$,

$$\gamma_{a, \lambda}(n) = \frac{1}{\mathcal{B}(n + 1, \lambda + 1)} \int_0^1 a(\sqrt{r}) (1 - r)^\lambda r^n dr;$$

in the hyperbolic case: $a = a(\theta)$, $\theta \in (0, \pi)$,

$\gamma_{a, \lambda} I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$,

$$\gamma_{a, \lambda}(\xi) = 2^\lambda (\lambda+1) \frac{|\Gamma(\frac{\lambda+2}{2} + i\xi)|^2}{\pi \Gamma(\lambda + 2) e^{\pi \xi}} \int_0^\pi a(\theta) e^{-2\xi \theta} \sin^\lambda \theta d\theta.$$
Spectra
Continuous symbols

Let $E$ be a subset of $\mathbb{R}$ having $+\infty$ as a limit point, and let for each $\lambda \in E$ there is a set $M_\lambda \subset \mathbb{C}$. Define the set $M_\infty$ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n \in \mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,
(ii) $\lim_{n \to \infty} \lambda_n = +\infty$,
(iii) $z = \lim_{n \to \infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda \to +\infty} M_\lambda,$$

and call $M_\infty$ the (partial) limit set of a family $\{M_\lambda\}_{\lambda \in E}$ when $\lambda \to +\infty$.

The a priori spectral information for $L_\infty$-symbols:

$$\text{sp} T_a^{(\lambda)} \subset \text{conv(ess-Range } a).$$
Given a symbol $a \in L_\infty(D)$, constant on cycles, the Toeplitz operator $T_{a}^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$. Thus

$$\text{sp } T_{a}^{(\lambda)} = \overline{M_{\lambda}(a)}, \quad \text{where } M_{\lambda}(a) = \text{Range } \gamma_{a,\lambda}.$$

**Theorem 10** Let $a$ be a continuous symbol constant on cycles. Then

$$\lim_{\lambda \to +\infty} \text{sp } T_{a}^{(\lambda)} = M_\infty(a) = \text{Range } a.$$

The set Range $a$ coincides with the spectrum $\text{sp } a I$ of the operator of multiplication by $a = a(y)$, thus the another form of the above is

$$\lim_{\lambda \to +\infty} \text{sp } T_{a}^{(\lambda)} = \text{sp } a I.$$
Two continuous symbol (both are hypocycloids)

\[ a_1(r) = \frac{3}{4}(r + i\sqrt{1 - r^2})^8 + (r - i\sqrt{1 - r^2})^4 \]

\[ a_2(\theta) = \frac{3}{4}e^{4i\theta} + e^{-2i\theta}. \]

The images of \( \gamma a_1,\lambda \) and \( \gamma a_2,\lambda \) for \( \lambda = 0 \).

The images of \( \gamma a_1,\lambda \) and \( \gamma a_2,\lambda \) for \( \lambda = 5 \).
The images of $\gamma_{a_1, \lambda}$ and $\gamma_{a_2, \lambda}$ for $\lambda = 12$.

The images of $\gamma_{a_1, \lambda}$ and $\gamma_{a_2, \lambda}$ for $\lambda = 200$. 
Piecewise continuous symbols

Let $a$ be a piecewise continuous symbol constant on cycles and having a finite number $m$ of jump points. Denote by $\bigcup_{j=1}^{m} I_j(a)$ the union of the straight line segments connecting the one-sided limit values of $a$ at the jump points. Introduce

$$\tilde{R}(a) = \text{Range } a \cup \left( \bigcup_{j=1}^{m} I_j(a) \right).$$

**Theorem 11** Let $a$ be a piecewise continuous symbol constant on cycles. Then

$$\lim_{\lambda \to \infty} \text{sp}_\lambda T^{(\lambda)}_a = M_\infty(a) = \tilde{R}(a).$$
Piecewise continuous symbol

\[ a(r) = \begin{cases} 
  e^{-i\pi r^2}, & r \in [0, 1/\sqrt{2}] \\
  e^{i\pi r^2}, & r \in (1/\sqrt{2}, 1]
\end{cases} \]

The sequence \( \gamma_{\alpha, \lambda} = \{\gamma_{\alpha, \lambda}(n)\} \) for \( \lambda = 0 \) and \( \lambda = 4 \).

The sequence \( \gamma_{\alpha, \lambda} = \{\gamma_{\alpha, \lambda}(n)\} \) for \( \lambda = 40 \) and \( \lambda = 200 \).
The symbol $a(\theta)$ and the function $\gamma_{a,\lambda}$ for $\lambda = 1$.

The function $\gamma_{a,\lambda}$ for $\lambda = 10$ and $\lambda = 70$.

The function $\gamma_{a,\lambda}$ for $\lambda = 500$ and the limit set $M_\infty(a)$. 
Oscillating symbols

\[ a_1(y) = (1+y)^i = e^{i \ln(1+y)} \quad \text{and} \quad a_2(y) = e^{iy}, \quad y \in [0, \infty) \]

The functions \( \gamma_{a_1, \lambda} \) and \( \gamma_{a_2, \lambda} \) for \( \lambda \) equals to 0, 10, and 1000.