KAM theory in the presence of monodromy

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- spherical pendulum
- monodromy
- local structure of a dynamical torus bundle
- Bohr quantisation
- KAM - Pöschel theory
- globalisation of KAM-Pöschel
- monodromy after perturbation

preprints: http://www.math.rug.nl/~broer/index.html#res

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Spherical pendulum

Classical mechanical system with

- configuration space $S^2 \subset \mathbb{R}^3$ or
  $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$
- state space $T(S^2)$ or
  $\{(p, v) \mid p \in S^2,\ v \perp p\}$
- Hamiltonian

$$H(p, v) = z(p) + \frac{1}{2}|v|^2$$

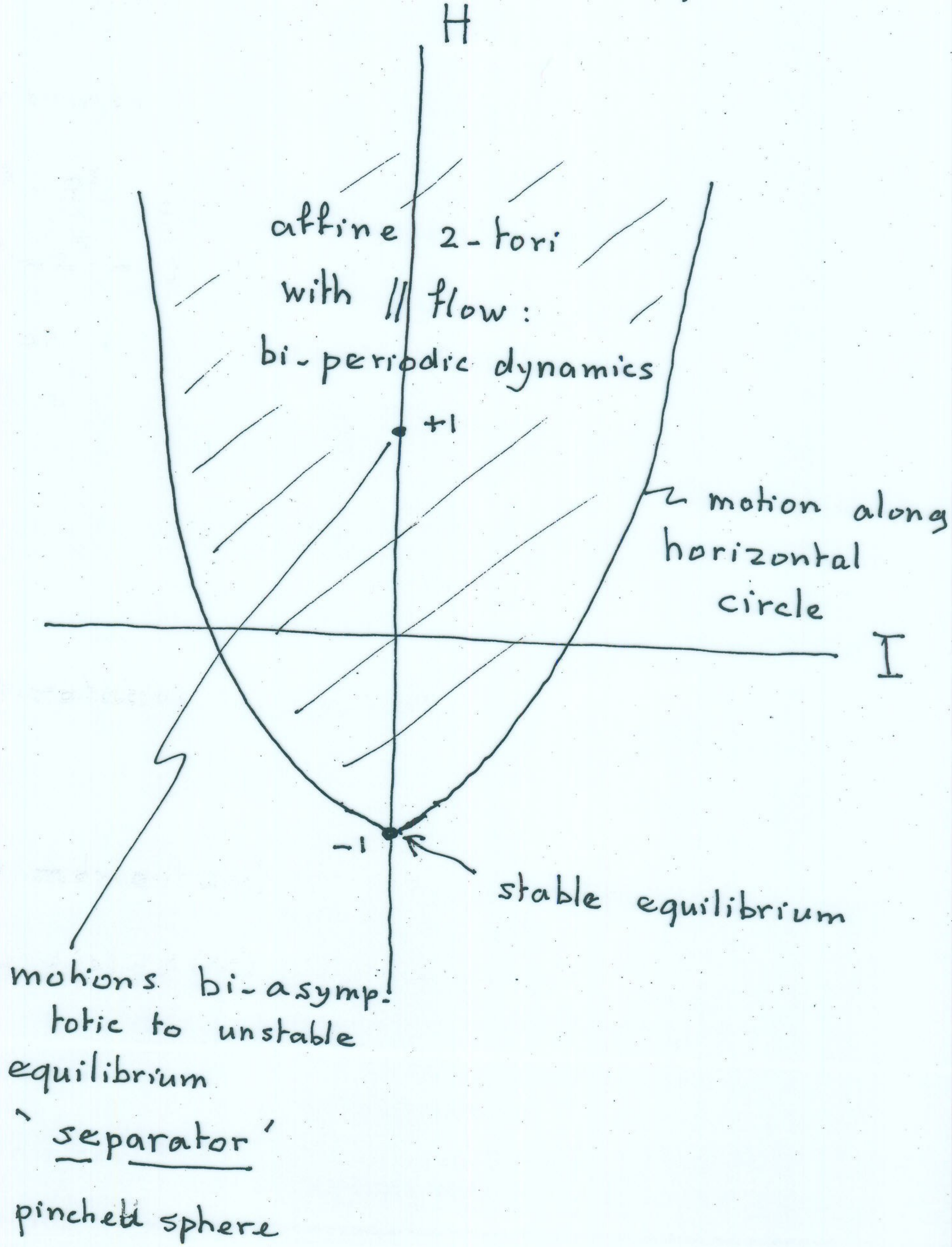
symmetry: with respect to rotation

around z-axis $\Longrightarrow$

first integral (angular momentum)

$$I(p, v) = x(p) \cdot y(v) - y(p) \cdot x(v)$$

Energy - momentum map



Monodromy (Duistermaat, 1980)

The T^2-bundle over the interior of

$$(I, H)(T(S^2)) \setminus \{0, 1\}$$

is not trivial, i.e. is not globally a product

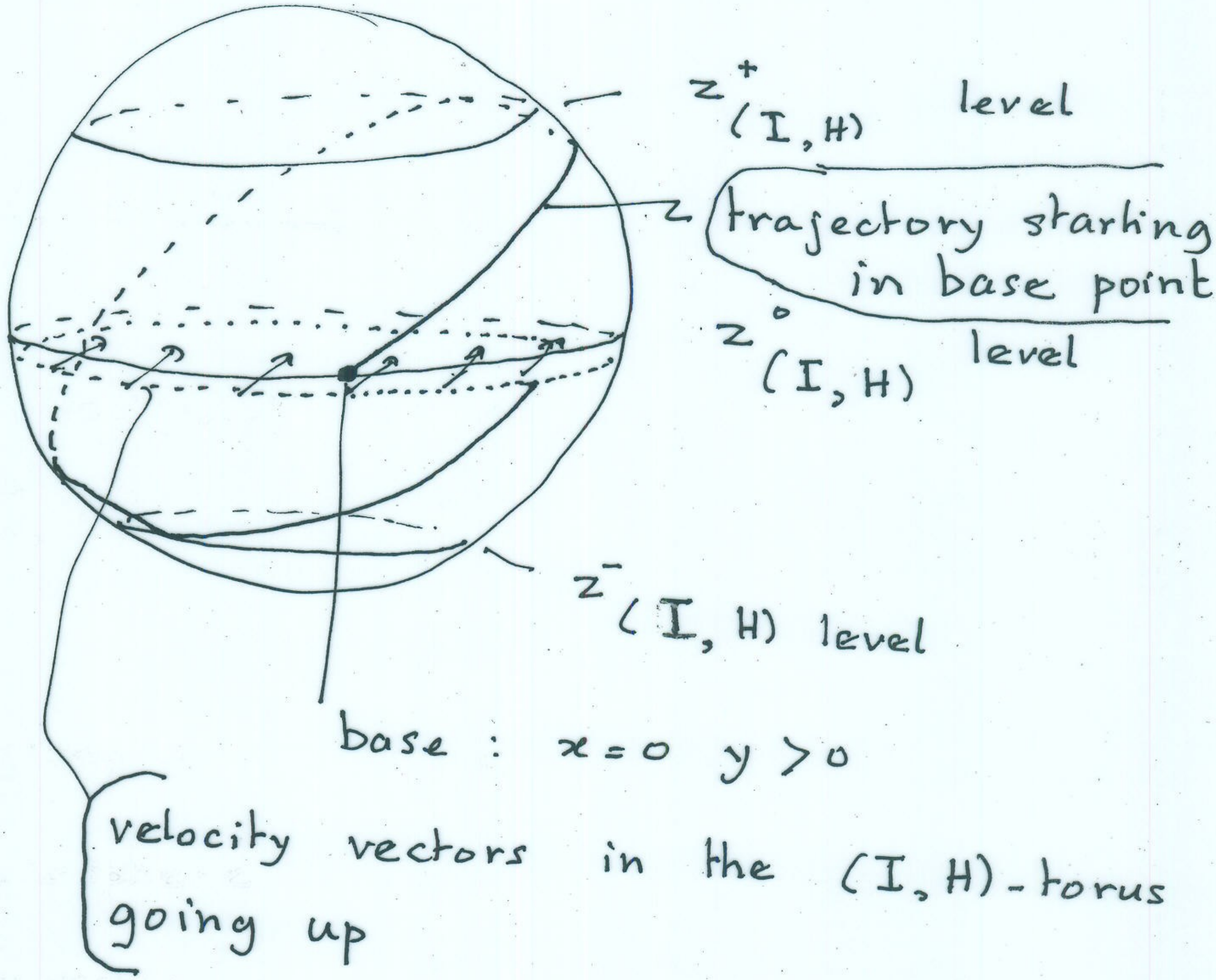
Try to construct a basis of π_1(tori) depending continuously on (I, H)"

For (I, H) in the torus-region the motions in the 2-sphere are going up and down between two levels $z^+_{(I,H)}$ and $z^-_{(I,H)}$;

$$z^-_{(I,H)} < z^+_{(I,H)}$$

define $z^0_{(I,H)}$ "=" $\frac{1}{2}\left(z^-_{(I,H)} + z^+_{(I,H)}\right)$

Monodromy (cont.)



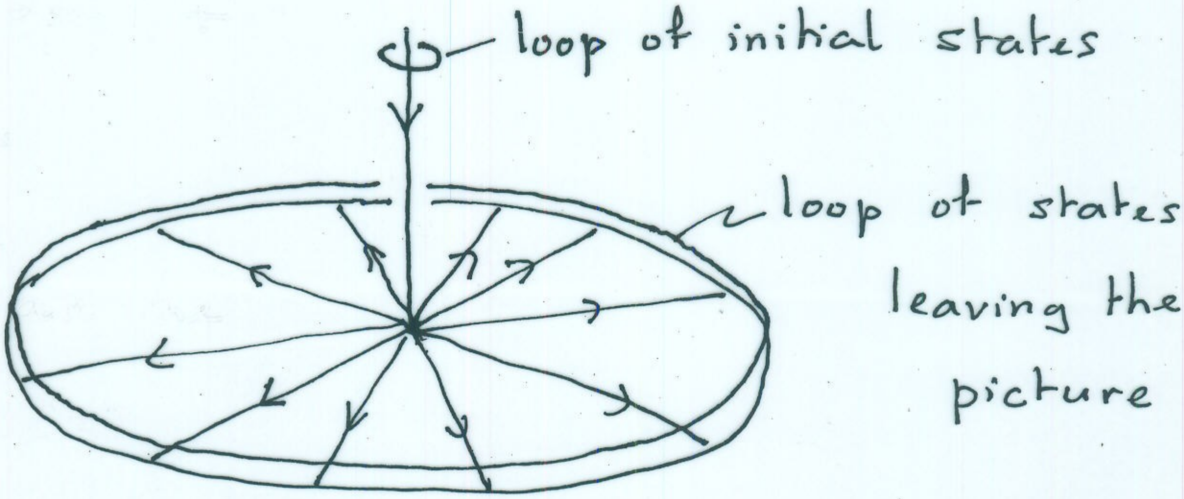
- Construction of one basic loop (+ base point) in (I,H)-torus
- second basic loop: take the curve, defined by the dynamics, starting at the base point and 'connect back' <u>non uniqueness</u>

Monodromy (cont.')

Do this construction for $\{(I(s), H(s))\}$ describing a small loop around $(0,1)$

Then the 'end-point' of the trajectory starting at the base point will go once around the equator (hence no 'connection back' possible depending continuously on (I, H) $\Rightarrow$ monodromy)

Reason: dynamics near saddle with 2-dim unstable manifold



Local structure of dynamical torus bundle[+]
(Liouville - Arnold)

$\exists$ local coordinates $p_1, p_2 \in \mathbb{R}/\mathbb{Z}$ and $q_1, q_2 \in \mathbb{R}$ such that

— the canonical 2-form is $\Omega = \sum_i dp_i \wedge dq_i$

— H and I are functions of q_1, q_2 alone ($\dfrac{\partial(H,I)}{\partial(q_1,q_2)}$ has max. rank

Dynamics is given by

$$\dot{p}_i = \frac{\partial H}{\partial q_i} = \omega_i(q) \quad \& \quad \dot{q}_i = 0$$

$\exists$ fundamental 1-form ϑ with $d\vartheta = \Omega$
(think of $\vartheta = -\sum q_i\, dp_i$)

Bohr quantisation conditions (1918)

A torus T_q satisfies the Bohr condition if for each closed loop γ in T_q

$$\frac{1}{h}\oint_\gamma \sigma \quad \text{is an integer}$$

There are $\alpha_1, \alpha_2 \in \mathbb{R}$ such that the Bohr-tori are at

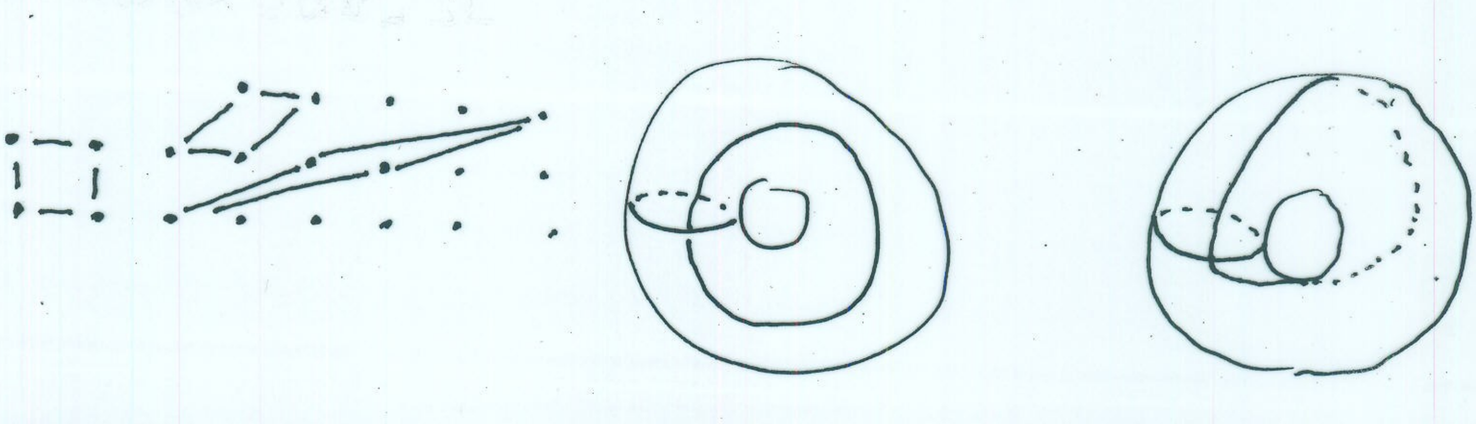
$$q_1 = n_1 h + \alpha_1 \quad \text{and} \quad q_2 = n_2 h + \alpha_2$$

$$n_1, n_2 \in \mathbb{N}$$

increasing q_1 or q_2 by h, increases $\frac{1}{h}\oint_\gamma \sigma$ by -1 if γ is the loop in the p_1 or p_2 direction

correspondence

fundamental domains



Monodromy as an imperfection of the eigenvalue spectrum of (H, I)

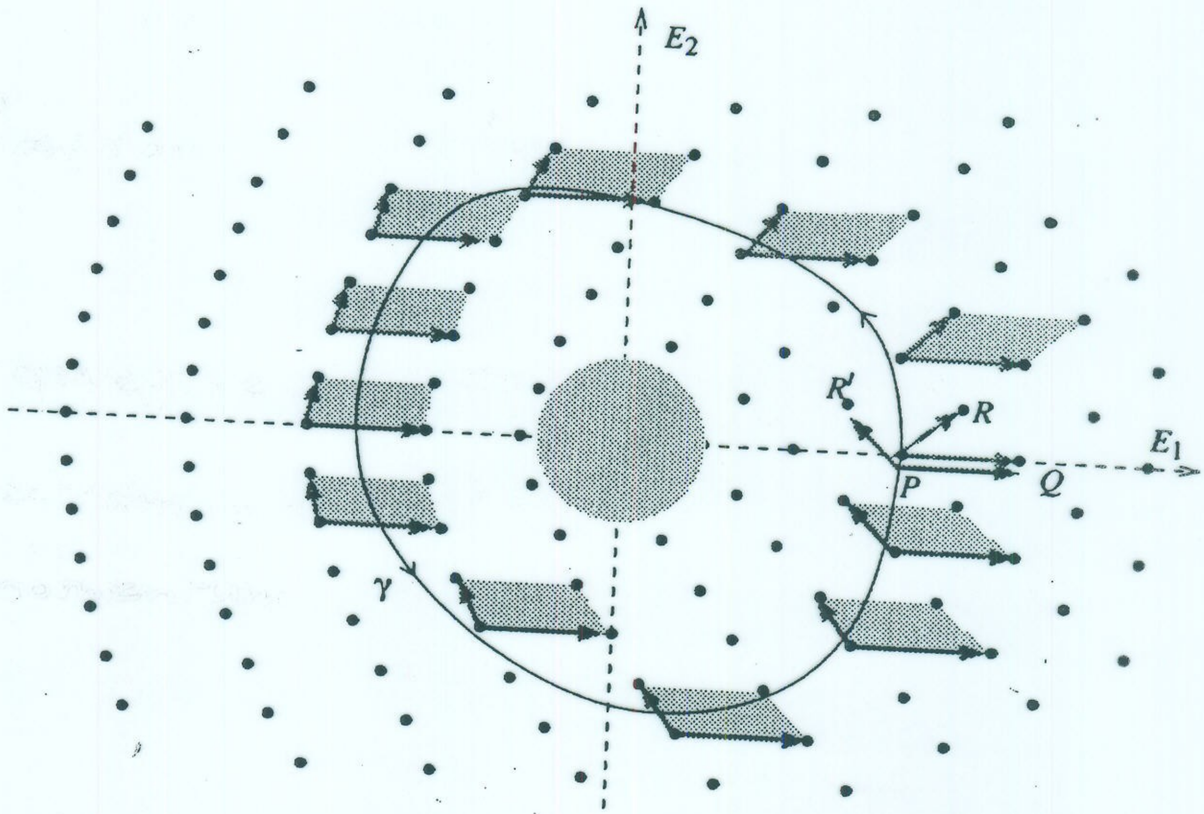
Bohr-tori define a 'deformed' latice in the 'torus part' of the H, I plane

Monodromy is visible as imperfection

For small h, the Bohr conditions give a good description of the joint spectrum of the commuting energy and momentum operators (in the new Q.M.)

details:

S. Vũ Ngọc, Comm. Math. Phys. 1999



E_2
E_1
R'
R
P
Q
γ

KAM – Pöschel theory (return to local structure of dynamical torus bundle)

coordinates $p_i \in \mathbb{R}/\mathbb{Z}$, $q_i \in \mathbb{R}$; $\Omega = \sum dp_i \wedge dq_i$

$H(q_1, q_2)$ $\quad \omega_i = \frac{\partial H}{\partial q_i}$

we consider small perturbations $\breve{H}$ of H (not preserving the rotation symmetry) $\Longrightarrow$

dynamics: $\quad \dot{p}_i = \frac{\partial \breve{H}}{\partial q_i}$ and $\dot{q}_i = -\frac{\partial \breve{H}}{\partial p_i}$

what remains of the conditionally periodic tori?

Diophantine condition

(ω_1, ω_2) is (γ, τ) – Diophantine iff

$$|k_1\omega_1 + k_2\omega_2| \geq \gamma \cdot (|k_1| + |k_2|)^{-\tau}$$

$\forall\ (0,0) \neq (k_1, k_2) \in \mathbb{Z}^2$

for $\tau > 1$ and γ small the set of (γ, τ) – Diophantine (ω_1, ω_2) have big measure in $\mathbb{R}^2$ (measure of complement, intersected with compactum, $\searrow 0$ for $\gamma \searrow 0$)

- Given $K \subset \mathbb{R}^2$, (γ, τ), C^∞-nbd $\mathcal{U}$ of identity in $\mathrm{Diff}^\infty(\mathbb{R}^2 \times T^2)$
- $\exists$ a C^∞-nbd V of H such that $\check{H} \in V \Rightarrow$
- $\exists$ $\Phi \in \mathcal{U}$ such that for each $q \in K$, with $(\omega_1(q), \omega_2(q))$ (γ, τ)-Diophantine, Φ conjugates T_q to an $\check{H}$-invariant torus

Globalization of KAM - Pöschel
(Rink (2004), Broer - Cushman - Fassò - Takens, to appear)

The Pöschel construction can only be done on (open) subsets with product structure — so for spherical pendulum one has to 'glue' local results

question: are the perturbed Diophantine tori independent of the coordinates used in the construction?

this turns out to be true, except possibly for a measure - zero set of these tori

$K \subset \mathbb{R}^n$ closed

Reconstruction of monodromy after perturbation

M is 4-manifold (phase space) with compact $K \subset M$ and metric ρ

$\{T_\gamma\}_{\gamma\in\Lambda}$ family of 2-tori in M such that:

- $\forall x \in K$ $\exists$ γ such that $\rho(x, T_\gamma) < \frac{\varepsilon}{2}$
- if $\rho(T_\gamma, T_{\gamma'}) < \varepsilon$ $\exists$ homeo $h_{\gamma'\gamma} : T_\gamma \to T_{\gamma'}$ with $\rho(x, h_{\gamma'\gamma}(x)) < 2\varepsilon$ $\forall$ $x \in T_\gamma$
- any homeo $h : T_\gamma \circlearrowleft$ with $\rho(x, h(x)) < 6\varepsilon$ $\forall$ x T_γ is homotopic to the identity

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Bundle over $\cup T_\gamma$ with fibres $F_x = H_1(T_\gamma)$ $(x \in T_\gamma)$ can be extended in a <u>unique</u> way over K such that for each $x \in (\cup T_\gamma) \cap K$ it's ε neighbourhood is trivial, compatible we the actions of $h_{\gamma'\gamma}$ on H_1

Monodromy corresponds to non-triviality of such a $\mathbb{Z}^2$ bundle