On Organizing Principles of Discrete Differential Geometry. Geometry of Spheres

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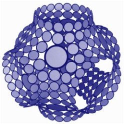
LMS Durham Symposium “Methods of Integrable Systems in Geometry", August 11-21, 2006



DFG Research Unit 565 “Polyhedral Surfaces”

Discrete Differential Geometry

- **Aim:** Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.
- **Question:** Which discretization is the best one?
  - (Theory): preserves fundamental properties of the smooth theory
  - (Applications): represent smooth shape by a discrete shape with just few elements; best approximation

This talk

- Discretization Principles
- Survey
- New results on discrete curvature line parametrized surfaces in Lie, Laguerre and Möbius geometry, joint with Yu.B. Suris `[arXiv:math.DG/0608291]`

Transformation Group Principle

Smooth geometric objects and their discretizations belong to the same geometry, i.e. are invariant with respect to the same transformation group

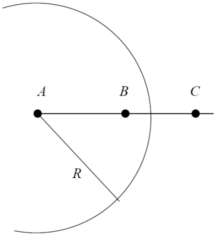
$\Rightarrow$ discrete Klein's Erlangen Program

Möbius transformations

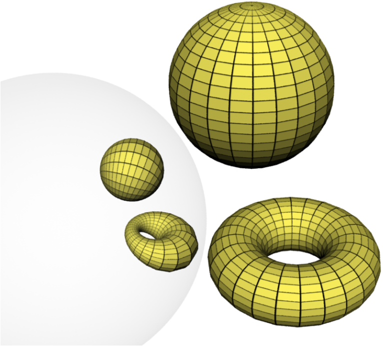
Definition. A Möbius transformation in $\mathbb{R}^3$ is a composition of reflections in spheres.

Properties.

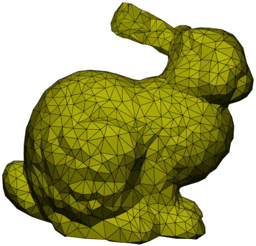
- Conformal
- preserve spheres
- Willmore energy $\mathcal{W} = \frac{1}{4}\int(k_1 - k_2)^2$



$|AB||AC| = R^2$



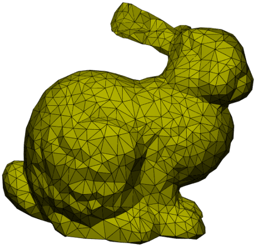
Discrete surfaces in Euclidean and Möbius geometries



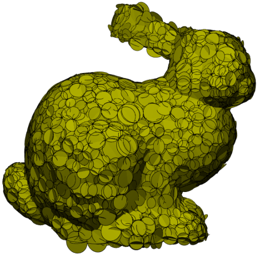
Euclidean geometry

Möbius geometry

Discrete surfaces in Euclidean and Möbius geometries

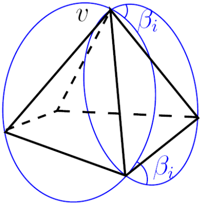


Euclidean geometry



Möbius geometry

Discrete Willmore energy for simplicial surfaces



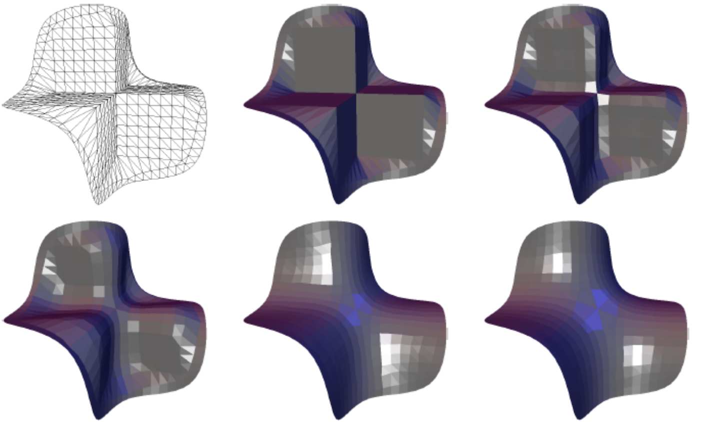
[B. '04] Theory
[B., Schröder '05] Applications

Definition. $W(v) = \sum_i \beta_i - 2\pi$

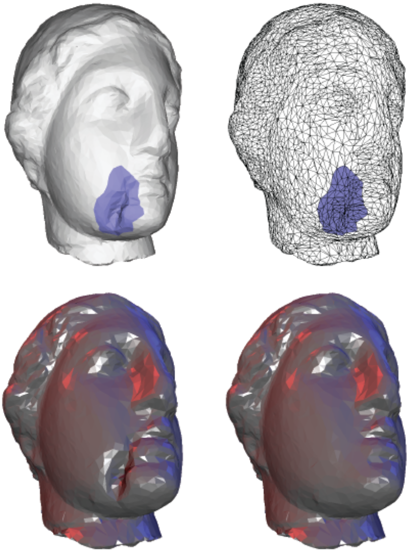
$$W(S) = \frac{1}{2}\sum_{v\in V} W(v) = \sum_{e\in E}\beta(e) - \pi|V|$$

- Möbius invariant
- $W(S) \geq 0$, and $W(S) = 0$ iff S is a spherical convex polyheder
- The discrete Willmore energy W approximates the smooth Willmore energy $\frac{1}{4}\int(k_1 - k_2)^2$; special smooth limit

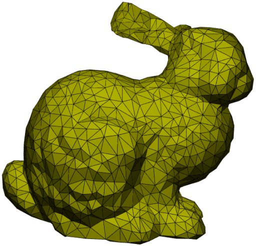
Application. Geometric fairing

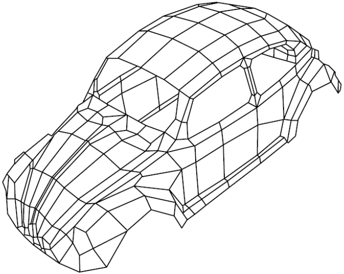


Application. Geometric C^1-surface restoration



Quadrilateral Surfaces

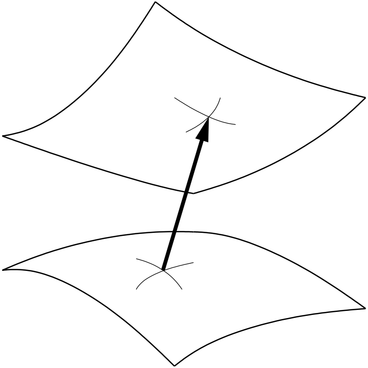




Quadrilateral surfaces as discrete parametrized surfaces

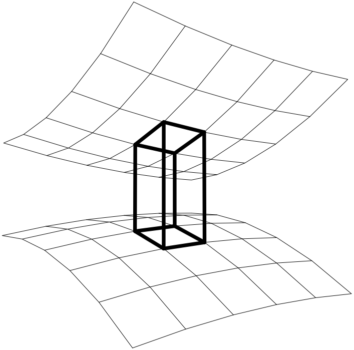
Surfaces and transformations

Classical theory of (special classes of) surfaces (constant curvature, isothermic, etc.)



special transformations (Bianchi, B"acklund, Darboux)

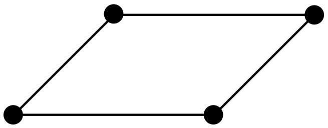
General and special Quad-surfaces



discrete $\rightarrow$ symmetric

Basic idea

Do not distinguish discrete surfaces and their transformations.
Discrete master theory.



Example - planar quadrilaterals as discrete conjugate systems.
Multidimensional Q-nets [Doliwa, Santini '97].

Basic idea

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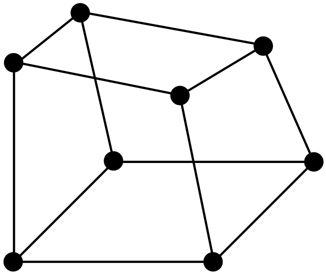
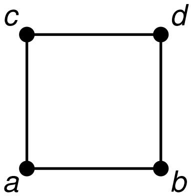
Discrete master theory.



Example - planar quadrilaterals as discrete conjugate systems.
Multidimensional Q-nets [Doliwa, Santini '97].

Integrability as Consistency

- Equation



$$f(a,b,c,d)=0$$

- Consistency

Integrability as Consistency

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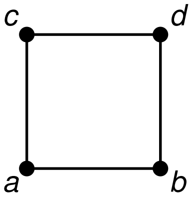
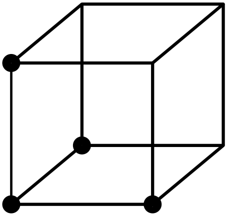


$$f(a, b, c, d) = 0$$

- Consistency



Integrability as Consistency

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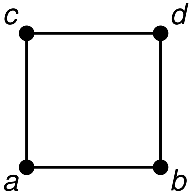

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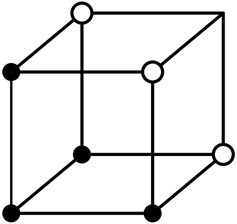
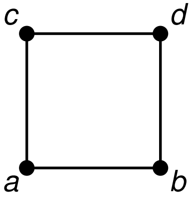


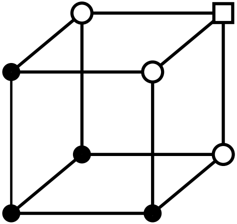
Integrability as Consistency

- Equation



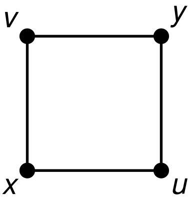
$$f(a, b, c, d) = 0$$

- Consistency



Discrete integrable systems 2D

- Equation



$Q(x, y, u, v) = 0$, Q affine with respect to all variables, square symmetry

- Consistency $\Rightarrow$ Integrability
- Lax representation, Darboux transformation [Adler, B., Suris '03], [Nijhoff '02]

Discrete integrable systems 2D. Classification

[Adler, B., Suris '03]

(Q1) $\alpha(x-v)(u-y)-\beta(x-u)(v-y)+\delta^2\alpha\beta(\alpha-\beta)=0,$

(Q2)

$$\alpha(x-v)(u-y)-\beta(x-u)(v-y)+\alpha\beta(\alpha-\beta)(x+y+u+v)$$
$$-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0,$$

(Q3)
$$\sin(\alpha)(xu+vy)-\sin(\beta)(xv+uy)-\sin(\alpha-\beta)(xy+uv)$$
$$+\delta^2\sin(\alpha-\beta)\sin(\alpha)\sin(\beta)=0,$$

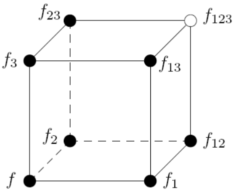
(Q4)
$$\mathrm{sn}(\alpha)(xu+vy)-\mathrm{sn}(\beta)(xv+uy)-\mathrm{sn}(\alpha-\beta)(xy+uv)$$
$$+\mathrm{sn}(\alpha-\beta)\mathrm{sn}(\alpha)\mathrm{sn}(\beta)(1+k^2xyuv)=0,$$

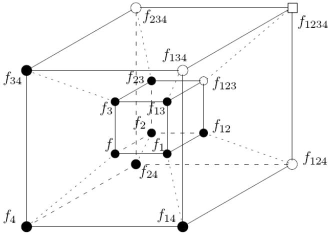
(H1) $(x-y)(u-v)+\beta-\alpha=0,$

(H2) $(x-y)(u-v)+(\beta-\alpha)(x+y+u+v)+\beta^2-\alpha^2=0,$

(H3) $\alpha(xu+vy)-\beta(xv+uy)+\delta(\alpha^2-\beta^2)=0$

Discrete integrable systems 3D





$Q(f, \ldots, f_{123}) = 0$, Q affine with respect to all variables, cube symmetry.
[Wolf, Tsarev, B.] (preliminary) Classification of 3D consistent systems.

Discretization Principles

- **Transformation Group Principle.** Smooth geometric objects and their discretizations belong to the same geometry, i.e. are invariant with respect to the same transformation group
  (discrete Klein's Erlangen Program)
- **Consistency Principle.** Discretizations of smooth parametrized geometries can be extended to multidimensional consistent nets
  (Integrability)

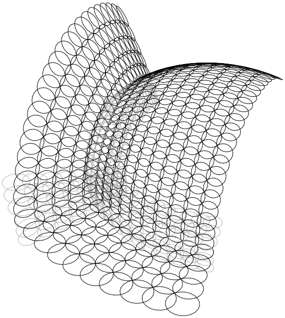
Discretization of classical geometries

Consistency principle can be imposed for discretization of classical geometries (Möbius, Laguerre, Lie,...):

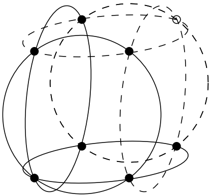
- transformation groups of various geometries (Möbius, Laguerre, Lie,...) are subgroups of the projective transformation group preserving absolute (distinguished quadric),
- multidimensional Q-nets (projective geometry) can be restricted to an arbitrary quadric [Doliwa '99].

Circular nets

Martin, de Pont, Sharrock ['86], Nutbourne ['96], B. ['96], Cieslinski, Doliwa, Santini ['97], Konopelchenko, Schief ['98], Akhmetishin, Krichever, Volvovski ['99], ...



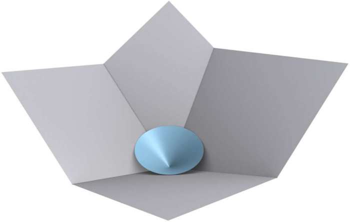
three “coordinate nets” of a discrete orthogonal coordinate system



elementary cube
→ Miquel theorem

Conical nets

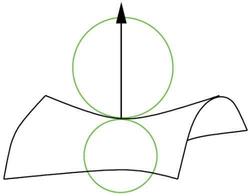
Conical nets as discrete curvature line parametrizations [Liu, Pottmann, Wallner, Yang, Wang '06]



- **Definition.** Neighboring quads touch a common cone of revolution (in particular intersect at the tip of the cone)
- Conical net $\Leftrightarrow$ circular Gauss map
- Normal shift
- Consistency

Curvature lines through spheres

Pencil of touching spheres



Principal directions are invariant with respect to:

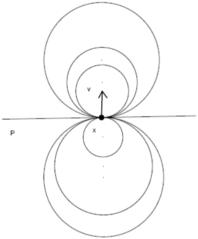
- Möbius transformations
- normal shift

Curvature lines belong to Lie geometry.

Möbius, Laguerre, Lie geometries

Lie geometry.

Lie sphere transformations: oriented spheres (including points and planes) mapped to oriented spheres preserving the oriented contact of sphere pairs



	distinguished	surfaces through	transformations
Möbius	points	points	Möbius
Laguerre	planes	tangent planes	normal shift ...
Lie	none	contact elements	Lie sphere

Lie quadric

$$\mathbb{L}^{N+1,2} = \{\xi \in \mathbb{R}^{N+1,2} : \langle \xi, \xi \rangle = 0\}.$$

Basis

$$\mathbf{e}_1, \dots, \mathbf{e}_N, \mathbf{e}_0, \mathbf{e}_\infty, \mathbf{e}_{N+3}, \|\mathbf{e}_i\| = 1, \|\mathbf{e}_{N+3}\| = -1, \|\mathbf{e}_0\| = \|\mathbf{e}_\infty\| = 0.$$

- Oriented hypersphere with center $c \in \mathbb{R}^N$ and signed radius $r \in \mathbb{R}$:

$$\hat{s} = c + \mathbf{e}_0 + (|c|^2 - r^2)\mathbf{e}_\infty + r\mathbf{e}_{N+3}.$$

- Oriented hyperplane $\langle v, x \rangle = d$ with $v \in \mathbb{S}^{N-1}$ and $d \in \mathbb{R}$:

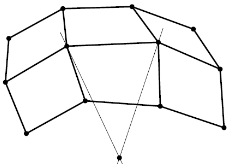
$$\hat{p} = v + 0 \cdot \mathbf{e}_0 + 2d\mathbf{e}_\infty + \mathbf{e}_{N+3}.$$

- Point $x \in \mathbb{R}^N$: $\hat{x} = x + \mathbf{e}_0 + |x|^2\mathbf{e}_\infty + 0 \cdot \mathbf{e}_{N+3}$.
- Infinity ∞: $\hat{\infty} = \mathbf{e}_\infty$.
- Contact element (x, p): $\mathrm{span}(\hat{x}, \hat{p}) = \ell \subset \mathbb{L}$, line.

Line congruence

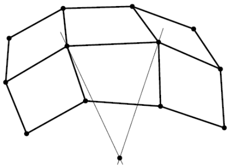
[Doliwa, Santini, Manas '00]

- neighboring lines intersect
- focal surfaces are Q-nets
- consistent



Discrete curvature line parametrization

- Line congruences can be restricted to (Lie) quadric
- Literal discretization of Blaschke's Lie geometric description of smooth curvature line parametrized surfaces

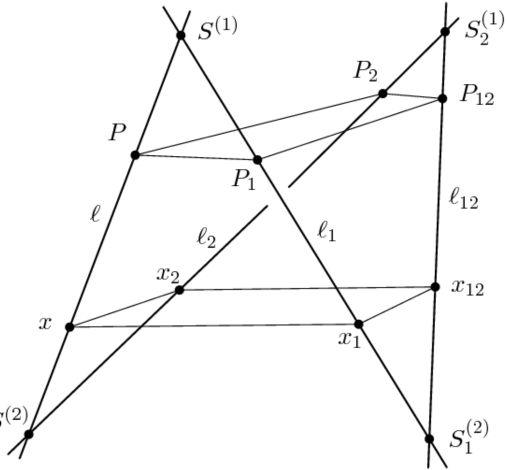


Definition. Discrete curvature line parametrization is a discrete congruence of isotropic lines

$$\ell : \mathbb{Z}^2 \to \{\text{isotropic lines in } \mathbb{L}\}$$

such that neighboring lines intersect

Curvature line net. Projective model

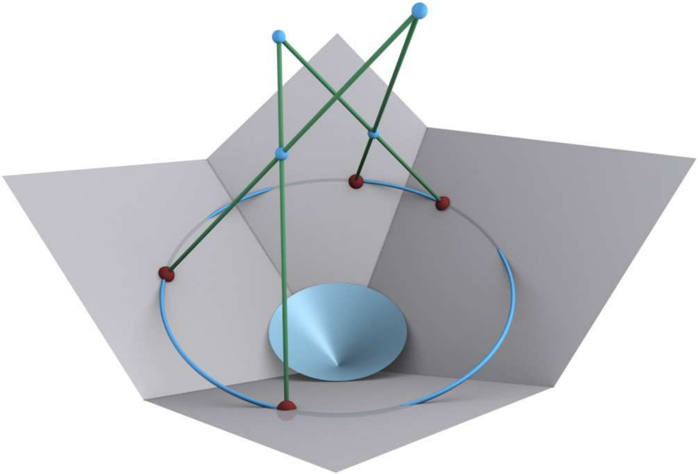


LAGUERRE

LIE

MÖBIUS

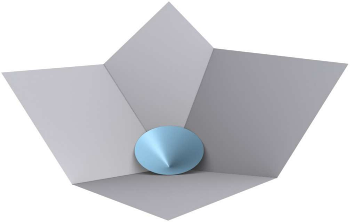
Curvature line net. Euclidean model



Circular and conical nets. Relation

[B., Suris '06], [Pottmann '06]

- Given a conical net p there exists a two-parameter family of circular nets x such that (x, p) is curvature line parametrized
- Given a circular net x there exists a two-parameter family of conical nets p such that (x, p) is curvature line parametrized



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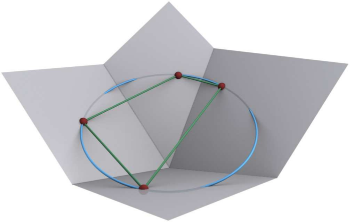

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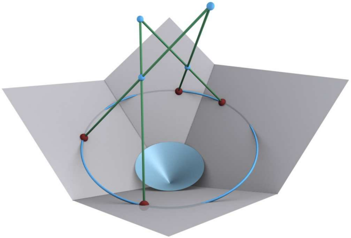
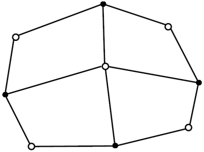


Ribaucour congruence

- a discrete **R-congruence** of spheres - Q-net in Lie quadric
- **Ribaucour transformation** of discrete curvature line parametrized surfaces - any two corresponding contact elements have a sphere in common
- Spheres of a Ribaucour transformation build an R-congruence
- To any elementary quadrilateral of a discrete R-congruence (whose spheres span a subspace of the signature (2,1)) there corresponds a Dupin cyclide
- Permutability of Ribaucour transformations

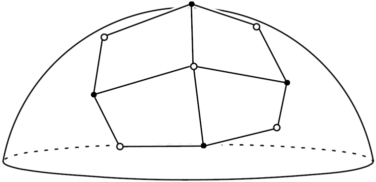
Permutability of Ribaucour transformations in Lie geometry of smooth surfaces [Burstall, Hertrich-Jeromin '05]

Special surfaces. Projective isothermic nets



- T-net (trapezoidal, parallel diagonals) in $\mathbb{R}^N$
- consistent
- (equivalent) Moutard nets [Nimmo, Schief '97] via Moutard equation $f_{ij} + f = a_{ij}(f_j + f_i)$
- Projective characterization: five "diagonal" points lie in a three-space ([Doliwa '05])

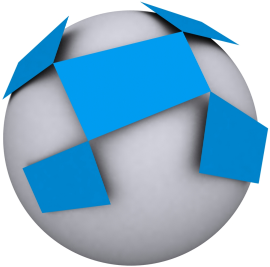
Special surfaces. Möbius isothermic nets



For discrete isothermic nets ($\mathbb{S}^2$ is a special case):

- (Möbius) isothermic net: Five “diagonal” points lie on a common sphere
- T-net in the light cone $\mathbb{L}^{N+1,1}$
- Equivalent to the cross-ratio definition of [B., Pinkall ’93]

Special surfaces. Laguerre isothermic nets



- **Definition.** Five “diagonal” planes have a common touching sphere
- Laguerre isothermic net $\Leftrightarrow$ isothermic Gauss map in $\mathbb{S}^2$