Integrable equations of the dispersionless Hirota type and hypersurfaces of the Lagrangian Grassmanian

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Collaboration

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$$\boxed{F(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt}) = 0}$$

Examples

$$u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy} \qquad \text{dKP}$$

$$u_{xx} + u_{yy} = e^{u_{tt}} \qquad \text{Boyer-Finley}$$

$$e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}}$$

$$(\alpha-\beta)e^{u_{xy}} + (\beta-\gamma)e^{u_{yt}} + (\gamma-\alpha)e^{u_{tx}} = 0 \qquad \text{dHirota}$$

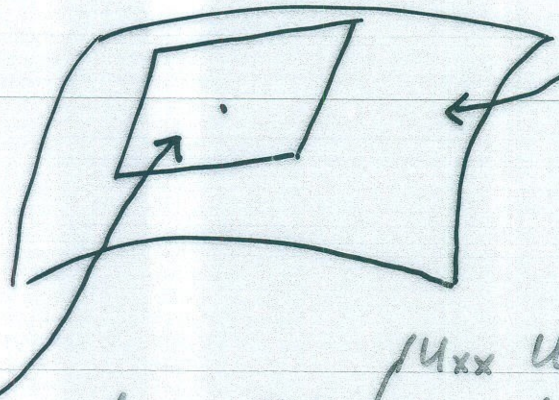
$$u_{xy}u_{zt} - u_{xt}u_{zy} = 1 \qquad \text{heavenly}$$

Integrability $\equiv$ existence of hydrodynamic reductions.

Classification? Geometry?

Geometric picture

$x = \begin{pmatrix} x \\ y \\ t \end{pmatrix}$, $p = \begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix}$ Lagrangian submanifold



$dp = V\,dx$, $V = \begin{pmatrix} u_{xx} & u_{xy} & u_{xt} \\ u_{xy} & u_{yy} & u_{yt} \\ u_{xt} & u_{yt} & u_{tt} \end{pmatrix}$ tangent plane

V – Gaussian image of the tangent plane in the Lagrangian Grassmanian Λ^6.

Equation $\boxed{F = 0}$ – hypersurface $M^5 \subset \Lambda^6$

Solutions – Lagrangian submanifolds whose Gaussian image belongs to M^5.

$Sp(6)$ – equivalence group of the problem

Classification of equations $\boxed{F=0}$ up to $Sp(6)$ - equivalence

$$|||$$

$Sp(6)$-geometry of hypersurfaces $M^5 \subset \Lambda^6$

Example: dKP

$$u_{xt} - \frac{1}{2} u_{xx}^2 = u_{yy}$$

New variables

$$u_{xx} = a, \quad u_{xy} = b, \quad u_{xt} = p, \quad u_{yy} = p - \frac{1}{2} a^2$$

Quasilinear form

$$a_y = b_x, \quad a_t = p_x, \quad b_t = p_y, \quad b_y = p_x - a a_x$$

Hydrodynamic reduction

$$a(R^1, \dots, R^n), \quad b(R^1, \dots, R^n), \quad p(R^1, \dots, R^n)$$

where

$$R^i_t = \lambda^i(R) R^i_x, \qquad R^i_y = \mu^i(R) R^i_x$$

commutativity conditions

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j$$

substitution implies

$$\partial_i b = \mu^i \partial_i a, \quad \partial_i p = \lambda^i \partial_i a, \qquad \lambda^i = {\mu^i}^2 + a$$

Equations for μ^i, a (Gibbons–Tsarev system)

$$\boxed{\partial_j \mu^i = \frac{\partial_j a}{\mu^j - \mu^i}, \qquad \partial_i \partial_j a = 2 \frac{\partial_i a \, \partial_j a}{(\mu^i - \mu^j)^2}}$$

In involution! General solution depends on n arbitrary functions of 1 variable.

Generalised dKP

$$u_{xt} - f(u_{xx}) = u_{yy}$$

$\cdots\cdots$

Generalized Gibbons-Tsarev system

$$\partial_j \mu^i = f''(a)\frac{\partial_j a}{\mu^j - \mu^i}, \quad \partial_i \partial_j a = 2f''(a)\frac{\partial_i a\, \partial_j a}{(\mu^j - \mu^i)^2}$$

Involutivity $\equiv \boxed{f''' = 0}$

In general:

$$u_{tt} = f(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt})$$

$$\Downarrow$$

Generalized Gibbons-Tsarev system

$$\partial_j \mu^i = (\dots)\,\partial_j a, \quad \partial_i \partial_j a = (\dots)\,\partial_i a\, \partial_j a$$

$$\Downarrow$$

Compatibility conditions

$$\boxed{f_{ijk} = (f, df, d^2 f)}$$

In involution!

Solutions depend on 21 parameters.

Partial classification results

Equations of the form $u_{tt} = f(u_{xx}, u_{yy})$

Integrability conditions

$$f_{aaa} = f_{aa}\left(\frac{f_{ac}}{f_c} + \frac{f_{aa}}{f_a}\right), \qquad f_{aac} = f_{aa}\left(\frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a}\right)$$

$$f_{acc} = f_{ac}\left(\frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_c}\right), \qquad f_{ccc} = f_{cc}\left(\frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a}\right)$$

$$f_{aa} f_{cc} = f_{ac}^2$$

Canonical forms

$$e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}}, \qquad u_{xx} + u_{yy} = e^{u_{tt}}$$

Equations of the form $u_{xy} = f(u_{xt}, u_{yt})$

Integrability conditions

$$f_{ppp} = f_{pp}\left(\frac{f_{pq}}{f_q} + \frac{f_{pp}}{f_p}\right), \qquad f_{ppq} = f_{pp}\left(\frac{f_{qq}}{f_q} + \frac{f_{pq}}{f_p}\right)$$

$$f_{pqq} = f_{qq}\left(\frac{f_{pq}}{f_q} + \frac{f_{pp}}{f_p}\right), \qquad f_{qqq} = f_{qq}\left(\frac{f_{qq}}{f_q} + \frac{f_{pq}}{f_p}\right)$$

Canonical forms

$$e^{u_{xt}} + e^{u_{xy}} + e^{u_{yt}} = e^{u_{xt} + u_{xy} + u_{yt}},$$

$$u_{xy} = u_{xt} + e^{u_{yt}}, \qquad u_{xy} = u_{xt} \tanh(u_{yt}),$$

$$u_{xy} = u_{xt} u_{yt}$$

Equations of the form $u_{tt} = f(u_{xx}, u_{xt}, u_{xy})$

Integrability conditions

$$f_{\beta\beta\beta} = 2\frac{f_{\beta\beta}^2}{f_\beta}, \quad f_{\alpha\beta\beta} = 2\frac{f_{\alpha\beta} f_{\beta\beta}}{f_\beta}, \quad f_{p\beta\beta} = 2\frac{f_{p\beta} f_{\beta\beta}}{f_\beta}$$

$$f_{\alpha\alpha\beta} = 2\frac{f_{\alpha\beta}^2}{f_\beta}, \quad f_{\alpha p\beta} = 2\frac{f_{\alpha\beta} f_{p\beta}}{f_\beta}, \quad f_{pp\beta} = 2\frac{f_{p\beta}^2}{f_\beta}$$

+ 4 more complicated equations.

Canonical forms

$$u_{tt} = u_{xy} + \frac{1}{4A}(Au_{xt} + 2Bu_{xx})^2 + Ce^{-Au_{xx}}$$

$$u_{tt} = \frac{u_{xy}}{u_{xx}} + \left(\frac{1}{u_{xx}} + \frac{A}{4u_{xx}^2}\right)u_{xt}^2 + \frac{B}{u_{xx}^2}u_{xt} + \frac{B^2}{Au_{xx}^2} + Ce^{A/u_{xx}}$$

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}\eta(u_{xx})u_{xt}^2,$$

here η solves the Chazy equation $\eta''' + 2\eta\eta' = 3\eta'$

$$u_{tt} = \ln u_{xy} - \ln\Theta(u_{xt}, u_{xx}) - \frac{1}{4}\int^{u_{xx}} \eta(\tau)\,d\tau$$

here Θ is a theta-function

Symplectic Monge-Ampere equations

$$U = \begin{pmatrix} u_{xx} & u_{xy} & u_{xt} \\ u_{xy} & u_{yy} & u_{yt} \\ u_{xt} & u_{yt} & u_{tt} \end{pmatrix}$$

$$M_3 + M_2 + M_1 + M_0 = 0$$

$M_3 = \det U, \; \dots, \; M_0 = \text{const.}$

3 orbits up to $Sp(6)$ action:

1. $\operatorname{Hess} u = 1$ (affine spheres)
2. $\operatorname{Hess} u = \Delta u$ (special Lagrangian 3-folds)
3. linear equations

Integrability $\equiv$ linearizability.

No longer true in dim 4!

$$u_{xt} u_{yz} - u_{xz} u_{yt} = 1$$

Geometry of the Lagrangian Grassmanian

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}$$

Action of $Sp(6)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(6) \Rightarrow \tilde{U} = (AU+B)(CU+D)^{-}$$

$$\det(d\tilde{U}) = (\cdots)\det(dU)$$

<u>Theorem</u>: The group of conformal automorphisms of the symmetric cubic form $\det(dU)$ is isomorphic to $Sp(6)$.

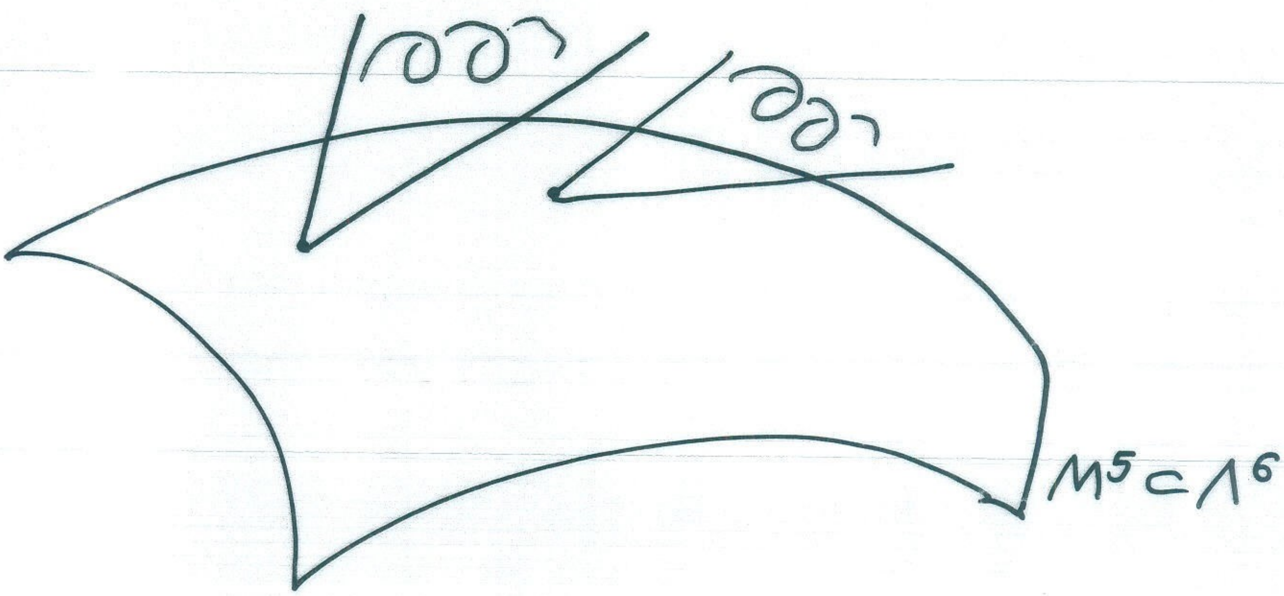
Objects in $PT(\Lambda^6) \equiv P^5$

cubic hypersurface $\det(dU) = 0$.

The set of its singular points is the Veronese surface $V^2 \subset P^5$.

Geometry of hypersurfaces $M^5 \subset \Lambda^6$

The intersection of TM^5 with the Veronese surface V^2 is a rational normal curve γ. $(1 : t : t^2 : t^3 : t^4)$



Each tangent space to M^5 carries a rational normal curve.

Conformal $SL(2)$-structure?

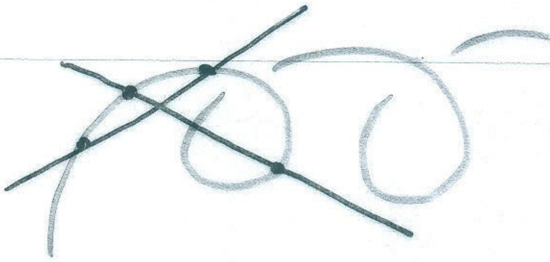
Veronese structure?

Paraconformal structure?

Geometry of a rational normal curve

$$(1 : t : t^2 : t^3 : t^4) \subset P^4$$
$$X^0 \quad X^1 \quad X^2 \quad X^3 \quad X^4$$

<u>Bisecant variety</u>

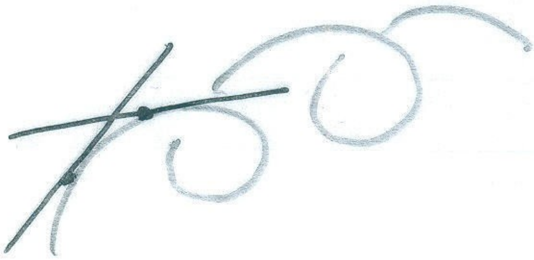


is a cubic defined by

$$\det \begin{pmatrix} X^0 & X^1 & X^2 \\ X^1 & X^2 & X^3 \\ X^2 & X^3 & X^4 \end{pmatrix} = 0$$

The tangent variety belongs to a unique <u>quadric</u> defined by

$$\frac{1}{3} X^0 X^4 - \frac{4}{3} X^1 X^3 + (X^2)^2 = 0$$



Let cubic $= \{C_{ijk}\}$, quadric $= \{g_{ij}\}$

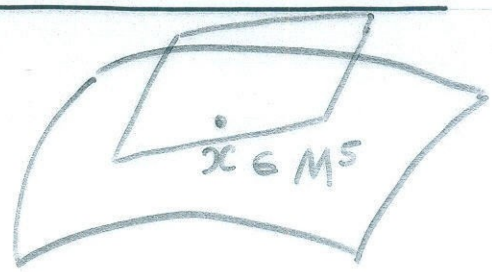
Then:

$$C_{ijk}\, g^{kj} = 0 \quad \text{(apolarity condition}$$

$$C_{jkr} g^{rs} C_{ens} + C_{ejr} g^{rs} C_{kns} + C_{ker} g^{rs} C_{jns} = g_{jk} g_{en} + g_{ej} g_{kn} + g_{ke} g$$

Geometry of a hypersurface $M^5 \subset \Lambda^6$

M^5, g_{ij}, C_{ijk}

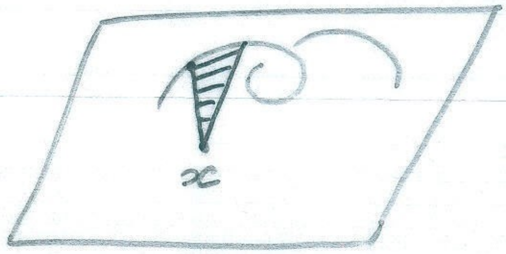


$$C_{ijk}\, g^{kj} = 0$$

$$C_{jkr}\, g^{rs} C_{lns} + C_{ljr}\, g^{rs} C_{kns} + C_{klr}\, g^{rs} C_{jns} =$$

$$= g_{jk} g_{ln} + g_{lj}\, g_{kn} + g_{kl}\, g_{jn}$$

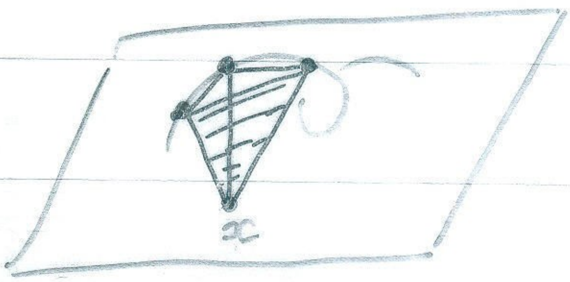
Bisecant planes



Bisecant surfaces $\equiv$ 2-component reductions

Any equation possesses 2-component reductions.
Thus, any hypersurface $M^5 \subset \Lambda^6$ possesses bisecant surfaces (depending on 2 f. of 1 var).

trisecant planes



trisecant surfaces

$\Updownarrow$

3-component reductions

$\Updownarrow$

integrability.