Hamiltonian stationary Lagrangian (HSL) tori in $\mathbb{R}^4$ (jt with P. Romon).

Background

Helein + Romon have shown:

A/ All HSL tori $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ can be constructed in <u>finite</u> Fourier series

$$f = \sum_{\gamma \in \Gamma^*} f_\gamma e^{2\pi i \langle \gamma, z \rangle}, \quad f_\gamma \in \mathbb{C}^4.$$

B/ HSL admits "loop of flat connexions" formulation

$$d\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

and "polynomial Killing fields" (pKf)

$$d\xi_\lambda = [\xi_\lambda, \alpha_\lambda].$$

Our aim: recreate A/ from spectral data derived from B/

1. Diff Geometry

$(\mathbb{R}^4, J, \omega)$ has gp of symplectic isometries

$$G = \{(g,u) \in SO(4) \ltimes \mathbb{R}^4 \mid gJg^{-1} = J\}$$
$$\simeq U(2) \ltimes \mathbb{C}^2$$

Lie alg. $\mathfrak{g} = \{(A,a) \in so(4) \ltimes \mathbb{R}^4 \mid [A,J] = 0\}$

$f: M \to \mathbb{R}^4$ is conformal & Lagrangian

$\iff \quad TM \oplus JTM = f^*T\mathbb{R}^4 =: E$

Define $S \in End(E)$ by $S = J_M \oplus JJ_MJ$.

Then $S^2 = -I$, $SJ = -JS$, $Sf_z = if_z$.

Hence
$$S(z) = e^{\beta(z)J} L$$

L fixed $LJ = -JL$

Lagrangian angle fn. $\beta: \tilde{M} \to [0, 2\pi]$.

In fact $S: M \to G/G^\tau \subset$ twistor bundle of $\mathbb{R}^4$

$$\tau(g,u) = (-LgL, -Lu)$$

4-symmetric space $\rightsquigarrow$ Burstall's talk

f is HSL iff $\Delta\beta = 0$

iff $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$

for $\alpha_\lambda = \left(\frac{1}{2}(\lambda^{-1}\partial\beta + \lambda\bar{\partial}\beta)J,\ e^{-J\beta/2}(\partial f + \lambda\bar{\partial} f)\right)$

cf. Hélein + Romon quadratic in λ.

2. HSL tori

$M = \mathbb{C}/\Gamma$, $\beta = 2\pi\langle\beta_0, z\rangle$ $\exists\, \beta_0 \in \Gamma^*$.

Hélein + Romon show:

i/ Let $u = e^{-J\beta/2} f_z \left[= \alpha_{\mathbb{R}^4}\left(\frac{\partial f}{\partial z}\right)\right]$.

Then $(\Delta + \pi^2|\beta_0|^2)u = 0$.

So $u = \sum_{\gamma \in \frac{1}{2}\Gamma^*} u_\gamma e^{2\pi i\langle\gamma, z\rangle}$ has $u_\gamma \neq 0$

$\Rightarrow \left|\frac{2\gamma}{\beta_0}\right| = 1$, $\gamma \neq \pm\frac{\beta_0}{2}$.

ii/ $\exists$ pKf $\xi_\lambda = (q(\lambda)J,\ g(\lambda, z))$ $\mathfrak{g}$-valued

$q(\lambda) \in \mathbb{C}[\lambda]$

pKf equations are closed under matrix mult.

So embed $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{p} = \left\{ \begin{pmatrix} A & a \\ 0 & b \end{pmatrix} : \begin{array}{l} A \in \mathfrak{gl}(4,\mathbb{C}) \\ a \in \mathbb{C}^4 \\ b \in \mathbb{C} \\ \boxed{AJ = JA} \end{array} \right.$

$$\mathfrak{p} \subset \mathfrak{gl}(5,\mathbb{C}).$$

Work with pKf's in $\Lambda\mathfrak{p} = \mathbb{C}[\lambda,\lambda^{-1}] \otimes \mathfrak{p}$

<u>Thm</u> ($\mathrm{M^cI}$ & Romon) Every pKf is generated over $\Lambda\mathfrak{p}$ by a fundamental pKf

$$(\, p(\lambda)\, I_4 \,,\; x(\lambda, z))$$

where :

i/ $\displaystyle p(\lambda) = \prod_{\substack{\gamma \text{ s.t.} \\ u_\gamma \neq 0}} \left(\lambda^2 - \frac{4\gamma^2}{\beta_0^2} \right)$, $\deg p := N$

$$\boxed{\text{ii/} \quad dx_\lambda + \tfrac{1}{2}\, d\beta_\lambda J x_\lambda = p_\lambda e^{-J\beta_\lambda/2} (\partial f + \lambda \bar\partial f)}$$

uniquely determines $\displaystyle x_\lambda = \sum_{j=1}^{N} x_j(z)\, \lambda^j$

$$d\beta_\lambda := \frac{\pi}{2} \left(\lambda^{-1} \bar\beta_0\, dz + \lambda \beta_0\, d\bar z \right)$$

Further

$$\overline{p(\bar{\lambda}^{-1})} = \frac{p(\lambda)}{\lambda^N p(0)}, \quad p(-\lambda) = p(\lambda)$$

$$\overline{x(\bar{\lambda}^{-1}, z)} = \frac{1}{\lambda^{N+1} p(0)} x(\lambda, z), \quad x(-\lambda, z) = iLx(\lambda, z).$$

Symmetries $\quad \bar{\rho}^* \qquad \tau^*$

3. Spectral Data

$\mathcal{A}$ = maximal abelian subalg. of pkf's in Λ_p.

$\simeq \mathbb{C}[\lambda, \lambda^{-1}, \xi_1, \ldots, \xi_4]$

s.t. $\xi_j^2 = p(\lambda)\xi_j$, $\xi_j \xi_k = 0$, $j \neq k$.

Hence

$$\mathcal{A} \simeq \mathbb{C}[X \setminus \lambda^{-1}\{0, \infty\}]$$

where $\quad X = C_1 \cup \ldots \cup C_4 \cup C_5$

$C_j \simeq \mathbb{CP}^1 \qquad C_j \cap C_5$ along $p(\lambda) = 0$.

Reducible rational curve of arithmetic genus $g = 4(N-1)$.

Symmetries $\quad \rho : X \to X$ real involution

$\qquad\qquad\quad \tau : X \to X$ hol. involution

$$\begin{tikzcd}
C_1 \arrow[rrrrrr, bend left] \arrow[rrrr, bend right] & \cup & C_2 \arrow[rr, bend left, "\tau"] \arrow[rrrr, bend right, "\rho"'] & \cup & C_3 & \cup & C_4 & \cup & C_5 \arrow[loop above, "\tau \quad \lambda \mapsto -\lambda"] \arrow[loop below, "\rho \quad \lambda \mapsto \bar{\lambda}^{-1}"']
\end{tikzcd}$$

$$\mathrm{Jac}(X) \simeq (\mathbb{C}^\times)^g$$

Define
$$J_{\mathbb{R}} = \mathrm{Jac}(X)^\rho \cap \mathrm{Jac}(X)^\tau$$
$$\simeq (\mathbb{R}^\times)^{N-1} \times (S^1)^{N-1}$$

Conj: movement along $J_{\mathbb{R}}$ gives Hamiltonian "homotopies" of the HSL torus.

$x_\lambda(z) \rightsquigarrow$ family of line bundles $\mathcal{L}(z) \to X$

s.t.: i) $\deg \mathcal{L} = (N, N, N, N, 0)$

ii) $\mathcal{L}(z) \otimes \mathcal{L}(0)^{-1} : \mathbb{C}/\Gamma \to J_{\mathbb{R}}$

"tangent to $\partial/\partial\lambda$".

<u>Note</u> $\mathrm{Pic}(X)/\mathrm{Jac}(X) \simeq \mathbb{Z}^5$. $\deg \mathcal{L} = (\deg \mathcal{L}|C_j)$.

Essentially, $\mathcal{L}(z)$ is (modulo common scaling) the data

$$\{ x(\lambda, z) \ : \ p(\lambda) = 0 \}$$

Elementary linear algebra can be used to reconstruct f from this since:

a) these values determine the full poly $x(\lambda, z)$,

b) at $\lambda = 1$ we have

$$f(z) - f(0) = \frac{1}{p(1)} \left(\exp\left(\tfrac{1}{2}\beta J\right) x(1, z) - x(1,0) \right).$$

Further, over $p(\lambda) = 0$ (where $\lambda = \pm \frac{2\gamma}{\beta_0}$)

$$x(\lambda, z) = \exp\left(-\tfrac{1}{2}\beta(\lambda) J\right) x(\lambda, 0)$$

$$\beta(\lambda) = \pi\langle \beta_0 \lambda, z\rangle = 2\pi \langle \gamma, z\rangle$$

$u_\gamma \neq 0$.

i.e. $\mathcal{L}(z)$ evolves linearly and $\text{Jac}(X)$ corresponds to indept. variation of the Fourier components of $x(\lambda, 0)$.