Gromov–Witten theory of orbifold- CP^1 and integrable hierarchies

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Frobenius structure

$$M = \left\{ x^k + \sum_{i=1}^{k} t_i x^{k-i} + \sum_{j=1}^{m-1} t_{k+j} \left(\frac{Qe^{t_N}}{x} \right)^j + \left(\frac{Qe^{t_N}}{x} \right)^m \right\},$$

where $N = k + m = \dim_C M$.

- multiplication $\bullet_f$ in $T_f M \cong C[x]/\langle f' \rangle$, $\partial/\partial t_i \mapsto [\partial_{t_i} f]$.
- metric on M

$$\eta_{ij} = \left(\partial_{t_i}, \partial_{t_j}\right)_f = -\left(\mathrm{res}_{x=0} + \mathrm{res}_{x=\infty}\right) \frac{\left(\partial_{t_i} f\omega\right)\left(\partial_{t_j} f\omega\right)}{df}$$

- Euler vector field $E_f = [f] \in T_f M$
- unity vector field $\partial/\partial t_k$

η is a flat metric, $H := T_0M$, $TM \cong M \times H$ via the Levi–Civita connection, i.e. we trivialize the tangent bundle by choosing flat coordinates τ_i $1 \le i \le N$. Let $\partial_i := \partial/\partial\tau_i$.

Define vector fields $J_{\mathcal{B}} : M \times C^* \to H$

$$(J_{\mathcal{B}}(\tau, z), \partial_i) = z\partial_i \int_{\mathcal{B}} e^{f_\tau/z}\omega,$$

where

$$\mathcal{B} \in \lim_{M} H_1(C^*, \mathrm{Re} f_\tau/z < -M; Z) \cong Z^N$$

Theorem 1. The vector fields $J_{\mathcal{B}}$ are horizontal sections of

$$\nabla = d - z^{-1}\sum_{i=1}^{N}(\partial_i \bullet) d\tau_i - \left(z^{-2}\mu - z^{-1}E\bullet\right) dz,$$

where $\mu = \nabla^{\mathrm{L.C.}}E - \frac{1}{2}Id$.

Theorem 2. M is isomorphic as a Frobenius manifold to the quantum cohomology of orbifold-CP^1 with two orbifold points of type C/Z_m and C/Z_k.

Fock space formalism

$$\mathcal{H} = H((z^{-1})) = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \Omega(f,g) = \mathrm{Res}_{z=0}\ (f(-z), g(z))dz$$
$$\mathcal{H} \ni f = \sum_{n,i} p_{n,i}\, d\tau_i\, (-z)^{-n-1} + q_n^i \partial_i z^n$$

- Fock space $B_H = \mathrm{Fun}(\mathcal{H}_+) = C[[q_n^i \mid n \geq 0, 1 \leq i \leq N]]$.
- $S(\tau, z)$ and $R(\tau, z)$ two symplectic transformations which describe the singularities of ∇ near $z = \infty$ and $z = 0$.
- Assume $\tau \in M$ is such that f_τ is a Morse function. $\mathcal{D}^M = \widehat{S}^{-1}\widehat{R}\mathcal{D}_{\mathrm{pt}}^{\otimes n}$, $\mathcal{A}_\tau^M = \widehat{R}\mathcal{D}_{\mathrm{pt}}^{\otimes n}$ – two vectors in the Fock space.

Conjecture 1: $\log \mathcal{D}^M$ and $\log \mathcal{A}_\tau^M$ are generating functions for the descendent and the ancestor GW invariants of orbifold-CP^1 with two orbifold points of type C/Z_m and C/Z_k.

Vertex operators

$f_\tau(x) = \lambda$ has N solutions $x_i(\tau, \lambda) \in C^*$, $1 \leq i \leq N$, except for $(\tau, \lambda) \in \Delta$ – hypersurface in $M \times C$, called discriminant.

(τ_0, λ_0) – reference point

- For each $\alpha \in H_1(C^*, f_{\tau_0}^{-1}(\lambda_0); C)$

$$\begin{aligned}
(I_\alpha^{(-n)}(\tau,\lambda), \partial_i) &= -\partial_i \int_{\alpha(\tau,\lambda)} \frac{(\lambda - f_\tau)^n}{n!}\omega, \quad 1 \leq i \leq N \\
I_\alpha^n(\tau,\lambda) &= \partial_\lambda^n I_\alpha^{(0)}(\tau,\lambda), \\
\mathbf{f}_\tau^\alpha(\lambda) &= \sum_n I_\alpha^{(n)}(\tau,\lambda)(-z)^n, \\
\Gamma_\tau^\alpha &= \exp \widehat{\mathbf{f}}_\tau^\alpha.
\end{aligned}$$

- $1 \le a \le N$

$$\begin{aligned} (I_a^{(-n)}(\tau,\lambda), \partial_i) &= -\partial_i \int_{[x_a]} d^{-1} \left(\frac{1}{n!} (\lambda - f_\tau)^n \omega \right), \\ I_a^{(n)} &= \partial_\lambda^n I_a^{(0)}, \\ \mathbf{f}_\tau^a(\lambda) &= \sum_n I_a^{(n)}(\tau,\lambda)(-z)^n, \\ \Gamma_\tau^a &= \exp \widehat{\mathbf{f}}_\tau^a. \end{aligned}$$

- Let $c_a(\tau,\lambda) = 1/f'_\tau(x_a)$, then for each critical value u_i of f_τ, in a neighborhood of $\lambda = u_i$ the operator

$$\sum_{a=1}^N c_a \Gamma_\tau^a \otimes \Gamma_\tau^{-a}$$

is $R-$equivalent, up to terms analytic in a neighbourhood of $\lambda = u_i$, to the operator which defines the HQE of KdV.

- the above operator is not monodromy invariant:

$$\sum_{a=1}^N c_a \left(\Gamma_\tau^{r_a \phi} \otimes \Gamma_\tau^{-r_a \phi} \right) \left(\Gamma_\tau^a \otimes \Gamma_\tau^{-a} \right),$$

where $r_a \in 2\pi i Z$ are some numbers depending on the monodromy transformation.

- to offset the complication, look at a larger Fock space $\mathcal{B}_H = \left\{ \sum a_I(x,\partial_x)\mathbf{q}^I \mid a_I \text{ differential operators} \right\}$ and introduce the following vertex operator

$$\Gamma_\tau^\delta = \exp\left((\mathbf{f}_\tau^\phi - w_\tau)\epsilon\partial_x \right) \ \exp\left(x v_\tau / \epsilon \right).$$

It satisfies the following commutation relation:

$$\left(\Gamma_\tau^{\delta\#} \otimes \Gamma_\tau^\delta \right) \left(\Gamma_\tau^{r\,\phi} \otimes \Gamma_\tau^{-r\,\phi} \right) = e^{(\hat{w}_\tau \otimes 1 - 1 \otimes \hat{w}_\tau)\, r} \Gamma_\tau^{\delta\#} \otimes \Gamma_\tau^\delta,$$

where w_τ, via our quantization formalism, is a linear function in $\mathbf{q}$ independent of λ.

Integrable hierarchies

Theorem 3: For each $n \in Z$ the one-form

$$\left(\Gamma_\tau^{\delta\#} \otimes \Gamma_\tau^\delta\right) \left(\sum_{i=1}^{N} c_\tau^i \Gamma_\tau^i \otimes \Gamma_\tau^{-i} \right) \left(\mathcal{A}_\tau^M \otimes \mathcal{A}_\tau^M\right) \ d\lambda,$$

computed at $\mathbf{q}'$, $\mathbf{q}''$ such that $\hat{w}_\tau' - \hat{w}_\tau'' = n$ is regular in λ.

Regular means: put $(q')_n^i = x_n^i - y_n^i$, $(q'')_n^i = x_n^i + y_n^i$ and expand in the powers of $y_0^i, y_1^i, \ldots$ (y_0^k excluded)

$$\sum_I \left(\sum_{j \le K} P_{I,j} \lambda^j \right) y^I,$$

where $P_{I,j}$ is a quadratic polynomial in $\partial^\bullet \mathcal{A}_\tau^M(x_0^k + x + n\epsilon, \ldots)$ and $\partial^\bullet \mathcal{A}_\tau^M(x_0^k + x, \ldots)$. The regularity means that $P_{I,j} = 0$ for $j < 0$.

The above system of PDE admits some kind of a limit

Theorem 4: For each $n \in Z$ the one-form

$$\left(\Gamma_\infty^{\delta\#} \otimes \Gamma_\infty^\delta\right) \left(\sum_a c_\infty^a \Gamma_\infty^a \otimes \Gamma_\infty^{-a} + \sum_b c_\infty^b \Gamma_\infty^b \otimes \Gamma_\infty^{-b} \right) \left(\mathcal{D}^{\mathcal{M}} \otimes \mathcal{D}^{\mathcal{M}}\right) \ d\lambda$$

computed at $\mathbf{q}'$, $\mathbf{q}''$ such that $(q')_0^k - (q'')_0^k = n\epsilon$ is regular in λ.

The vertex operators can be computed explicitly:

$$\mathbf{f}_\infty^a = \frac{1}{k}\mathbf{g}_\infty^a - \sum_{i=1}^{k-1}\sum_{n\in Z}\frac{\prod_{l=-\infty}^{-n-1}(i/k-l)}{\prod_{l=-\infty}^{0}(i/k-l)}\,\lambda^{i/k+n}\partial_i(-z)^{-n-1},$$

where

$$\mathbf{g}_\infty^a = \sum_{n\geq 0}\frac{\lambda^n}{n!}(\log\lambda - C_n)d\tau_k(-z)^{-n-1} + \sum_{n\geq 0} n!\lambda^{-n-1}d\tau_k z^n,$$

and $C_0 := 0$, $C_n := 1 + 1/2 + \ldots + 1/n$ are the Harmonic numbers.

$$\mathbf{f}_\infty^b = -\frac{1}{m}\mathbf{g}_\infty^b - \sum_{j=1}^{m}\sum_{n\in Z}\frac{\prod_{l=-\infty}^{-n-1}(j/m-l)}{\prod_{l=-\infty}^{0}(j/m-l)}\,\lambda^{j/m+n}\partial_{k+m-j}(-z)^{-n-1},$$

where

$$\mathbf{g}_\infty^b = \sum_{n\geq 0}\frac{\lambda^n}{n!}\left[\log(\lambda Q^{-m}) - C_n\right]d\tau_k(-z)^{-n-1} + \sum_{n\geq 0} n!\lambda^{-n-1}d\tau_k z^n.$$

$L = \Lambda^k + u_1\Lambda^{k-1} + \ldots + u_{k+m}\Lambda^{-m},\ \Lambda = e^{\epsilon\partial_x}$

$$\begin{aligned} A_{k-i,n} &= \frac{\Gamma(1+i/k)}{\Gamma(n+1+i/k)}\left(L^{n+i/k}\right)_+,\ \text{for } i = 1,\ldots,k-1 \\ A_{k+j,n} &= -\frac{\Gamma(1+j/m)}{\Gamma(n+1+j/m)}\left(L^{n+j/m}\right)_-,\ \text{for } j = 1,\ldots,m \\ A_{k,n} &= \frac{2}{n!}\left[L^n\left(\log L - \frac{1}{2}\left(\frac{1}{k}+\frac{1}{m}\right)c_n\right)\right]_+ \end{aligned}$$

Extended bi-graded Toda hierarchy (G. Carlet)

$$\epsilon\frac{\partial L}{\partial q_n^i} = [A_{i,n}, L].$$

Conjecture 2: The HQE from Theorem 4 charactarize the tau-functions of the Extended bi-graded Toda hierarchy.