Abelian functions as a $\mathcal{D}$-module

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in Methods of Integrable
Systems in Geometry
at University of Durham
19th August, 2006

1. Theta function

$\omega_1, \omega_2, \eta_1, \eta_2 \cdots\ g \times g$ complex matrices

s.t.

- $\det(\omega_1) \neq 0$,
- ${}^t\tau = \tau$, $\operatorname{Im}\tau > 0$, if $\tau = \omega_1^{-1}\omega_2$,
- $M \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} {}^tM = -\frac{\pi i}{2} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, if $M = \begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix}$.

Such matrices arise as period matrices of a compact Riemann surfaces of genus g.

Then (F.Klein 1888, Buchstaber-Enolski-Leykin 1997)

$u = (u_1, ..., u_g)$, $\delta = {}^t(\delta', \delta'') \in \mathbb{R}^{2g}$,

$\sigma[\delta](u)$: a holomorphic function on $\mathbb{C}^g$,

s.t.

$$\sigma[\delta](u + \Omega(m_1, m_2)) = e^{-\pi i {}^t m_1 m_2 + 2\pi i {}^t(\delta' m_1 - \delta'' m_2)} \times e^{{}^tE(m_1,m_2)(u + \frac{1}{2}\Omega(m_1,m_2))}\sigma[\delta](u),$$

$\Omega(m_1, m_2) = 2\omega_1 m_1 + 2\omega_2 m_2$,
$E(m_1, m_2) = 2\eta_1 m_1 + 2\eta_2 m_2$,
$m_1, m_2 \in \mathbb{Z}^g$.

It is known that, for each δ, the function $\sigma[\delta](u)$ exists and is unique up to constant multiples. It is explicitly written using Riemann's theta function $\theta[\delta](z,\tau)$ as

$$\sigma[\delta](u) = C\exp(\frac{1}{2}{}^t u\eta_1\omega_1^{-1}u)\ \theta[\delta]((2\omega_1)^{-1}u,\tau).$$

The matrices ω_1, ω_2 determine an abelian variety

$$X = \mathbb{C}^g/2\omega_1\mathbb{Z}^g + 2\omega_2\mathbb{Z}^g$$

.

Fix a δ and define the theta divisor as the zero set of the sigma function:

$$\Theta = (\sigma[\delta](u) = 0) \subset X$$

.

2. Abelian function

The function

$$\zeta_{ij}(u) = \frac{\partial^2}{\partial u_i \partial u_j} \log\, \sigma[\delta](u)$$

satisfies

$$\zeta_{ij}(u + \Omega(m_1, m_2)) = \zeta_{ij}(u).$$

This is an example of an abelian function of order 2.

Introduce the sapce A as

$$A = \{\text{meromorphic functions on } X \text{ which are regular on } X - \Theta\}$$

$$= \cup_{n=0}^{\infty} \left\{ f(u) = \frac{F(u)}{\sigma[\delta](u)^n} \mid f(u + \Omega(m_1, m_2)) = f(u) \right\}$$

$$= \cup_{n=0}^{\infty} A(n).$$

Notice that

$$a(u+\Omega(m_1, m_2)) = a(u) \Longrightarrow \frac{\partial a}{\partial u_i}(u+\Omega(m_1, m_2)) = \frac{\partial a}{\partial u_i}(u).$$

If we set

$$\mathcal{D} = \mathbb{C}[\partial_{u_1}, ..., \partial_{u_g}], \quad \partial_{u_i} = \frac{\partial}{\partial u_i},$$

then

$$A \text{ becomes a } \mathcal{D}\text{-module.}$$

3. Example– $g = 1$ –

In this case we take $\delta = {}^t(1/2, 1/2)$. Then

$\sigma \cdots$ Weierstrass' σ- function

$\wp(u) = -\zeta_{11}(u) = -\frac{\partial^2}{\partial u^2} \log\, \sigma(u)$,

$\mathcal{D} = \mathbb{C}[\partial_u], \quad \Theta = \{u = 0\} \subset X = \mathbb{C}/2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}.$

and

$$
\begin{aligned}
A(n) &= \mathbb{C} \oplus \mathbb{C}\wp(u) \oplus \mathbb{C}\wp'(u) \oplus \cdots \oplus \mathbb{C}\wp^{(n-2)}(u), \\
A \quad &= \mathbb{C} \oplus \mathbb{C}\wp(u) \oplus \mathbb{C}\wp'(u) \oplus \cdots, \\
&= \mathcal{D}1 + \mathcal{D}\wp.
\end{aligned}
$$

As a $\mathcal{D}$-module

- generators $\cdots$ $1, \wp$
- relations $\cdots$ $\partial_u(1) = 0.$

This structure can conveniently be described in a $\mathcal{D}$-free resolution:

$$
0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}^2 \longrightarrow A \longrightarrow 0.
$$

Problem Determine

1. generators $\longrightarrow$ cohomologies
2. relations $\longrightarrow$ a free resolution
3. a linear basis of A.

We study this problem for $g = 3$ hyperelliptic Jacobians.

4. $g = 3$ hyperelliptic Jacobian

Consider the hyperelliptic curve

$$C : y^2 = 4x^7 + \lambda_2 x^6 + \lambda_4 x^5 + \cdots + \lambda_{14}.$$

Take a canonical homology basis $\{\alpha_i, \beta_i\}$ and a canonical cohomology basis $\{du_i, dv_j\}$ such that $du_i = x^{3-i}dx/y$ and dv_i is a 2nd kind differential. Define

$$\omega_1 = \left(\int_{\alpha_j} du_i\right), \quad \omega_2 = (\left(\int_{\beta_j} du_i\right), \quad \eta_1 = \left(\int_{\alpha_j} dv_i\right), \quad \eta_2 = (\left(\int_{\beta_j} dv_i\right).$$

We specify δ in the definition of σ-function as

$$\delta = {}^t(\delta', \delta''), \quad \delta' = {}^t(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad \delta'' = {}^t(\frac{3}{2}, \frac{2}{2}, \frac{1}{2}).$$

This δ corresponds to the Riemann constant for the base point ∞ and some choice of canonical homology basis.

Set

$$\sigma(u) = \sigma[\delta](u),$$

$$u = (u_1, u_2, u_3) = (t_1, t_3, t_5),$$

$$\wp_{ij}(u) = -\partial_i \partial_j \log \sigma(u), \quad \partial_i = \frac{\partial}{\partial t_i}.$$

To study the $\mathcal{D}$-module structure of A it is important to introduce a filtration on A and to consider the associated graded module. We consider two filtrations on A.

5. Pole filtration

We have already defined an increasing filtration in defining A.

$$A = \cup_{n=0}^{\infty} A(n),$$

$$A(n) = \{ f \in A \,|\, \text{the order of poles on } \Theta \leq n \},$$

The associated graded space is

$$\mathrm{gr}^{pol} A = \oplus_n A(n)/A(n-1).$$

The relations

$$\partial_i A(n) \subset A(n+1),$$

imply that

$$\mathrm{gr}^{pol} A \text{ becomes a } \mathcal{D}\text{-module.}$$

A minimal set of generators of $\mathrm{gr}^{pol} A$ is given by a basis of

$$\frac{\mathrm{gr}^{pol} A}{\sum_{i=1}^{3} \partial_i \, \mathrm{gr}^{pol} A} \simeq H^3(\mathrm{gr}^{pol} A \otimes \Omega^{\bullet}).$$

Arguments using the grading show that

$$\dim\, H^3 < \infty \Longrightarrow \mathrm{gr}^{pol} A \text{ is finitely generated.}$$

$$\Longrightarrow A \text{ is finitely generated.}$$

6. KP filtration

In general set

$$\wp_{i_1 \ldots i_n} = -\partial_{i_1} \cdots \partial_{i_n} \log \sigma(u).$$

It is known that A is described as

$$A = \mathbb{C}\,[\,\wp_{i_1 \ldots i_n}\,|\, n \geq 2, \quad i_j \in \{1,3,5\}\,]\,.$$

Define a filtration $\{A_n\}$ by specifying

$$\wp_{i_1 \ldots i_k} \in A_n \quad \text{for any } n \geq i_1 + \cdots + i_k.$$

Then

$$\begin{aligned}
&A = \cup_{n=0}^{\infty} A_n, \\
&A_0 = A_1 = \mathbb{C}, \\
&A_2 = \mathbb{C} + \mathbb{C}\wp_{11}, \\
&A_3 = \mathbb{C} + \mathbb{C}\wp_{11} + \mathbb{C}\wp_{111}, \\
&A_4 = \mathbb{C} + \mathbb{C}\wp_{11} + \mathbb{C}\wp_{111} + \mathbb{C}\wp_{1111} + \mathbb{C}\wp_{11}^2 + \mathbb{C}\wp_{13}.
\end{aligned}$$

etc.

The associated graded space is

$$\mathrm{gr}^{kp}\, A = \oplus_n\, A_n / A_{n-1}.$$

Then

$\partial_i A_n \subset A_{n+i} \Longrightarrow \mathrm{gr}^{kp}\, A$ becomes a $\mathcal{D}$-module.

Generators are given by a basis of

$$\frac{\mathrm{gr}^{kp} A}{\sum_{i=1}^{3} \partial_i \, \mathrm{gr}^{pol} A} \simeq H^3(\mathrm{gr}^{kp} A \otimes \Omega^\bullet).$$

Arguments using the grading show that

$$\dim\ H^3 < \infty \Longrightarrow \mathrm{gr}^{kp} A \text{ is finitely generated.}$$

$$\Longrightarrow A \text{ is finitely generated.}$$

7. Algebraic de Rham complex

Set

$$\Omega^k = \sum_{i_1 < \cdots < i_k} \mathbb{C} dt_{i_1} \wedge \cdots \wedge dt_{i_k}.$$

The operator

$$d = \sum_{i=1}^{3} \partial_i \otimes dt_i : A \otimes \Omega^k \longrightarrow A \otimes \Omega^{k+1}$$

determines a complex, called algebraic de Rham complex,

$$(A \otimes \Omega^\bullet, d).$$

algebraic de Rham theorem

$$H^i(A \otimes \Omega^\bullet) \simeq H^i(X - \Theta, \mathbb{C}).$$

Similarly the following two complexes are defined.

$$(\mathrm{gr}^{pol} A \otimes \Omega^\bullet, d), \quad (\mathrm{gr}^{kp} A \otimes \Omega^\bullet, d).$$

The highest cohomology group becomes

$$H^3(\mathrm{gr}^{pol} A \otimes \Omega^\bullet) = \frac{(\mathrm{gr}^{pol} A) dt_1 \wedge dt_3 \wedge dt_5}{d(\sum_{i<j} (\mathrm{gr}^{pol} A) dt_i \wedge dt_j)} \simeq \frac{\mathrm{gr}^{pol} A}{\sum \partial_i (\mathrm{gr}^{pol} A)}.$$

8. Predictions on Euler characteristic – pole filtration –

In general, for a graded vector space

$$V = \oplus_d V_d$$

define its character by

$$\mathrm{ch}\, V = \sum_d q^d \dim\, V_d.$$

Now

$$\deg dt_i = -1 \Longrightarrow \mathrm{gr}^{pol} A \otimes \Omega^k \text{ is graded.}$$

$$\Longrightarrow \mathrm{ch}\left(\mathrm{gr}^{pol} A \otimes \Omega^k\right) \text{ is defined.}$$

These definitions are straightforwardly generalized for g-dimensional case.

Using the well known formula

$$\dim\, \mathrm{gr}_n^{pol} A = \dim A(n) - \dim\, A(n-1) = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ n^g - (n-1)^g & n \geq 2 \end{cases}$$

we get

$$\begin{aligned} \mathrm{ch}(\mathrm{gr}^{pol} A) &= (1-q)\left(1 + (q\frac{d}{dq})^g \frac{1}{1-q}\right), \\ \mathrm{ch}\Omega^k &= \binom{g}{k} q^{-k}. \end{aligned}$$

The q-Euler characteristic is calculated as

$$
\begin{aligned}
\chi_q^{pol} &:= \sum_{i=0}^{g} (-1)^i \mathrm{ch}\left(\mathrm{gr}^{pol} A \otimes \Omega^i\right) \\
&= (-1)^g q^{-g} (1-q)^{g+1} \left(1 + (q\frac{d}{dq})^g \frac{1}{1-q}\right) \\
&\stackrel{q \longrightarrow 1}{\longrightarrow} (-1)^g\, g!.
\end{aligned}
$$

The following is known (can be easily proved).

Proposition

Suppose that Θ is non-singular. Then

$$\chi(X - \Theta) = (-1)^g \, g!.$$

This suggests that the pole filtration works well for generic abelian varieties. In fact we can prove the following theorem.

Theorem (Cho-N, '06)

Suppose that Θ is non-singular.

(1) $H^i(\mathrm{gr}^{pol} A \otimes \Omega^\bullet) \simeq H^i(X - \Theta)$. In particular it is finite dimensional.

(2) It is possible to construct a $\mathcal{D}$-free resolution of both $\mathrm{gr}^{pol} A$ and A explicitly.

Notice that

- the theta divisor of 3-dimensional hyperelliptic Jacobian is singular,
- $\chi(X - \Theta) = -5 \neq (-1)^3 3! = -6$.

9. Predictions on Euler characteristic – KP filtration –

The q-Euler characteristic is already known (Smirnov-N, '01) as

$$\chi_q^{KP} = -q^{-9} \frac{[\frac{1}{2}]_{q^2}[7]_{q^2}!}{[3]_{q^2}![4]_{q^2}![\frac{7}{2}]_{q^2}},$$

where

$[x]_p = 1 - x^p,$

$[k]_p! = [k]_p[k-1]_p \cdots [1]_p,$

We have

$$\lim_{q \longrightarrow 1} \chi_q^{KP} = -5 = \chi(X - \Theta).$$

It seems that the KP-filtration works well for our case. In fact we can prove the following.

Proposition

(1) $\dim H^i(\mathrm{gr}^{kp} A \otimes \Omega^\bullet) = \dim\ H^i(X - \Theta, \mathbb{C})$

$$= \begin{pmatrix} 6 \\ i \end{pmatrix} - \begin{pmatrix} 6 \\ i-2 \end{pmatrix}.$$

(2) $\dim H^3(\mathrm{gr}^{pol} A \otimes \Omega^\bullet) = \infty.$

This means that $\mathrm{gr}^{pol} A$ is not finitely generated but A is finitely generated.

Set

$$(i_1, ..., i_k; j_1, ..., j_k) = \det(\wp_{i_r j_s}).$$

Theorem

(1) As a $\mathcal{D}$-module A is generated by

$$1, \quad \wp_{ij}, \quad (i_1 i_2; j_1 j_2), \quad (123; 123).$$

(2) We have the explicitly described minimal $\mathcal{D}$-free resolution of A of the form

$$0 \longrightarrow \mathcal{D} \otimes W^1 \longrightarrow \mathcal{D} \otimes W^2 \longrightarrow \mathcal{D} \otimes W^3 \longrightarrow 0.$$

Remark The theorem solves the conjecture in [Smirnov-N '01] for $g = 3$ case.

10. The singularity of Θ

$$\begin{aligned} \Theta &= \{\sigma(u) = 0\} \\ \mathrm{Sing}\Theta &= \{(0,0,0)\}. \end{aligned}$$

The function σ has the following expansion (H.F.Baker 1898, [BEL] 1999):

$$\sigma(u) = \sum a_\alpha(\lambda) t^\alpha, \quad t^\alpha = t_1^{\alpha_1} t_3^{\alpha_3} t_5^{\alpha_5}, \quad a_\alpha(\lambda) \in \mathbb{C}[\lambda_2, ..., \lambda_{14}]$$

$$\begin{aligned} \sigma(u)|_{\lambda_i \longrightarrow 0} &= s_{(3,2,1)}(t) \qquad \cdots \text{ Schur function} \\ &= t_5 t_1 - t_3^2 - \tfrac{1}{3} t_3 t_1^3 + \tfrac{1}{45} t_1^6 \\ &= t_1 (t_5 - \tfrac{1}{3} t_3 t_1^2 + \tfrac{1}{45} t_1^5) - t_3^2 \\ &= xy - z^2 \qquad \cdots A_1 \text{ singularity.} \end{aligned}$$

Similarly

$$\sigma(u) = XY - Z^2 \qquad \text{near}(0,0,0).$$

Example $g = 2$

$$u = (t_1, t_3)$$

$$1, \qquad \wp_{ij}(u), \qquad (13;13) = \begin{vmatrix} \wp_{11} & \wp_{13} \\ \wp_{13} & \wp_{33} \end{vmatrix}.$$

$$\sharp = 1 \qquad + \qquad 3 \qquad + \qquad 1 \qquad = 5.$$

Example $g = 3$ hyperelliptic $y^2 = x^7 + \cdots$

$$u = (t_1, t_3, t_5)$$

$$1, \qquad \wp_{ij}(u), \qquad (i_1 i_2; j_1 j_2), \qquad (135; 135)$$

$$\sharp = 1 \qquad + \qquad 6 \qquad + \qquad 6 \qquad + \qquad 1 \qquad = 14.$$

Example $g = 3$ non-hyperelliptic $y^3 = x^4 + \cdots$

$$u = (t_1, t_2, t_5)$$

$$1, \qquad \wp_{ij}(u), \qquad (i_1 i_2; j_1 j_2), \qquad (125; 125)$$

$$v = \wp_{2222} - 6\wp_{22}^2.$$

$$\sharp = 14 \qquad + \qquad 1 \qquad = 15.$$