Esquisse d'une synthèse/Schizzo d'una sintesi

Germ of a synthesis: space-time is spinorial, extra dimensions are time-like

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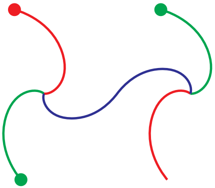
Outline

The impossible triangle

The triality identity



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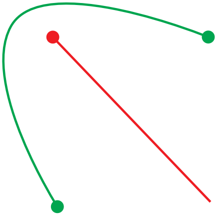
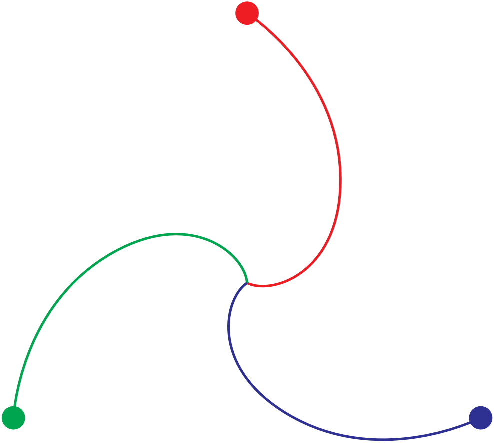
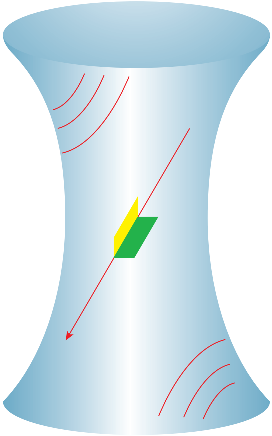


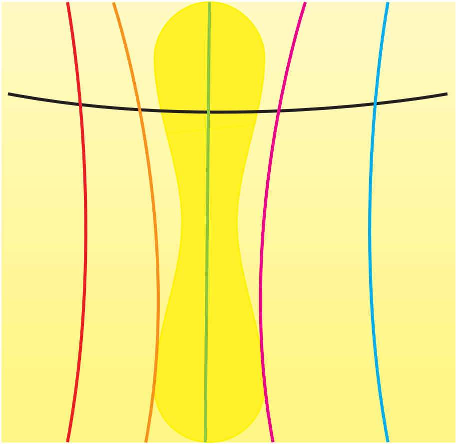
Cartan's triality symbol linking space-time and two twistor spaces



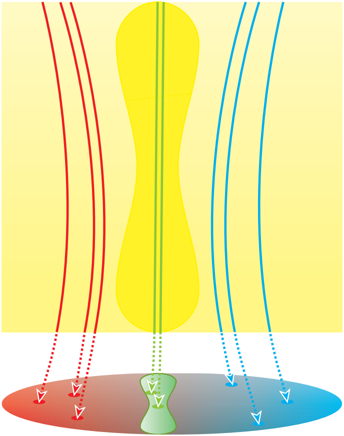
The Klein Quadric



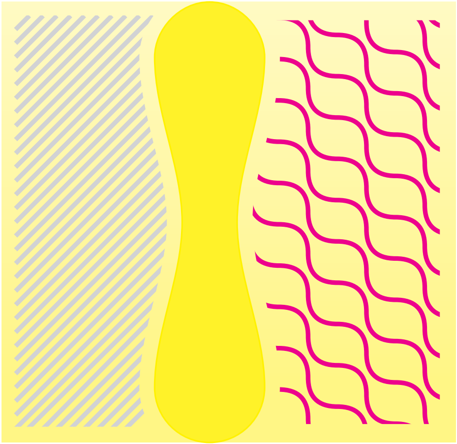
The various kinds of complex space-time points



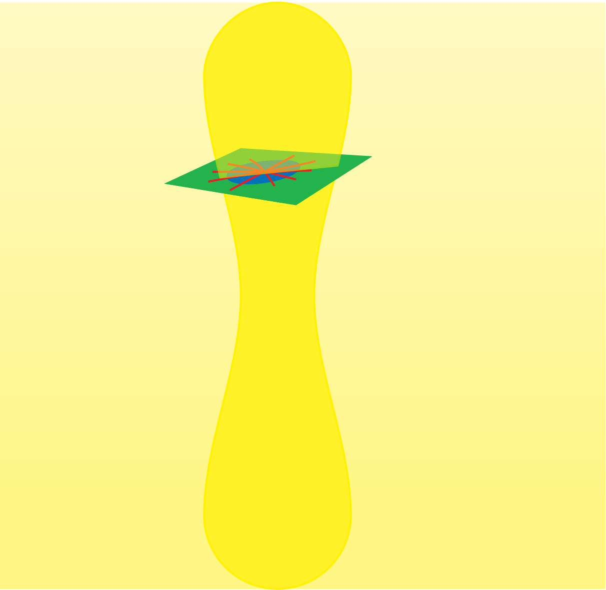
The $\mathbb{S}^4$ projection



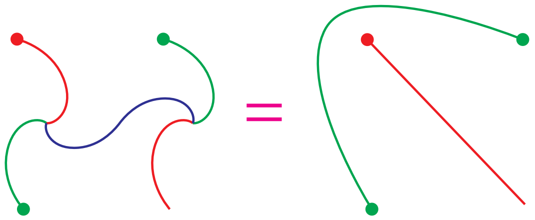
The Zhang-Hu fluid



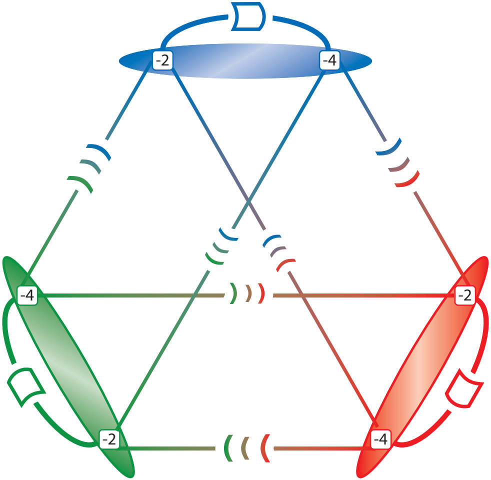
Null twistors



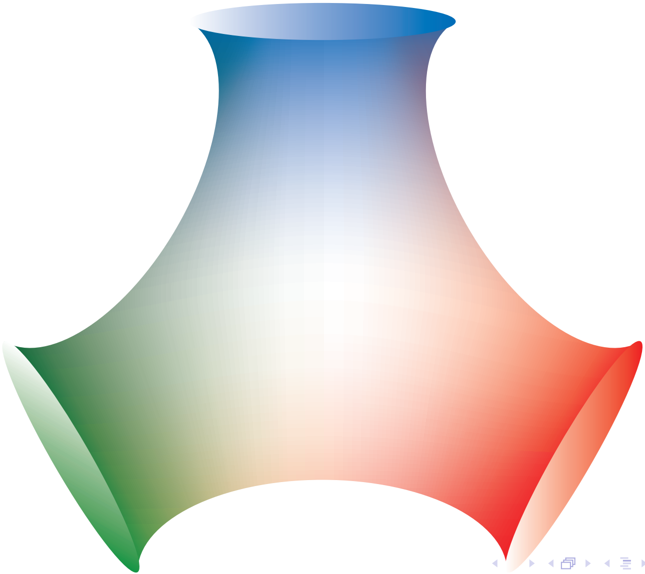
The triality identity



The concinnity



The String Picture



Spinors and the null cone

- Penrose's twistor theory tried to incorporate spinorial structures into general relativity in a non-local way.
  - Spinors, particularly for spin one-half, are naturally complex and link to quantum mechanics as the space of states at a point of a quantum (fermionic) particle.
  - Two-component spinors parametrize the future null cone, so are directly tied to ideas of causality. Explicitly, we represent a space-time point $X = (t, x, y, z) \in \mathbb{C}^4$ by a 2×2-matrix:

$$X = \begin{vmatrix} t+z & x-iy \\ x+iy & t-z \end{vmatrix}, \quad \det(X) = t^2 - x^2 - y^2 - z^2, \ \mathrm{tr}(X) = 2t.$$

  - X is real iff $X = X^*$, where * denotes hermitian conjugation.
  - X is real null and future pointing iff $0 \neq \pi = (p, q) \in \mathbb{C}^2$ exists with:
$$X = 2\pi\pi^*.$$
  - Thus we may parametrize the future null cone as follows:

$$\pi \to (|p|^2 + |q|^2, \ p\bar{q} + q\bar{p}, \ i(p\bar{q} - q\bar{p}), \ |p|^2 - |q|^2) = (t, x, y, z),$$

$$t^2 - x^2 - y^2 - z^2 = 0$$

The Lorentz group and $\mathbb{SL}(2, \mathbb{C})$

- π is called a spinor and the two-complex dimensional vector space of all spinors is called spin space $\mathbb{S}$.
  - The formula $X = 2\pi\pi^*$ is phase invariant: π and $t\pi$ give the same null ray if $t \neq 0$ and the same null vector if $|t| = 1$.
  - The group $\mathbb{GL}(2, \mathbb{C})$ acts naturally on X by $X \to LXL^*$, for $L \in \mathbb{GL}(2, \mathbb{C})$, inducing a real linear transformation, $\Lambda(L)$ of space-time, which preserves the metric up to a dilation.
  - Then $\mathbb{GL}(2, \mathbb{C})$ acts on the spin space by $\pi \to L\pi$.
  - If $\det(L) = 1$, $\Lambda(L) \in \mathbb{O}_+^+(1, 3, \mathbb{R})$, the identity component of the real Lorentz group. The map $\mathbb{SL}(2, \mathbb{C}) \to \mathbb{O}_+^+(1, 3, \mathbb{R})$, $L \to \Lambda(L)$ is 2:1 and onto, with kernel $\mathbb{Z}_2 = \{\pm I\}$.
  - For a vector space $\mathbb{V}$ of dimension n, and for $0 \leq k \leq n$, $\mathbb{G}(k, \mathbb{V})$ denotes the Grassmanian of all subspaces of $\mathbb{V}$ of dimension k. Then $\mathbb{G}(k, \mathbb{V})$ has dimension $k(n - k)$.
  - The space of null rays at a space-time point is naturally a Riemann sphere, $\mathbb{PS} = \mathbb{G}(1, \mathbb{S})$. $\mathbb{PS}$ has as its symmetry group $\mathbb{PSL}(2, \mathbb{C}) = \mathbb{SL}(2, \mathbb{C})/\mathbb{Z}_2 = \mathbb{O}_+^+(1, 3, \mathbb{R})$ and the group actions on spinors, vectors and null rays all mesh.

The four kinds of spinors

- There are four kinds of two-component spinors, distinguished by how they transform under a Lorentz transformation $L \in \mathbb{SL}(2, \mathbb{C})$:
  - $\mathbb{S}$, the unprimed spin space, with typical element α^A and transformation matrix L.
  - $\mathbb{S}^*$, the unprimed co-spin space, with typical element β_A and transformation matrix L^{-1}.
  - $\mathbb{S}'$, the primed spin space, with typical element $\gamma^{A'}$ and transformation matrix $\overline{L}$.
  - $(\mathbb{S}')^*$, the primed co-spin space, with typical element $\delta_{A'}$ and transformation matrix $\overline{L}^{-1}$.
- The spaces $\mathbb{S}$ and $\mathbb{S}^*$ are naturally dual, as are the spaces $\mathbb{S}'$ and $(\mathbb{S}')^*$.
  - Conjugation interchanges $\mathbb{S}$ with $\mathbb{S}'$ and $\mathbb{S}^*$ with $(\mathbb{S}')^*$.
  - The tensor product $\mathbb{S} \otimes_{\mathbb{C}} \mathbb{S}'$ transforms as a complex Lorentz vector, whereas $\mathbb{S}^* \otimes_{\mathbb{C}} (\mathbb{S}')^*$ transforms as a complex Lorentz co-vector.
- The spinor algebra subsumes the Lorentz tensor algebra.

The metric and the spinor symplectic forms

We use lower case Latin indices for Lorentz tensors.

- A real Lorentz vector x^a has a spinor description as $x^a = x^{AA'}$, where $x^{AA'}$ is self-conjugate: $\overline{x}^{A'A} = x^{AA'}$.
- Then the metric $g_{ab} = g_{ABA'B'}$ factorizes as $g_{ab} = \epsilon_{AB}\epsilon_{A'B'}$, where $\epsilon_{AB} \neq 0$ is skew and has complex conjugate $\epsilon_{A'B'}$.
- Then ϵ_{AB} and $\epsilon_{A'B'}$ are complex symplectic structures for the unprimed and primed spin-spaces, respectively.
- Abstractly we have $\epsilon \in \Omega^2(\mathbb{S}^*)$, such that $g = \epsilon \otimes_{\mathbb{C}} \overline{\epsilon}$.
- The inverse symplectic form is ϵ^{AB}, normalized by the relation $\epsilon_{AB}\epsilon^{AC} = \delta_B{}^C$.
- Spinor indices are raised and lowered according to the rules:
$$\alpha^A\epsilon_{AB} = \alpha_B, \quad \beta_B\epsilon^{AB} = \beta^A.$$
- A spin frame $\{\alpha^A, \beta^B\}$ is normalized iff $\alpha^A\beta^B - \alpha^B\beta^A = \epsilon^{AB}$.

Relating the abstract and matrix approaches

- We relate the position vector x^a of a space-time point to its Minkowskian co-ordinates (t, x, y, z) using the decomposition:

  $$x^a\sqrt{2} = (t+z)\alpha^A\overline{\alpha}^{A'} + (t-z)\beta^A\overline{\beta}^{A'} + (x+iy)\alpha^A\overline{\beta}^{A'} + (x-iy)\beta^A\overline{\alpha}^{A'}.$$

  Here $\{\alpha^A, \beta^A\}$ is a normalized spin frame.
  Then we have:

  $$g_{ab}x^a x^b = x^a x_a = t^2 - x^2 - y^2 - z^2.$$

- Abstractly, using X for the position vector, this decomposition is:

  $$X\sqrt{2} = (t+z)\alpha\otimes_{\mathbb{C}}\overline{\alpha} + (t-z)\beta\otimes_{\mathbb{C}}\overline{\beta} + (x+iy)\alpha\otimes_{\mathbb{C}}\overline{\beta} + (x-iy)\beta\otimes_{\mathbb{C}}\overline{\alpha}.$$

- A complex vector v^a is null iff $v^{AA'} = \gamma^A\delta^{A'}$ for some spinors γ^A and $\delta^{A'}$ (abstractly $v = \gamma \otimes_{\mathbb{C}} \delta$).
  - Further v^a is real, null and future-pointing iff $v^{AA'} = \gamma^A\overline{\gamma}^{A'}$, for some non-zero spinor γ^A (unique up to a phase).

Twistors for flat space-time

- The basic twistor is a light ray in space-time; in compactified complexified Minkowski space-time, this generalizes naturally to a completely null self-dual complex two-plane. The space of such twistors is a complex projective three-space $\mathbb{PT}$, whose real dimension is six. The space of light rays forms a real hypersurface $\mathbb{PN}$, of $\mathbb{PT}$, with a regular $\mathcal{CR}$-structure of Levi signature $(1, 1)$.
- At the projective level there are three intertwined spaces: $\mathbb{PT}$, the dual space, $\mathbb{PT}^*$, and the Klein quadric, $\mathbb{M}$.
- There is a four-complex dimensional vector space, twistor space $\mathbb{T}$, such that $\mathbb{PT} = \mathbb{G}(1, \mathbb{T})$, $\mathbb{PT}^* = \mathbb{G}(3, \mathbb{T})$ and $\mathbb{M} = \mathbb{G}(2, \mathbb{T})$. Note that $\mathbb{M}$ is also the projective space of the space of elements X of $\Omega^2(\mathbb{T})$, such that $X \wedge X = 0$, so $\mathbb{M}$ has a natural conformally flat conformal structure.

Twistors and reality

- The twistor space carries a conjugation $Z \in \mathbb{T} \to \overline{Z} \in \mathbb{T}^*$, such that the quadratic form $\overline{Z}(Z)$ has signature $(2, 2)$.
- If we also preserve a volume form on the twistor space, the symmetry group is $\mathbb{SU}(2, 2, \mathbb{C})$ which is fifteen-dimensional.
- The three groups $\mathbb{SU}(2, 2, \mathbb{C})$, $\mathbb{SO}(2, 4, \mathbb{R})$ and $\mathbb{C}(1, 3, \mathbb{R})$, the conformal group of compactified Minkowksi space-time are real Lie groups of fifteen dimensions and are naturally locally isomorphic.
- The fifteen dimensions are accounted for by:
    - Spatial rotations: three dimensions
    - Lorentz boosts: three dimensions
    - Spatial translations: three dimensions
    - Time translations: one dimension
    - Dilations: one dimension
    - Special conformal transformations: four dimensions

Twistor groups

We use Greek indices for twistors.

- The group $\mathbb{SU}(2,2,\mathbb{C})$ acts as $Z^\alpha \to L^\alpha_\beta Z^\beta$, for each twistor Z^α and each $L^\alpha_\beta \in \mathbb{SU}(2,2,\mathbb{C})$.
  - The twistor pseudo-hermitian form, $Z^\alpha \overline{Z}_\alpha$, is preserved, so we have $L^\alpha_\gamma \overline{L}^\gamma_\beta = \delta^\alpha_\beta$; also $\det(L) = 1$.
- The induced action on any $X^{\alpha\beta} = -X^{\beta\alpha} \in \Omega^2(\mathbb{T})$ is:
$$X^{\alpha\beta} \to L^\alpha_\gamma L^\beta_\delta X^{\gamma\delta}.$$
  - This action preserves $X^{\alpha\beta}\overline{X}_{\alpha\beta}$ and gives an action of the group $\mathbb{SO}_e(2,4,\mathbb{R})$ (the subscript e denotes identity component). The transformations $\pm L^\alpha_\beta$ are identified.
- The action on $\Omega^2(\mathbb{T})$ preserves the null cone $X^{[\alpha\beta}X^{\gamma\delta]} = 0$, so induces a conformal transformation of space-time.
  - Here the transformations tL^α_β with $t^4 = 1$ are identified, since $X^{\alpha\beta}$ and $-X^{\alpha\beta}$ represent the same space-time point.
- Summarizing, we have group epimorphisms with kernel $\mathbb{Z}_2$:
$$\mathbb{SU}(2,2,\mathbb{C}) \to \mathbb{SO}_e(2,4,\mathbb{R}) \to \mathbb{C}_e(1,3,\mathbb{R}).$$
Here the subscript e signifies identity component. The

Twistors and reality

We parametrize twistors by pairs of spinors as follows:

- $Z^\alpha = (\omega^A, \pi_{A'}) = (p, q, r, s)$, $\overline{Z}_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'}) = (\overline{r}, \overline{s}, \overline{p}, \overline{q})$. Here $\omega^A = (p, q)$ and $\pi_{A'} = (r, s)$. Then the twistor form is:

$$Z^\alpha \overline{Z}_\alpha = p\overline{r} + r\overline{p} + q\overline{s} + s\overline{q} = \omega^A \overline{\pi}_A + \pi_{A'}\overline{\omega}^{A'}.$$

  - We embed space-time in $\Omega^2(\mathbb{T})$ as follows:

$$X^{\alpha\beta} = \begin{vmatrix} 0 & u & ix + y & -it - iz \\ -u & 0 & it - iz & -ix + y \\ -ix - y & -it + iz & 0 & -v \\ it + iz & ix - y & v & 0 \end{vmatrix}.$$

  - Then we have:

$$\overline{X}_{\alpha\beta} = \begin{vmatrix} 0 & -\overline{v} & i\overline{x} - \overline{y} & i\overline{t} - i\overline{z} \\ \overline{v} & 0 & -i\overline{t} - i\overline{z} & -i\overline{x} - \overline{y} \\ -i\overline{x} + \overline{y} & i\overline{t} + i\overline{z} & 0 & \overline{u} \\ -i\overline{t} + i\overline{z} & i\overline{x} + \overline{y} & -\overline{u} & 0 \end{vmatrix},$$

  - So $X^{\alpha\beta}\overline{X}_{\alpha\beta} = -2u\overline{v} - 2v\overline{u} + 4|t|^2 - 4|x|^2 - 4|y|^2 - 4|z|^2$.

Twistors and reality

Dualizing $\overline{X}_{\alpha\beta}$, we get:

$$\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\overline{X}_{\gamma\delta} = \begin{vmatrix} 0 & \overline{u} & i\overline{x}+\overline{y} & -i\overline{t}-i\overline{z} \\ -\overline{u} & 0 & i\overline{t}-i\overline{z} & -i\overline{x}+\overline{y} \\ -i\overline{x}-\overline{y} & -i\overline{t}+i\overline{z} & 0 & -\overline{v} \\ i\overline{t}+i\overline{z} & i\overline{x}-\overline{y} & \overline{v} & 0 \end{vmatrix}.$$

Here $\epsilon^{\alpha\beta\gamma\delta}$ is totally skew and $\epsilon^{1234} = 1$.

- The incidence of a space-time point $X = (t, x, y, z) \in \mathbb{C}^4$ with a twistor $Z = (\omega, \pi) = (p, q, r, s) \in \mathbb{C}^4$, where $\omega = (p, q)$ and $\pi = (r, s)$, is given by: the formula:

$$\left|\begin{array}{c} p \\ q \end{array}\right| = i \left|\begin{array}{cc} t - z & x - iy \\ x + iy & t + z \end{array}\right| \left|\begin{array}{c} r \\ s \end{array}\right|, \quad \omega = iX\pi.$$

  Then the twistor quadratic form is:

$$\overline{Z}(Z) = \overline{r}p + \overline{s}q + \overline{p}r + \overline{q}s = \pi^*\omega + \omega^*\pi = i\pi^*(X - X^*)\pi.$$

  - Reality of the space-time point $X = X^*$ gives $\overline{Z}(Z) = 0$.
  - Conversely if $\overline{Z}(Z) = 0$, for all r and s, then X is real.
  - As r and s vary, for fixed X, we trace out a two-dimensional subspace of $\mathbb{T}$, so an element of $\mathbb{G}(2, \mathbb{T})$.
  - Given $Z \neq 0$, the incidence relation has a two parameter set of solutions, X, giving the self-dual null two-plane of complex space-time, corresponding to Z. Z and tZ have the same two-plane for any complex number $t \neq 0$.

ODE questions

An ODE is $\Psi(y, y', y'', \dots, y^{(n)}, x) = 0$.
Solution depends on n constants of the motion:

$$\Phi(x, y, a_1, a_2, \dots, a_n) = 0.$$

- Case $n = 2$; $\Phi(x, y, a_1, a_2) = 0$.
  Get ODE by $y = f(x, a_1, a_2)$, $y' = g(x, a_1, a_2)$, $y'' = h(x, a_1, a_2)$: eliminate (a_1, a_2) between these three relations get $\Psi(x, y, y', y'') = 0$.
- Example: $\Phi(x, y, a_1, a_2) = y^2 - ax^2 - bx - c = 0$.

$$2yy' - 2ax - b = 0,$$

$$yy'' = a - (y')^2,$$

$$4y^3y'' = 4ay^2 - (2ax + b)^2$$

$$= 4a(ax^2 + bx + c) - (2ax + b)^2 = 4ac - b^2,$$

$$y^3y'' + F = 0, \quad F = b^2 - 4ac.$$

- Symmetry $(a, b) \leftrightarrow (x, y)$.
    - Dual differential equation
    - Bifoliate structure
    - Associated Cartan-Tanaka-Chern-Moser structure.
    - Associated Fefferman conformal structure.
- Dual equation for $y'' + Fy^{-3} = 0$.

$$y^2 = ax^2 + bx + c, \quad F = b^2 - 4ac,$$

$$y^2 = a'x^2 + b'x + c',$$

$$y^2 = a''x^2 + b''x + c''.$$

$$[P, Q, R] = [a - a, b - b', c - c'] \times [a - a'', b - b'', c - c''].$$

$$0 = Q^2 - PR,$$

$$0 = bb' - 2a'c - 2ac' = 0,$$

$$0 = (b')^2 + bb'' - 2a''c - 2ac'' - 4a'c' = 0.$$

Algebraic differential equation, quadratic in second derivatives, degree six in first derivatives.

- Case $n = 3$: $\Phi(x, y, a, b, c) = 0$:
  Strange asymmetry between the (x, y)-space and the (a, b, c)-space.
  - Wunschmann invariant zero: information coded in a conformal three-geometry: standard twistor construction gives an associated Fefferman conformal structure in six dimensions, (3, 3)-signature.
  - Wunschmann non-zero: no conformal three-geometry; but generalized Fefferman conformal structure still persists. Believe it is a (split) G_2-structure.

  Problem: understand this from first principles.
- Case $n = 4$: ????
- Case $n > 4$: ?????

What is a quantum massless particle?

- The Hilbert space $\mathcal{H}$ carries the standard representation of the
  Heisenberg algebra for a system of n degrees of freedom:
  $$[k_a, k_b] = 0, \quad [k_a, x^b] = i\hbar\delta_a^b, \quad [x^a, x^b] = 0.$$
  Here x^a is the quantum operator $x^a = -i\hbar\dfrac{\partial}{\partial k_a}$.
- We have $\mathcal{H} = \mathcal{H}^+ + \mathcal{H}^-$, where $f \in \mathcal{H}^\pm$ iff $f(-k) = \pm f(k)$.
- Then the second quantization is straightforward:
  - $\mathcal{H}^+$ is the one-particle boson Hilbert space
  - $\mathcal{H}^-$ is the one-particle fermion Hilbert space.
  - The second quantization of the operators k_a and x^b gives the superalgebra:
    $$[k_a, k_b]_+ = E_{ab}, \quad [k_a, x^b] = E_a^b, \quad [x^a, x^b]_+ = E^{ab}, \ldots.$$
    Here the E operators generate the symplectic algebra.
  - The algebra has $2n$ odd generators
  - The algebra has $n(2n+1)$ even generators.
  - The algebra is called the orthosymplectic algebra.

The case $n = 4$

- The case $n = 4$ gives the massless particles of relativity. We write k_a as a complex two-component spinor: $k_{A'}$. The expansion of X is then written as:

$$X = x + x^{A'} k_{A'} + \overline{x}^{A} \overline{k}_{A} + \overline{x}^{AB} \overline{k}_{A} \overline{k}_{B} + x^{AA'} k_{A'} \overline{k}_{A} + x^{A'B'} k_{A'} k_{B'} + \ldots$$

If we suppress the other components of X, $\phi(X, f)$ becomes a function on the spin bundle of space-time:

$$\phi(X, f) = \phi(x^{AA'}, \overline{x}^{A}, \overline{x}^{A'}).$$

The field equation is just:

$$\partial_{AA'} \phi = \overline{\partial}_{A} \partial_{A'} \phi.$$

Note that in particular we have the massless Klein-Gordon equation:

$$\partial_{AA'} \partial_{BB'} \phi = \partial_{BA'} \partial_{AB'} \phi.$$

Basic twistor formulas

- A twistor Z^α is a pair $(\omega^A,\ \pi_{A'})$ of Weyl spinors.
- The relativistic twistor "norm squared" is:

$$Z^\alpha \overline{Z}_\alpha = \omega^A \overline{\pi}_A + \pi_{A'} \overline{\omega}^{A'}.$$

- Here $\overline{Z}_\alpha = (\overline{\pi}_A,\ \overline{\omega}^{A'})$ is the conjugate of Z^α.
- The invariance group is $\mathbb{SU}(2,2)$ (real dimension 15). This group covers the conformal group of spacetime.
- The incidence relation of a twistor Z^α with a point $x^a = x^{AA'}$ in complex Minkowski space-time is: $\omega^A = ix^{AA'}\pi_{A'}$. Then $Z^\alpha \overline{Z}_\alpha = -2y^a \pi_{A'}\overline{\pi}_A$. Here y^a is the imaginary part of x^a.
  - If y^a is past pointing timelike then $Z^\alpha \overline{Z}_\alpha > 0$.
  - If y^a is future pointing timelike then $Z^\alpha \overline{Z}_\alpha < 0$.
  - If y^a is zero then $Z^\alpha \overline{Z}_\alpha = 0$ (these are called *null* twistors).
- A null twistor corresponds to a real null geodesic:

$$x^a = -i(\overline{\omega}^{B'}\pi_{B'})^{-1})\omega^A \overline{\omega}^{A'} + s\pi^{A'}\overline{\pi}^A.$$

Geometrical relations

- A point in space-time is a 2-D plane in twistor space.
  - If Y^α and Z^α lie in the plane, then the skew tensor: $X^{\alpha\beta} = Y^\alpha Z^\beta - Z^\alpha Y^\beta$ represents the point.
  - Conversely $X^{\alpha\beta}$ represents a point iff: $X^{[\alpha\beta}X^{\gamma\delta]} = 0$. This defines a quadric in $\mathbb{CP}^5$, the Klein quadric. It has a natural conformally flat conformal structure.
  - Z^α is incident with $X^{\alpha\beta}$ iff $Z^{[\alpha}X^{\beta\gamma]} = 0$.
  - A dual twistor W_α is incident with $X^{\alpha\beta}$ iff $W_\alpha X^{\alpha\beta} = 0$.
- The $X^{\alpha\beta}$ incident with a given Z^α form a two-plane.
  - This is completely null and is called an α-plane.
- The $X^{\alpha\beta}$ incident with a given W_α form a two-plane.
  - This is completely null and is called a β-plane.
  - Two α-planes meet in a unique point. Two β-planes meet in a unique point.
  - An α-plane (Z^α) and a β-plane (W_α) meet iff $Z^\alpha W_\alpha = 0$. Then they meet in a projective line: a null geodesic. Conversely each null geodesic gives rise to a unique projective pair (Z^α, W_α) with $Z^\alpha W_\alpha = 0$.

Anatomy of a relativistic massless particle

The phase space of a classical zero rest mass particle may be described by:

- $\underline{n}$ a real unit three-vector.
- $\underline{J}$ the angular momentum three-vector.
- These obey the Poisson bracket relations:

$$\{\underline{J}.\underline{a}, \underline{J}.b\} = (\underline{a} \times \underline{b}).\underline{J}, \;\; \{\underline{J}.\underline{a}, \underline{n}.b\} = (\underline{a} \times \underline{b}).\underline{n}, \;\; \{\underline{n}.\underline{a}, \underline{n}.b\} = 0.$$

- In addition there are two canonically conjugate scalars t and u, which each Poisson commute with both $\underline{J}$ and $\underline{n}$ and which obey the relation:

$$\{t, u\} = 1.$$

- The particle four-momentum is then $p = e^u[1, \underline{n}]$.
- The boost operator is $\underline{K} = t\underline{n} + \underline{J} \times \underline{n}$.
- The helicity Casimir operator is $S = \underline{J}.\underline{n}$.

Anatomy of a relativistic massless particle

Altogether there are six degrees of freedom for a particle with a given helicity. This information is neatly packaged in a classical twistor:

$$Z^\alpha = (\omega^A, \pi_{A'}).$$

- Energy-momentum four-vector: $p_a = \pi_{A'}\pi_A$
- Lorentz generators: $M_{ab} = i\epsilon_{AB}\pi_{(A'}\overline{\omega}_{B')} - i\epsilon_{A'B''}\overline{\pi}_{(A}\overline{\omega}_{B)}$
- Helicity: $2S = Z^\alpha \overline{Z}_\alpha = \pi_{A'}\overline{\omega}^{A'} + \omega^A\overline{\pi}_A$.

Here Z^α and $e^{i\theta}Z^\alpha$ represent the same particle for any real θ. The commutation relations (Poisson brackets) are simple:

$$\{Z^\alpha, Z^\beta\} = 0, \quad \{\overline{Z}_\alpha, \overline{Z}_\beta\} = 0, \quad \{Z^\alpha, \overline{Z}_\beta\} = i\delta^\alpha_\beta.$$

What is a quantum massless particle?

Fix n a positive integer.

- The Hilbert space is $\mathbb{L}^2(\mathbb{R}^n)$.
- A state is $f(k)$, a complex square-integrable function of $k \in \mathbb{R}^n$.
- The field $\Phi(X, f)$ is given by the Fourier integral formula:

$$\Phi(X, f) = \int e^{\frac{i}{\hbar}X(k)} f(k) d^n k.$$

  Here $X(k)$ is a real function.

- For the field equations we may use functional differentiation techniques, or more simply, we expand $X(k)$ as a power series:

$$X(k) = x + x^a k_a + x^{ab} k_a k_b + x^{abc} k_a k_b k_c + \dots.$$

  Here each coefficient tensor $x^{a_1 a_2 \dots a_k}$ is real and symmetric. Then the field equations are (one for each positive integer k): $\partial_{a_1 a_2 \dots a_k} \Phi(X, f) = \partial_{a_1} \partial_{a_2} \dots \partial_{a_k} \Phi(X, f)$.

The second quantization

- The Hilbert space $\mathcal{H}$ carries the standard representation of the Heisenberg algebra for a system of n degrees of freedom:

$$[k_a, k_b] = 0, \quad [k_a, x^b] = i\hbar\delta_a^b, \quad [x^a, x^b] = 0.$$

Here x^a is the quantum operator $x^a = -i\hbar\frac{\partial}{\partial k_a}$.

- We have $\mathcal{H} = \mathcal{H}^+ + \mathcal{H}^-$, where $f \in \mathcal{H}^\pm$ iff $f(-k) = \pm f(k)$.
- Then the second quantization is straightforward:
  - $\mathcal{H}^+$ is the one-particle boson Hilbert space
  - $\mathcal{H}^-$ is the one-particle fermion Hilbert space.
  - The second quantization of the operators k_a and x^b gives the superalgebra:

$$[k_a, k_b]_+ = E_{ab}, \quad [k_a, x^b] = E_a^b, \quad [x^a, x^b]_+ = E^{ab}, \ldots.$$

    Here the E operators generate the symplectic algebra.
  - The algebra has $2n$ odd generators
  - The algebra has $n(2n+1)$ even generators.
  - The algebra is called the orthosymplectic algebra.

The case $n = 4$ gives massless particles

- When $n = 4$, we write the four-vector k_a as a complex two-component Lorentzian spinor: $k_{A'}$.
- The expansion of X is then written as:

$$X = x + x^{A'}k_{A'} + \overline{x}^{A}\overline{k}_{A} + \overline{x}^{AB}\overline{k}_{A}\overline{k}_{B} + x^{AA'}k_{A'}\overline{k}_{A} + x^{A'B'}k_{A'}k_{B'} + \dots$$

Here $x^{A'B'} = x^{B'A''}$ and $x^{AA'} = \overline{x}^{AA'}$.
If we suppress the other components of X, the field $\phi(X, f)$ becomes a function on the spin bundle of space-time:

$$\phi(X, f) = \phi(x^{AA'}, \overline{x}^{A}, \overline{x}^{A'}).$$

The field equation is just:

$$\partial_{AA'}\phi = \overline{\partial}_{A}\partial_{A'}\phi.$$

In particular the massless Klein-Gordon equation holds:

$$\partial_{AA'}\partial_{BB'}\phi = \partial_{BA'}\partial_{AB'}\phi.$$

The twistor operators

- $Z^\alpha = (\hbar\overline{\partial}^A,\ k_{A'})$ and $\overline{Z}_\alpha = (\overline{k}_A,\ -\hbar\partial^{A'})$ are the twistor operators. These obey the Heisenberg algebra:

  $$[Z^\alpha,\ Z^\beta] = 0,\ \ [\overline{Z}_\alpha,\ \overline{Z}_\beta] = 0,\ \ [Z^\alpha,\ \overline{Z}_\beta] = \hbar\delta^\beta_\alpha.$$

  The helicity operator S is given by the formula:

  $$4S = Z^\alpha\overline{Z}_\alpha + \overline{Z}_\alpha Z^\alpha = 2\hbar(\overline{k}_A\overline{\partial}^A - k_{A'}\partial^{A'}).$$

  The E operators include the generators of $\mathbb{U}(2,2)$: $2E^\alpha_\beta = Z^\alpha\overline{Z}_\beta + \overline{Z}_\beta Z^\alpha$. In particular, writing $Z^\alpha = (\omega^A, \pi_{A'})$, we have the fundamental relation of supersymmetry:

  $$\pi_{A'}\overline{\pi}_A + \overline{\pi}_A\pi_{A'} = 2P_{AA'}.$$

  Here P_a is the standard Poincare energy-momentum operator. *Note however that the second supersymmetric relation that is usually imposed also, namely the formula*:

  $$\pi_{A'}\pi_{B'} + \pi_{A'}\pi_{B'} = 0$$

  does not hold here.

The twistor quantization

- The twistor quantization quantizes Z^α as a multiplication operator:
$$f(Z) \to Z^\alpha f(Z).$$
Then $\overline{Z}_\alpha$ is quantized as a derivative: $\overline{Z}_\alpha = -\hbar \dfrac{\partial}{\partial Z^\alpha}$.
- The quantization uses locally defined *holomorphic* functions
on the twistor space to represent the particle states.
These are encoded into the first sheaf cohomology group of the twistor space, with holomorphic coefficients.
- The helicity operator is:
$$S = -\frac{\hbar}{2}(Z^\alpha \partial_\alpha + 2).$$

Helicity eigen-states

So homogeneous holomorphic twistor functions represent helicity eigenstates:

- Degree -6: helicity 2 (graviton)
- Degree -5: helicity $\frac{3}{2}$ (gravitino??)
- Degree -4: helicity 1 (photon)
- Degree -3: helicity $\frac{1}{2}$ (neutrino?)
- Degree -2: helicity 0 (scalar massless ??)
- Degree -1: helicity $-\frac{1}{2}$ (neutrino?)
- Degree 0: helicity -1 (photon)
- Degree 1: helicity $-\frac{3}{2}$ (gravitino??)
- Degree 2: helicity -2 (graviton)

The nice unitary representations of $\mathbb{SU}(2,2)$

- The group $\mathbb{SU}(2,2)$ is a rank three non-compact Lie group.
- Its nicest unitary representations are those of the discrete series. These may be characterized in terms of their decomposition into unitary Poincare representations, relative to a chosen Poincare subgroup of the group.
- There is a lowest spin j_1 and a highest spin j_2 (which differ by an integer).
  Each spin in between occurs exactly once, with the intermediate spins j differing from j_1 by an integer.
- The allowed particle spectrum may be described as a free assemblage of:
  - spin one half quarks
  - spin one half anti-quarks
  - spin zero di-quarks
  - spin zero anti-di-quarks.

  Each of these ingredients is treated as a boson for the sake of constructing the representation.

The interpretation; the discrete series boundaries

- We think of the generic discrete series representations as describing an idealized hadron. In this language the three degrees of freedom corresponding to the rank of the Lie group are the lowest and highest spins of the representation and the "baryon number." They require three-twistors for their description ($n = 12$ above).
- The first boundary of the discrete series are representations that can be described by functions of two twistors ($n = 8$ above). These form the "walls" of the three-dimensional representation space. They are each of a fixed Poincare spin. We think of these boundary discrete series representations as describing an idealized lepton.
- The second boundary of the discrete series is located where the walls of the representation space meet. This gives the standard one-twistor massless particles.