Nonlinear eigenvalue problems in practice:

Analysis and numerical methods

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Outline

The analysis of the dynamical/acoustic behavior of
structures, vehicles, or molecules
needs the numerical solution of nonlinear eigenvalue problems.

- ▷ Such systems have been solved for decades!
- ▷ The mathematics is well-known and used in industrial engineering every day!
- ▷ The numerical methods are available in (commercial) software!
- ▷ We just buy bigger computers to handle the higher complexity?
- ▷ Do we still need to talk about it?
- ▷ Do we need improved numerical methods?
- ▷ **What are the challenges?**

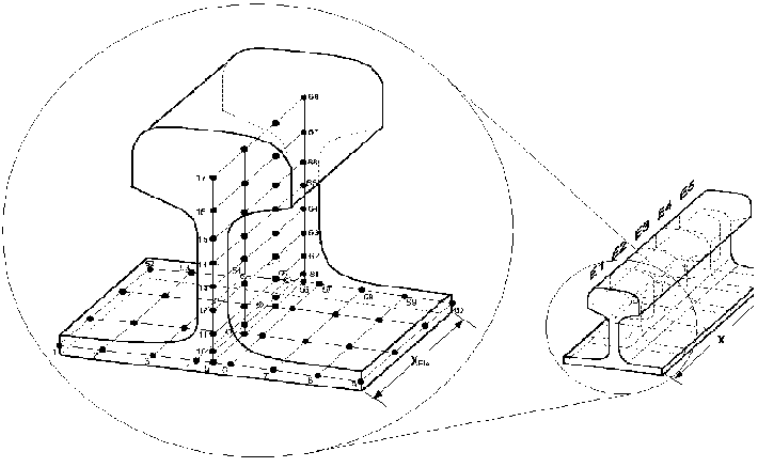
Optimality through mathematics

- Society is increasingly sensitive to inconveniences that come with modern technologies such as pollution and noise.
- There is an increasing demand for optimal solutions. Minimal energy consumption, minimal noise, pollution, waste.
- Optimal solutions need mathematical techniques, such as model based optimization/optimal control.
- We need better mathematical models, faster and more accurate numerical methods, robust implementations on modern computer architectures.
- **The progress through better methods exceeds the progress through better hardware by large factors.**

Outline


DB

FEM in space



Infinite dimensional second order system

Under the assumption of an infinite rail, FEM in space leads to the second order system

$$\mathcal{M}\ddot{z} + \mathcal{D}\dot{z} + \mathcal{K}z = \mathcal{F},$$

with symmetric infinite block tridiagonal coefficient matrices (operators) $\mathcal{M}, \mathcal{D}, \mathcal{K}$, where

$$\mathcal{M} = \begin{bmatrix} \ddots & \ddots & 0 & \dots & 0 \\ \ddots & M_{j-1,0} & M_{j,1} & 0 & \dots \\ 0 & M_{j,1}^T & M_{j,0} & M_{j+1,1} & 0 \\ \vdots & \ddots & M_{j+1,1}^T & M_{j+1,0} & M_{j+2,1} \\ 0 & \dots & 0 & \ddots & \ddots \end{bmatrix} z = \begin{bmatrix} \vdots \\ z_{j-1} \\ z_j \\ z_{j+1} \\ \vdots \end{bmatrix},$$

Operators $\mathcal{D}, \mathcal{K}$ have the same structure. Furthermore, $M_{j,0} > 0$, $D_{j,0}$, $K_{j,0} \geq 0$.

Fourier expansion

$$F_j = \hat{F}_j e^{i\omega t}, \; z_j = \hat{z}_j e^{i\omega t},$$

where ω is the excitation frequency.
Using periodicity and combining ℓ parts into one vector

$$y_j = \begin{bmatrix} \hat{z}_j^T & \hat{z}_{j+1}^T & \dots & \hat{z}_{j+\ell}^T \end{bmatrix}^T$$

gives a (singular) difference equation

$$A_1(\omega) y_{j+1} + A_0(\omega) y_j + A_1(\omega)^T y_{j-1} = G_j.$$

with $A_0(\omega) = A_0^T(\omega)$ block tridiagonal and $A_1(\omega)$ singular of rank smaller than $n/2$.

The associated eigenvalue problem

Ansatz $y_{j+1} = \kappa y_j$, leads to the large scale rational eigenvalue problem

$$R(\kappa)x = (\kappa A_1(\omega) + A_0(\omega) + \frac{1}{\kappa} A_1(\omega)^T)x = 0.$$

Alternative representation as so called palindromic polynomial eigenvalue problem

$$P(\lambda)x = (\lambda^2 A_1(\omega) + \lambda A_0(\omega) + A_1(\omega)^T)x = 0.$$

Summary of railtrack problem

- ▷ Large scale nonlinear structured eigenvalue problem.
- ▷ All (commercial/non-commercial) methods failed (no correct digits in double precision).
- ▷ Many infinite and zero eigenvalues, structured deflation necessary (second talk).
- ▷ Effective use of structure (second talk).
- ▷ $\rightarrow$ film.

Outline

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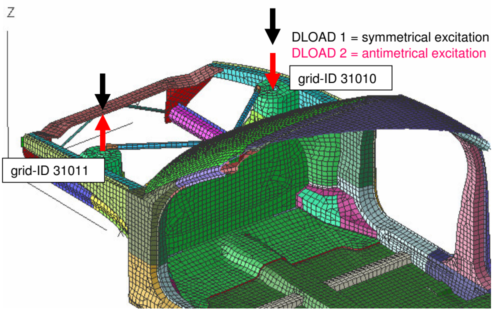
SFE AKUSMOD

FE Model: Excitation

Unit force = 1 N mm



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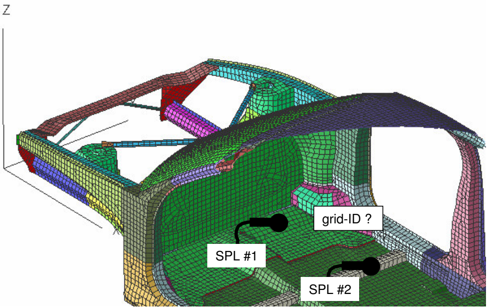
FE Model: SPL for two microphone positions



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- Parameterized FEM model for car body, as well as air in car.
- Geometry and topology changes lead ad hoc to new linear systems/eigenvalue problems (up to size $10,000,000$).
- Goal: Minimize noise in important regions in car interior.

Tasks: Numerical methods for large scale (complex symmetric) linear systems (frequency response) and eigenvalue problems (model reduction, modal analysis, optimization of frequencies).

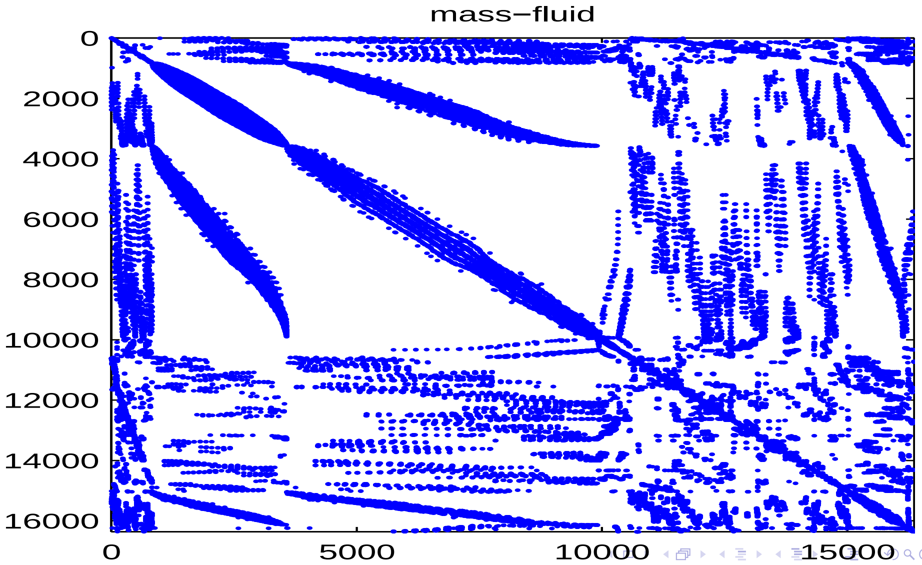
Solve $P(\omega)u(\omega) = f(\omega)$, for $\omega = 0 - 1000hz$, where

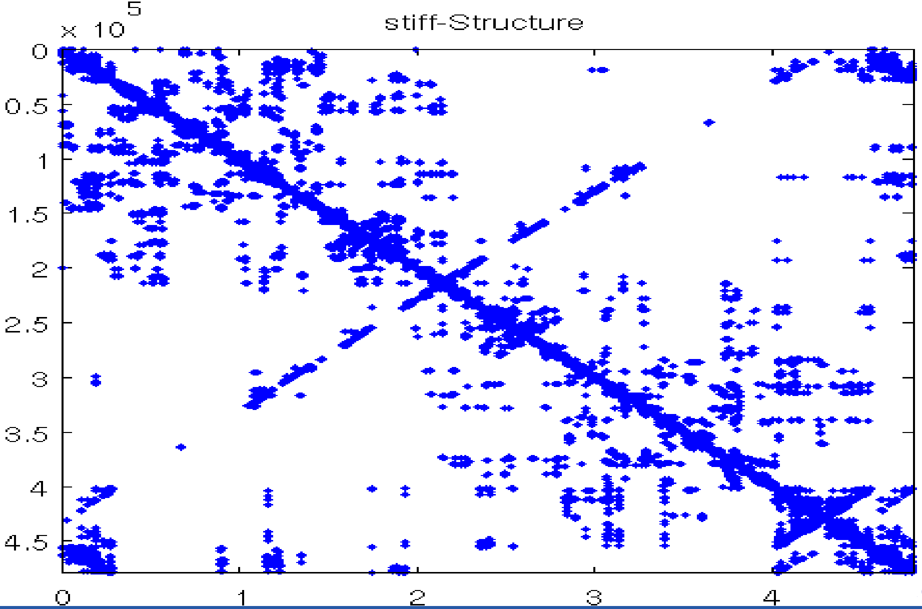
$$P(\omega) := -\omega^2 \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} + \imath\omega \begin{bmatrix} D_s & D_{as}^T \\ D_{as} & D_f \end{bmatrix} + \begin{bmatrix} K_s(\omega) & 0 \\ 0 & K_f \end{bmatrix},$$

is complex symmetric of dimension up to $10,000,000$ and

- M_s, M_f, K_f are real symm. pos. semidef. mass/stiffness matrices of structure and air, M_s is singular and diagonal, M_f is sparse.
- $K_s(\omega) = K_s(\omega)^T = K_1(\omega) + \imath K_2$.
- D_s is a real damping matrix, D_f is complex symmetric.
- D_{as} is real coupling matrix between structure and air.
- All matrices depend on geometry, topology and material parameters.

Sparsity of fluid mass matrix M_f




x 10^5
stiff-Structure
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
0
1
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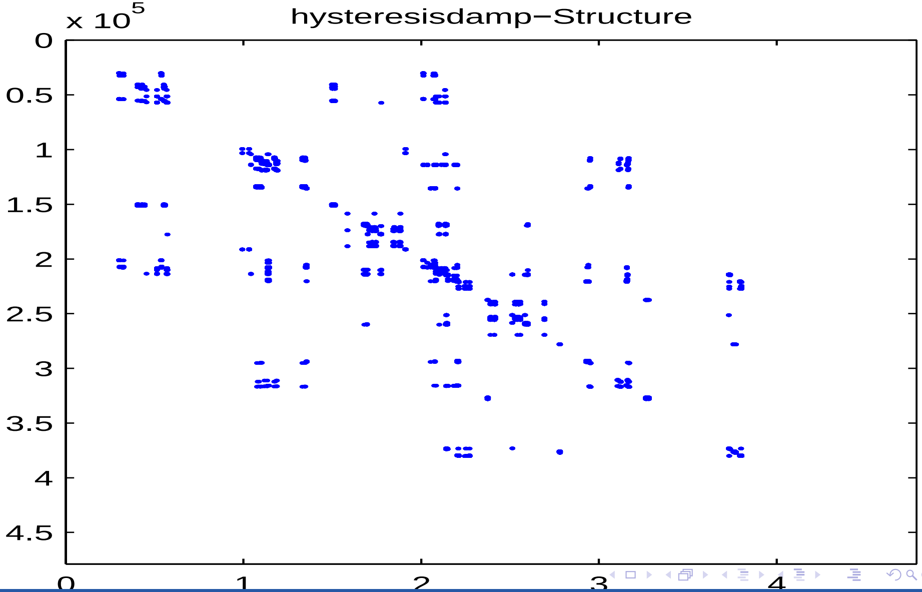
3
4

Sparsity of K_2



Math. model: Eigenvalue problem

Compute smallest real eigenvalues and associated eigenvectors of $P(\lambda)x = 0$, where the matrix polynomial

$$P(\lambda) := \lambda^2 \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} + \lambda \begin{bmatrix} D_s & D_{as}^T \\ D_{as} & D_f \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & K_f \end{bmatrix},$$

is **complex symmetric** and has dimension up to $10,000,000$.

Tasks:

- ▷ Project the problem into the subspace spanned by these eigenvectors.
- ▷ Solve the second order differential-algebraic system (DAE).
- ▷ Optimize the eigenfrequencies w.r.t. the set of parameters.

Summary of car acoustic problem

- Large scale nonlinear (complex symmetric) eigenvalue problem arising from coupled FEM models for structure/fluid.
- Problem contains parameters to be optimized.
- Eigenvalue path following.
- Homotopy from undamped to damped problem.
- Shift-and-invert Lanczos/Arnoldi/Jacobi Davidson.
- Subspace recycling (warm restarts).
- Model reduction for optimization (third talk).
- We should really use adaptive FEM for eigenvalue problem. (Not much theory and no code for non-elliptic problems).

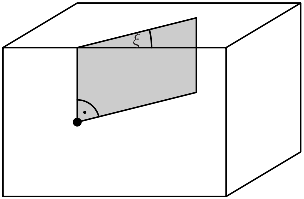
Outline

Near a singularity, the displacement field U of an elastic body can be expanded (Kondrat'ev 1967), as

$$\gamma r^{\alpha} u(\phi, \theta),$$

where γ is the stress intensity factor and $u(\phi, \theta)$ is angular part of U in spherical coordinates.

Example: Crack in 3D Domain Ω



The Operator Eigenvalue Problem

The singular exponent α, (changed to $\lambda = \alpha + 1/2$) satisfies the operator evp:

$$\lambda^2 m(u,v) + \lambda g(u,v) - k(u,v) = 0,$$

with sesquilinear forms

$$\begin{aligned} m(u,v) &= m(v,u), \\ g(u,v) &= -g(v,u), \\ k(u,v) &= k(v,u). \end{aligned}$$

Finite element subspace V_{0h}.

Take continuous functions from V_0 which are piecewise linear on a triangular finite element mesh $\mathcal{T}_h$. With

$$\lambda_h \in \mathbb{C}, \quad u_h \in V_{0h} \setminus \{0\}$$

we have a finite dimensional problem

$$\lambda_h^2\, m(u_h, v_h) + \lambda_h\, g(u_h, v_h) - k(u_h, v_h) = 0 \quad \forall v_h \in V_{0h}$$

and with appropriate FEM bases, a so called even quadratic evp

$$P(\lambda)u = \lambda^2 Mu + \lambda Gu - Ku = 0,$$

with $M = M^T > 0$, $G = -G^T$, $K = K^T > 0$.

Theorem (Karma 1996)

Consider an eigenpair (λ, u).
Let κ *be the maximal size of an associated Jordan block. For a sequence of eigenpairs* $\{(\lambda_h, u_h)\}_{h\to 0}$ *with* $\lambda_h \to \lambda_0$ *the estimates*

$$\begin{aligned} |\lambda_0 - \lambda_h| &\leq Ch^{2/\kappa}, \\ \|u_0 - u_h\|_V &\leq Ch^{\nu}, \quad \nu = \min\{1, 2/\kappa\} \end{aligned}$$

hold.

Approximation results II

Theorem (Karma 1996)

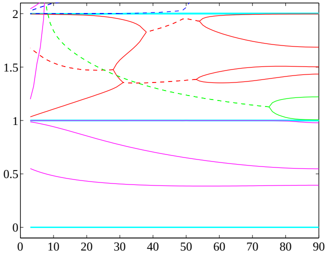
For an eigenvalue λ_0 with algebraic multiplicity m there exist m disjoint sequences $\{\lambda_{h,i}\}$ with $\lambda_{h,i} \to \lambda_0$, $i = 1, \dots, m$. Then for the arithmetic mean $\hat{\lambda}_h := \frac{1}{m}\sum_{i=1}^{m} \lambda_{h,i}$ the improved estimate

$$|\lambda_0 - \hat{\lambda}_h| \leq Ch^2$$

holds.

Results of structured method SHIRA

Ev's with real part in $(0.1, 2.1)$. Dashed: nonreal eigenvalues. Triple ev's $\alpha = 0$ and $\alpha = 1$, 3 simple real ev's.



Numerical methods for nonlinear evps

- For polynomial/rational eigenvalue problems, use **linearization** to turn problem into a larger linear evp.
- Apply the known methods for the linear evp, Lanczos, Arnoldi, Jacobi-Davidson, inverse iteration, ...

- For genuine nonlinear problems there are several approaches, all variations of Newton's method. None of them is global and robust.
- **Many open problems**

For surveys see M./Voss 2005 or Dissertation Schreiber 2008.

Methods directly for nonlinear problem

- Second order Arnoldi method Bai 2006
- Rational Krylov method Ruhe 1998, 2000
- Residual iteration method Neumaier 1985
- Newton methods Schreiber/Schwetlick 2006, 2008,
- Rayleigh quotient iterations Schreiber 2008, Freitag/Spence 2007, 2008
- Jacobi-Davidson method Sleijpen/Van der Vorst et al 1996, Betcke/Voss 2004, Hochstenbach 2007
- Arnoldi type methods Voss 2003
- ...

Only few of these methods make use of structure (second talk), convergence theory and preconditioning needs more analysis.

For nonlinear EVP $\mathcal{F}(\lambda)x = 0$ consider nonlinear problem

$$\begin{bmatrix} \mathcal{F}(\lambda)x \\ v^H x - 1 \end{bmatrix} = 0,$$

with some normalization vector v. One Newton step gives

$$\begin{bmatrix} \mathcal{F}(\lambda_k) & \mathcal{F}'(\lambda_k)x_k \\ v^H & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} - x_k \\ \lambda_{k+1} - \lambda_k \end{bmatrix} = - \begin{bmatrix} \mathcal{F}(\lambda_k)x_k \\ v^H x_k - 1 \end{bmatrix}.$$

The first component yields

$$x_{k+1} = -(\lambda_{k+1} - \lambda_k)\mathcal{F}(\lambda_k)^{-1}\mathcal{F}'(\lambda_k)x_k,$$

i.e. the direction of the new approximation is
$u_{k+1} := \mathcal{F}(\lambda_k)^{-1}\mathcal{F}'(\lambda_k)x_k$. Assuming that $v^H x_k = 1$, we get

$$\lambda_{k+1} = \lambda_k - \frac{v^H x_k}{v^H u_{k+1}}.$$

Analysis of Newton's method

- For simple eigenvalues Newton's method converges locally quadratically Anselone/Rall 68,Osborne 64.
- One needs very good starting values.
- No guarantee that one gets the desired eigenvalues.
- No direct use of special structures.
- Many matrix factorizations are needed.

Variation of Newton's method

Change normalization vector v in each step to $v_k = \mathcal{F}(\lambda_k)^H y_k$, where y_k is an approximation of a left eigenvector. Then one gets

$$\lambda_{k+1} = \lambda_k - \frac{y_k^H \mathcal{F}(\lambda_k) x_k}{y_k^H \mathcal{F}'(\lambda_k) x_k},$$

which is a generalized Rayleigh functional Lancaster 2002. This is a Newton step for

$$f_k(\lambda) := y_k^H \mathcal{F}(\lambda) x_k = 0.$$

- For linear Hermitian eigenproblems, cubic convergence Crandall 51, Ostrowski 58.
- Analysis for nonlinear symmetric eigenproblems Rothe 1989.
- More analysis in Schreiber 2008, Schreiber/Schwetlick 2008.

Residual inverse iteration

To avoid large number of factorizations, Neumaier 1995 suggested the residual inverse iteration.
If $\mathcal{F}(\lambda)$ is twice continuously differentiable, then

$$\begin{aligned} x_k - x_{k+1} &= x_k + (\lambda_{k+1} - \lambda_k)\mathcal{F}(\lambda_k)^{-1}\mathcal{F}'(\lambda_k)x_k \\ &= \mathcal{F}(\lambda_k)^{-1}(\mathcal{F}(\lambda_k) + (\lambda_{k+1} - \lambda_k)\mathcal{F}'(\lambda_k))x_k \\ &= \mathcal{F}(\lambda_k)^{-1}\mathcal{F}(\lambda_{k+1})x_k + \mathcal{O}(|\lambda_{k+1} - \lambda_k|^2). \end{aligned}$$

Neglect second order term and replace λ_k by fixed shift σ:

$$x_{k+1} = x_k - \mathcal{F}(\sigma)^{-1}\mathcal{F}(\lambda_{k+1})x_k.$$

Another variation of this idea is the method of successive linear problems of Ruhe 1973 which converges locally quadratically.

Set $U_1 = [u]$ for starting vector u.

For $k = 1, 2, \ldots,$

Solve nonlinear evp $U_k^H F(\lambda) U_k c = 0$ for (λ, c).

Set $u = U_k c$, $r = F(\lambda) u$.

If $\frac{\|r\|}{\|u\|} < \epsilon$ STOP

Compute s orthogonal to u from correction eq.

$$\left(I - \frac{\dot{F}(\lambda) u u^H}{u^H \dot{F}(\lambda) u} \right) F(\lambda) \left(I - u u^H \right) s = -r.$$

Set U_{k+1} to be the result of modified Gram-Schmidt applied to span(U_k, s).

The advantage of JD is that it is often sufficient to solve the correction equation approximately.

Evaluation of Newton type methods

- No guarantee that all desired eigenvalues are obtained.
- No guarantee to obtain desired relative residual?
- Methods are very sensitive to changes of parameters.
- Erratic convergence behavior?
- Locking and purging or deflation of converged eigenvalues?
- Code implementation ?

Detailed analysis and comparison: Dissertation Schreiber 2008.

In general, the situation is not satisfactory! If at all possible, linearization seems to be a more robust approach.

The classical companion linearization for polynomial eigenvalue problems

$$P(\lambda)x = \sum_{i=0}^{k} \lambda^i A_i x$$

is to introduce new variables

$$y^T = \left[\ y_1, y_2, \ldots, y_k\ \right]^T = \left[\ x, \lambda x, \ldots, \lambda^{k-1} x\ \right]^T$$

and to turn it into a generalized linear eigenvalue problem

$$L(\lambda)y := (\lambda \mathcal{E} + \mathcal{A})y = 0$$

of size $nk \times nk$.

Damped mechanical system:

$$(\lambda^2 M + \lambda D + K)x = 0$$

Introduce (velocity) $v = \lambda x$ and obtain companion form

$$\left(\begin{bmatrix} M & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} D & K \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} v \\ x \end{bmatrix} = 0.$$

Definition

For a matrix polynomial $P(\lambda)$ of degree k, a matrix pencil $L(\lambda) = (\lambda\mathcal{E} + \mathcal{A})$ is called linearization of $P(\lambda)$, if there exist nonsingular unimodular matrices (i.e., of constant nonzero determinant) $S(\lambda)$, $T(\lambda)$ such that

$$S(\lambda)L(\lambda)T(\lambda) = \mathrm{diag}(P(\lambda), I_{(n-1)k}).$$

A linearization is called strong if also $\mathrm{rev}L$ is a linearization of $\mathrm{rev}P$.

Properties of companion linearization

- Companion linearization preserves the algebraic and geometric multiplicities of all finite eigenvalues.
- There are some difficulties with multiple eigenvalues including ∞ and the singular part, Byers/M./Xu 2007.
- The geometric multiplicity of the eigenvalue ∞ and the sizes of singular blocks are not invariant under unimodular transformations.
- Companion linearization destroys the structure.

The matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 + \lambda + 1 & 1 \\ 1 & 0 \end{bmatrix}$$

has only the eigenvalue ∞. Multiplying from the left with

$$Q(\lambda) = \begin{bmatrix} 1 & -(\lambda^2 + \lambda + 1) \\ 0 & 1 \end{bmatrix}$$

we obtain

$$T(\lambda) = Q(\lambda)P(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is not necessary to perform a linearization.
Is this a matrix polynomial of degree $1, 2$, or one of degree 0 with leading coefficients 0?

Consider the Euler-Lagrange equations of a linear, constrained and damped mechanical system

$$\begin{array}{rcl} \hat{M}\ddot{x} + \hat{D}\dot{x} + \hat{K}x + \hat{G}^T\mu & = & f(t) \\ \hat{G}x & = & g. \end{array}$$

The associated matrix polynomial is

$$P(\lambda) = \lambda^2 \begin{bmatrix} \hat{M} & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} \hat{D} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{K} & \hat{G}^T \\ \hat{G} & 0 \end{bmatrix}.$$

If $\hat{M}$ is positive definite and $\hat{G}$ has full row rank, then the companion form has a Kronecker block associated with ∞ of size 4.

Modified first order form

The first order formulation used in multibody dynamics only introduces $y = \dot{x}$ and not $\gamma = \dot{\mu}$.

$$\begin{array}{rcl} M\dot{y} + D\dot{x} + Kx + G^T\mu & = & f(t), \\ \dot{x} & = & y, \\ Gx & = & 0 \end{array}$$

and the associated linear matrix pencil

$$\tilde{L}(\lambda) = \lambda \begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} D & K & G^T \\ -I & 0 & 0 \\ 0 & G & 0 \end{bmatrix},$$

has a Kronecker block at ∞ of size 3.

Pros and cons, companion linearization

Pros

- Simpler analysis for first order systems and linear evp's..
- Not much analysis methods for matrix polynomials.
- No generalization of Jordan/Kronecker canonical form for matrix polynomials.
- Locking, deflation and restart very difficult in nonlinear case.

Cons

- The condition number (sensitivity) may increase. Tisseur 2000, Higham/Mackey/Tisseur 2007.
- The size of the problem is increased.
- Symmetry structures may be lost.
- Approach only works for polynomial or rational nonlinearities.

Goal: Find a large class of linearizations for which:

- ▷ the linear pencil is easily constructed;
- ▷ structure preserving linearizations exist;
- ▷ the conditioning of the linear problem can be optimized; Higham/Mackey/Tisseur 06, Higham/Li/Tisseur 06.
- ▷ eigenvalues/vectors of the original problem are easily read off;
- ▷ we have structure preserving numerical methods;
- ▷ a structured perturbation analysis is possible.

Notation: $\Lambda := [\lambda^{k-1}, \lambda^{k-2}, \ldots, \lambda, 1]^T$, $\otimes$ - Kronecker product.

Definition (Mackey[2]/Mehl/M. 2006.)

For a given $n \times n$ matrix polynomial $P(\lambda)$ of degree k define the sets:

$$
\begin{aligned}
\mathcal{V}_P &= \{v \otimes P(\lambda) \,:\, v \in \mathbb{F}^k\}, \quad v \text{ is called right ansatz vector}, \\
\mathcal{W}_P &= \{w^T \otimes P(\lambda) : w \in \mathbb{F}^k\}, \quad w \text{ is called left ansatz vector}, \\
\mathbb{L}_1(P) &= \left\{L(\lambda) = \lambda\mathcal{E} + \mathcal{A} : \mathcal{E}, \mathcal{A} \in \mathbb{F}^{kn\times kn}, L(\lambda)\cdot(\Lambda \otimes I_n) \in \mathcal{V}_P\right\}, \\
\mathbb{L}_2(P) &= \left\{L(\lambda) = \lambda\mathcal{E} + \mathcal{A} :\, \mathcal{E}, \mathcal{A} \in \mathbb{F}^{kn\times kn}, (\Lambda^T \otimes I_n)\cdot L(\lambda) \in \mathcal{W}_P\right\} \\
\mathbb{DL}(P) &= \mathbb{L}_1(P) \cap \mathbb{L}_2(P).
\end{aligned}
$$

For $P(\lambda) = \lambda^2 M + \lambda D + K$, we have

$$\mathbb{L}_1(P) = \left\{ \lambda\mathcal{E} + \mathcal{A} : (\lambda\mathcal{E} + \mathcal{A}) \begin{bmatrix} \lambda I_n \\ I_n \end{bmatrix} = \begin{bmatrix} v_1 P(\lambda) \\ v_2 P(\lambda) \end{bmatrix} \right\}.$$

We have the freedom to choose the vector v. How can we use this freedom?

Proposition

For any $n \times n$ matrix polynomial $P(\lambda)$ of degree k,

- *$\mathbb{L}_1(P)$ is a vector space of dimension $k(k-1)n^2 + k$,*
- *$\mathbb{L}_2(P)$ is a vector space of dimension $k(k-1)n^2 + k$,*
- *$\mathbb{DL}(P)$ is a vector space of dimension k.*

These are not all linearizations but they form a large class.

The first and second companion forms

$$C_1(\lambda) := \lambda \begin{bmatrix} A_k & 0 & \cdots & 0 \\ 0 & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_n \end{bmatrix} + \begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix}$$

$$C_2(\lambda) := \lambda \begin{bmatrix} A_k & 0 & \cdots & 0 \\ 0 & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_n \end{bmatrix} + \begin{bmatrix} A_{k-1} & -I_n & \cdots & 0 \\ A_{k-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \cdots & 0 \end{bmatrix}.$$

are linearizations in $\mathbb{L}_1(P)$, $\mathbb{L}_2(P)$, respectively.

Theorem (Mackey, Mackey, Mehl, M. 2006)

Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k, and let $L(\lambda)$ be any pencil in $\mathbb{L}_1(P)$ with ansatz vector $v \neq 0$.

Then $x \in \mathbb{C}^n$ is a right eigenvector for $P(\lambda)$ with finite eigenvalue $\lambda \in \mathbb{C}$ if and only if $\Lambda \otimes x$ is a right eigenvector for $L(\lambda)$ with eigenvalue λ.

If in addition P is regular, i.e. $\det P(\lambda) \not\equiv 0$, and $L \in \mathbb{L}_1(P)$ is a linearization, then every eigenvector of L with finite eigenvalue λ is of the form $\Lambda \otimes x$ for some eigenvector x of P.

Similar results hold for $\mathbb{L}_2(P)$.

Lemma (Mackey, Mackey, Mehl, M. 2006)

Consider an $n \times n$ *matrix polynomial* $P(\lambda)$ *of degree* k*. Then, for* $v = (v_1, \ldots, v_k)^T$ *and* $w = (w_1, \ldots, w_k)^T$ *in* $\mathbb{F}^k$*, the associated pencil satisfies* $L(\lambda) = \lambda\mathcal{E} + \mathcal{A} \in \mathbb{DL}(P)$ *if and only if* $v = w$*.*

Theorem (Mackey, Mackey, Mehl, M. 2006)

Consider an $n \times n$ *matrix polynomial* $P(\lambda)$ *of degree* k*. Then for given ansatz vector* $v = w = [v_1, \ldots, v_k]^T$ *the associated linear pencil in* $\mathbb{DL}(P)$ *is a linearization if and only if no root of the* *v-polynomial*

$$p(v; x) := v_1 x^{k-1} + \ldots + v_{k-1} x + v_k$$

is an eigenvalue of P*.*

Strong Linearization Property

Theorem (Mackey, Mackey, Mehl, M. 2006)

Let $P(\lambda)$ be a regular matrix polynomial and $L(\lambda) \in \mathbb{L}_1(P)$ (or $L(\lambda) \in \mathbb{L}_2(P)$). Then the following statements are equivalent.

(i) *$L(\lambda)$ is a linearization for $P(\lambda)$.*

(ii) *$L(\lambda)$ is a regular pencil.*

(iii) *$L(\lambda)$ is a strong linearization for $P(\lambda)$.*

Example

Consider the eigenvalue problem.

$$P(\lambda) = \lambda^2 M + \lambda G + K$$

with K singular and take $v = e_1$.
Then $p(v; x) = 1x^1 + 0x^0$ has an eigenvalue 0, which is an eigenvalue of P if K is singular.

Are the classes large enough?

Theorem (Mackey, Mackey, Mehl, M. 2006)

For any regular $n \times n$ matrix polynomial $P(\lambda)$ of degree k, almost every pencil in $\mathbb{L}_1(P)$ ($\mathbb{L}_2(P)$) is a linearization for $P(\lambda)$.

For any regular matrix polynomial $P(\lambda)$, pencils in $\mathbb{DL}(P)$ are linearizations of $P(\lambda)$ for almost all $v \in \mathbb{F}^k$.

'Almost every' means for all but a closed, nowhere dense set of measure zero.

- Perturbation analysis Tisseur 00, Higham/Mackey/Tisseur 06, Higham/Li/Tisseur 06.
- Computation of a simple eigenvalue $\hat{\lambda}$ via the linearized eigenvalue problem is very ill-conditioned if $p(v, \hat{\lambda})$ is small.
- Proper scaling is necessary.
- **Open problem. Does the solution via a good, properly scaled, structure preserving linearization produce generally better results than the direct solution of the original structured problem.**

Summary linearization theory

- Companion linearization has some problems for infinite eigenvalue and singular parts.
- Companion linearization destroys the structure.
- If no 'bad eigenvalues' occur, then there exist structured linearizations in $\mathbb{DL}(P)$.
- Linearizations are ill-conditioned if eigenvalues near the bad eigenvalues occur. Higham/Mackey/Tisseur 2006.
- Unified theory for pseudospectra of matrix pencils/polynomials Ahmad 2008.
- We need to deflate bad eigenvalues/near bad eigenvalues?

How about the industrial problems?

- The railtrack problem has a majority of infinite eigenvalues (all interior FEM nodes). It also has a few eigenvalues near -1. Structured and nonstructured linearizations are very ill-conditioned.
- The problem in crack following has multiple eigenvalues and eigenvalues at 0.
- The complex symmetric car acoustic problem is sometimes singular and has a lot of infinite eigenvalues.

To get accurate numerical results **it is essential to deflate 'bad' evs in a structured way.** ($\longrightarrow$ second talk)

Outline

Num. Methods for linear problems

Many classical methods are available for generalized linear evp's

$$(\lambda E - A)x = 0.$$

- Inverse iteration.
- Generalized Rayleigh quotient method.
- Implicitly restarted (shift-and-invert) Arnoldi method, ARPACK.
- (Non)symmetric (shift-and-invert) Lanczos method.
- Quasi-minimal residual method QMR
- Jacobi-Davidson method.
- ...

Much better understanding, convergence analysis, implementations.

There are still many challenges for large scale linear problems, More talks this week.

- ▷ Preconditioning.
- ▷ Inner-outer iterations.
- ▷ Guaranteed convergence of all eigenvalues in a given region.
- ▷ Preservation of structure.
- ▷ Subspace recycling, warm starts.
- ▷ Adaptivity of discretization and ev. solver.
- ▷ ...

Outline

Summary

- Industrial PDE applications lead to nonlinear eigenvalue problems.
- These sometimes have extra structure that reflects the physical properties.
- The analysis and numerical solution methods for genuinely nonlinear eigenvalue problems are not well-enough understood.
- The eigenvalue methods should be intertwined with the discretization methods. (Adaptivity of discretization and eigenvalue iteration).
- The numerical methods need to reflect structures of the physical problems.
- Deflation of 'bad' parts of the spectrum is necessary for good numerical solutions.

information, papers, codes etc

`http://www.math.tu-berlin.de/~mehrmann`

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