Henselian fields with analytic structure

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Notation

K is a valued field

K° its valuation ring

$K^{\circ\circ}$ its maximal ideal

$\widetilde{K} = K^\circ / K^{\circ\circ}$ its residue field.

$|\cdot|$ is the norm,

$||\cdot||$ the corresponding gauss-norm.

$$T_n(K) = K\langle\xi\rangle = \{\sum a_\mu \xi^\mu : |a_\mu| \to 0 \text{ as } |\mu| \to \infty\}$$

is the ring of strictly convergent powerseries. For K complete they define analytic functions $(K^\circ)^n \to K$.

Background

•Ax-Kochen, and later Macintyre, studied the valued fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ in the algebraic language $\mathcal{L} = \langle +, \cdot, ^{-1}, \ldots \rangle$. The "uniformity" result is $\prod \mathbb{Q}_p/\mathcal{U} \equiv \prod \mathbb{F}_p((t))/\mathcal{U}$.

•Denef and van den Dries extended Q.E. to $\mathbb{Q}_p$ in an analytic language with function symbols for the elements of $T_n(\mathbb{Q}_p)$. The ring of functions depends on p.

•van den Dries, 1992: Consider $T_n(\mathbb{Z}[[t]]) = \mathbb{Z}[[t]]\langle \xi \rangle =$

$$\{\sum a_\mu \xi^\mu : Z[[t]] \ni a_\mu \to 0 \text{ t-adically as } |\mu| \to \infty\}.$$

Under the mappings $t \mapsto p$ and $t \mapsto t$ we have homomorphisms

$$T_n(\mathbb{Z}[[t]]) \to T_n(\mathbb{Q}_p) \quad \text{and} \quad T_n(\mathbb{Z}[[t]]) \to T_n(\mathbb{F}_p((t))),$$

so the elements of $T_n(\mathbb{Z}[[t]])$ naturally define functions on all the $\mathbb{Q}_p^\circ$ and $\mathbb{F}_p((t))^\circ$. Enrich the language $\mathcal{L}$ with symbols for these functions and get $\mathcal{L}_{an}$ and an analytic Ax-Kochen theorem

$$\prod \mathbb{Q}_p/\mathcal{U} \equiv \prod \mathbb{F}_p((t))/\mathcal{U}.$$

in $\mathcal{L}_{an}$.

•The Q.E. and other results of Denef-van den Dries for $\mathbb{Q}_p$ can be carried over to (complete) algebraically closed valued fields, e.g. $K = \mathbb{C}_p$. Here the appropriate rings of analytic functions are the rings of *separated power series*

$$S_{m,n}(K) = \bigcup_B B\langle\xi\rangle[[\rho]].$$

$B \in \{$subrings of K° generated by null-sequences$\}$.

The ξ_i vary over K° and the ρ_j vary over $K^{\circ\circ}$.

•For results uniform in p (Robinson, L) one uses

$$S_{m,n}(\mathbb{Z}[[t]]) = \mathbb{Z}[[t]]\langle\xi\rangle[[\rho]].$$

•van den Dries, Haskell, Macpherson studied the model theory of nonstandard models of $Th(\mathbb{Q}_p)$ in the analytic language for $\mathbb{Q}_p$ i.e. the language with symbols for the elements of $T_m(\mathbb{Q}_p)$. Crucial is the understanding of one-variable terms with parameters in an arbitrary model.

•(Cluckers, Robinson, L) considerably extended the notion of *fields with analytic structure* to prove a Denef-Pas type cell decomposition in this context and give uniformity results for analytic motivic integrals.

E is any noetherian ring complete and separated in its $I-$adic topology. I is a arbitrary nontrivial ideal of E. The "ring of analytic functions" is

$$S_{m,n}(E) = E\langle\xi\rangle[[\rho]].$$

The analytic structure on a valued field K is given by homomorphisms

$$E\langle\xi\rangle[[\rho]] \to \text{ functions } \big((K^\circ)^m \times (K^{\circ\circ})^n \to K^\circ\big).$$

•The algebra and model theory of this situation has been further developed by Çelikler.

•Key machinery

(i)Two notions of regularity and corresponding Weierstrass Preparation and Division Theorems in the two kinds of variables:

$$[\text{lower powers of } \xi_m] + \xi_m^s + [\text{higher powers of } \xi_m]$$

$$||\cdot|| \leq 1 \qquad\qquad\qquad ||\cdot|| < 1$$

$$[\text{lower powers of } \rho_n] + \rho_n^s + [\text{higher powers of } \rho_n]$$

$$||\cdot|| < 1 \qquad\qquad\qquad ||\cdot|| \leq 1$$

(ii) Noetherianness is used to see that for

$$\Sigma a_{\mu\nu}\xi^\mu\rho^\nu \in E\langle\xi\rangle[[\rho]]$$

the "dominant" coefficient is always among a finite number of the $a_{\mu\nu}$.

Indeed we get the **Strong Noetherian Property.**

Let $f \in A_{m+m',n+n'}$ and write $f = \sum_{\mu,\nu} \overline{f}_{\mu\nu}(\xi,\rho)(\xi')^{\mu}(\rho')^{\nu}$, where the $\overline{f}_{\mu\nu} \in A_{m,n}$. There is a finite set $J \subset \mathbb{N}^{m'+n'}$ and units of the form $1 + g_{\mu\nu}$ with $g_{\mu\nu} \in A^{\circ}_{m+m',n+n'}$, such that

$$f = \sum_{(\mu,\nu)\in J} \overline{f}_{\mu\nu}(\xi,\rho)(\xi')^{\mu}(\rho')^{\nu}(1 + g_{\mu\nu}).$$

A is an arbitrary ring and I is an ideal of A. $\widetilde{A} = A/I$. A **separated Weierstrass system** is a system $\mathcal{A} = \{A_{m,n}\}_{m,n\in\mathbb{N}}$ of A-algebras $A_{m,n}$, satisfying, for all $m \leq m'$ and $n \leq n'$,

(i) $A_{0,0} = A$,

(ii) $A_{m,n} \subset A[[\xi, \rho]]$,

(iii) $A_{m,n}[\xi'', \rho''] \subset A_{m',n'}$,

(iv) $\widetilde{A_{m,n}} \subset \widetilde{A}[\xi][[\rho]]$,

(v) if $f \in A_{m+m',n+n'}$, say $f = \sum_{\mu\nu} \overline{f}_{\mu\nu}(\xi, \rho)(\xi')^{\mu}(\rho')^{\nu}$, then the $\overline{f}_{\mu\nu}$ are in $A_{m,n}$,

(vi) the two Weierstrass Division Theorems hold,

(vii) a weak technical condition saying roughly that one has the strong noetherian property for $\Sigma a_{\mu\nu}\xi^{\mu}\rho^{\nu} \in A_{m,n}$. From this together with (vi) we get a "piecewise" Strong Noetherian Property.

Let $\mathcal{A} = \{A_{m,n}\}$ be a separated Weierstrass system, and let K be a valued field. A **separated analytic $\mathcal{A}$-structure** on K is a collection of homomorphisms $\{\sigma_{m,n}\}_{m,n\in\mathbf{N}}$, such that, for each $m, n \geq 0$,

$$\sigma_{m,n} : A_{m,n} \to \text{ functions } \big((K^\circ)^m \times (K^{\circ\circ})^n \to K^\circ\big)$$

and such that:

(1) $\sigma_{0,0}(I) \subset (K^{\circ\circ})$,

(2) $\sigma_{m,n}(\xi_i)$ = the i-th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$, $i = 1, \ldots, m$, and $\sigma_{m,n}(\rho_j)$ = the $(m+j)$-th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$, $j = 1, \ldots, n$, and

(3) $\sigma_{m,n+1}$ extends $\sigma_{m,n}$, where we identify in the obvious way functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ with functions on $(K^\circ)^m \times (K^{\circ\circ})^{n+1}$ that do not depend on the last coordinate, and $\sigma_{m+1,n}$ extends $\sigma_{m,n}$ similarly.

• The language of fields with $\mathcal{A}$–analytic structure is $\mathcal{L}_{\mathcal{A}}$, the language of valued fields with function symbols for the elements of $\bigcup A_{m,n}$.

• If we want to consider fields with **strictly convergent analytic structure** we (roughly speaking) omit the variables of the second kind (ρ), require the valuation ring to have a prime, and choose an element $\pi \in I$ to denote the prime.

Examples

1. $A_{m,n} = S_{m,n}(\mathbb{Z}[[t]]) = \mathbb{Z}[[t]]\langle\xi\rangle[[\rho]]$.

or $A_{m,n} = \bigcup_k \mathbb{Z}[\frac{1}{k}][[t]]\langle\xi\rangle[[\rho]]$ if we are only interested in models of equicharacteristic 0, for examples ultraproducts of the $\mathbb{Q}_p$.

2. $A_m = \mathbb{Z}_p\langle\xi\rangle$ if we are interested in fields elementarily equivalent to $\mathbb{Q}_p$ in the analytic language.

3. $A_{m,n} = S_{m,n}(K)$
Here $A = K^\circ$ and $I = K^{\circ\circ}$. If K is not discretely valued then A is not $I-$adically separated.

4. Let K be maximally complete. We can take $A_{m,n} = \bigcup_B B\langle\xi\rangle[[\rho]]$, where B varies over all subsets of K° with well-ordered support.

5. F trivially valued and $A_{m,n} = F[\xi][[\rho]]$.

6. Overconvergent powerseries.
The $\Sigma a_{\mu\nu}\xi^{\mu}\rho^{\nu} \in S_{m,n}(K)$ such that for some $\gamma > 1$ we have $|a_{\mu\nu}| < \gamma^{-|\mu|}$ or $|\nu| > (\gamma - 1)|\mu|$.

7. $A = \bigcup_k \mathbb{Z}[[t_1, \ldots, t_k]]$, which is not noetherian, $I = (t_1, t_2, \ldots)$, so A is not I–adically complete, and $A_{m,n} = \bigcup_k \mathbb{Z}[[t_1, \ldots, t_k]]\langle\xi\rangle[[\rho]]$.

8. There are many examples of countable Weierstrass systems.

9. **The Algebraic Case.** The theory of henselian valued fields in the usual algebraic language is also included in our formalism:

Every henselian field K carries a (uniformly existentially definable) separated analytic structure.

Let $R \subset K^\circ$ be an excellent DVR, $\widehat{R}$ the completion of R, and $A_{m,n} = S_{m,n}(R)_{alg}$, the algebraic closure of $R[\xi, \rho]$ in $S_{m,n}(\widehat{R}) = \widehat{R}\langle\xi\rangle[[\rho]]$. Then $\mathcal{A} = \{A_{m,n}\}$ is a separated Weierstrass system, and the functions $(K^\circ)^m \times (K^{\circ\circ})^n \mapsto K^\circ$ naturally represented by the powerseries in $A_{m,n}$ are uniformly existentially definable in the algebraic language.

The proof is fairly complicated, using the machinery of Artin Approximation

Elementary Properties

Let $\mathcal{A}$ be a Weierstrass system and let K have $\mathcal{A}$–analytic structure. Then:

- K is henselian.

- The analytic functions on K° are closed under (meaningful) compositions.

- The $\mathcal{A}$–analytic structure extends naturally to any finite algebraic extension of K, hence also to K_{alg} the algebraic closure of K.

- If K is p–adic, or algebraically closed, then we have QE in $\mathcal{L}_{\mathcal{A}}$.

• (Extension of parameters) For $f \in A_{m+m',n+n'}$ we can write

$$f = \sum_{\mu\nu} \overline{f}_{\mu\nu}(\xi', \rho')\xi^\mu \rho^\nu$$

where the $\overline{f}_{\mu\nu}$ are in $A_{m',n'}$, Then for

$$\overline{a} \in (K^\circ)^{m'}, \quad \overline{b} \in (K^{\circ\circ})^{n'},$$

we have

$$\sum_{\mu\nu} \overline{f}_{\mu\nu}(\overline{a}, \overline{b})\xi^\mu \rho^\nu \in K^\circ[[\xi, \rho]]$$

and

$$\sigma_{m+m',n+n'}(f)(\overline{a}, \xi, \overline{b}, \rho) : (K^\circ)^m \times (K^{\circ\circ})^n \to K^\circ$$

Define $A_{m,n}(K) =$

$$\{f(\xi, \overline{a}, \rho, \overline{b}) : f \in A_{m+m',n+n'}, \overline{a} \in (K^\circ)^{m'}, \overline{b} \in (K^{\circ\circ})^{n'}\}.$$

Then $\{A_{m,n}(K)\}$ is a Weierstrass system and K has $\{A_{m,n}(K)\}$–analytic structure. (This property was lacking in previous formalisms).

Advanced Properties

- One variable terms

Let K be a Henselian valued field.

(a) A K-*annulus formula* is a formula φ of the form

$$|p_0(x)|\square_0\varepsilon_0 \wedge \bigwedge_{i=1}^{L} \varepsilon_i\square_i|p_i(x)|,$$

where the $p_i \in K^\circ[x]$ are monic and irreducible, the $\varepsilon_i \in \sqrt{|K \setminus \{0\}|}$ and the $\square_i \in \{<, \leq\}$. Define $\overline{\square}_i$ by $\{\square_i, \overline{\square}_i\} = \{<, \leq\}$. We require further that the sets

$$\mathcal{H}_i := \{x \in K_{alg} : |p_i(x)|\overline{\square}_i\varepsilon_i\},\ i = 1, \ldots, L,$$

be disjoint and contained in $\{x \in K_{alg} : |p_0(x)|\square_0\varepsilon_0\}$.

(b) The corresponding K-*annulus* is

$$\mathcal{U}_\varphi := \{x \in K_{alg} : \varphi(x)\}$$

(If $K_1 \supset K_{alg}$ is a field then φ also defines a subset of K_1).

(c) A K-annulus formula φ and the K-annulus $\mathcal{U}_\varphi$ are called *linear* if the p_i are all linear and the $\varepsilon_i \in |K| \setminus \{0\}$.

(d) A K-annulus fomula φ and the K-annulus $\mathcal{U}_\varphi$ are called *closed* (resp. *open*) if all the $\Box_i$ are $\leq$ (resp. $<$).

(e) A K-annulus formula is called *good* if the p_i are of lowest possible degrees among all K-annulus formulas defining the same K-annulus.

Rings of analytic functions. Let K have separated analytic $\mathcal{A}$-structure, and let φ be a K-annulus formula. Define the corresponding *generalized rings of fractions* over K°, resp. K, by

$$\mathcal{O}_K(\varphi) := A_{m+1,n}(K)/(p_0^{\ell_0}(x)-a_0z_0, p_1^{\ell_1}(x)z_1-a_1, \ldots, p_L^{\ell_L}(x)z_L-a_L),$$

and

$$\mathcal{O}_K^\dagger(\varphi) := K \otimes_{K^\circ} \mathcal{O}_K(\varphi),$$

where $a_i \in K^\circ$, $|a_i| = \varepsilon_i^{\ell_i}$, $m+n = L+1$, $\{z_0, \ldots, z_L\}$ is the set $\{\xi_2, \ldots, \xi_{m+1}, \rho_1, \ldots, \rho_n\}$ and x is ξ_1 and z_i is a ξ or ρ variable depending, respectively, on whether $\square_i$ is $\leq$ or $<$. By Weierstrass Division, each $f \in \mathcal{O}_K^\dagger(\varphi)$ defines a function $\mathcal{U}_\varphi \to K_{alg}$ via the analytic structure on K_{alg}. Denote this function by f^σ. Let $\mathcal{O}_K^\sigma(\varphi)$ be the image of $\mathcal{O}_K^\dagger(\varphi)$ under $f \mapsto f^\sigma$ and call $\mathcal{O}_K^\sigma(\varphi)$ the *ring of analytic functions on $\mathcal{U}_K(\varphi)$*.

Mittag-Leffler Decomposition. Let

$$\varphi := |p_0(x)|\Box_0\varepsilon_0 \wedge \bigwedge_{i=1}^{n} \varepsilon_i\Box_i|p_i(x)|$$

be a good K-annulus formula and let $f \in \mathcal{O}_K^\dagger(\varphi)$. Then, if $f^\sigma \neq 0$, there exist a monic polynomial $P(x)$ with zeros only in $\mathcal{U}_\varphi$, integers n_i and a strong unit $E \in \mathcal{O}_K^\dagger(\varphi)$ such that

$$f^\sigma = P(x) \cdot \prod_{i=1}^{n} p_i(x)^{n_i} \cdot E^\sigma. \tag{1}$$

P, E and the n_i are uniquely determined by f (and φ).

The following Theorem is what we need for model-theoretic applications.

Theorem *Let K be a valued field with separated analytic $\mathcal{A}$-structure, and let $\mathcal{L}_{\mathcal{A}(K)}$ be the language of valued fields, $\langle 0, 1, +, \cdot, (\cdot)^{-1}, |\cdot| \rangle$, augmented with function symbols for all the elements of $\bigcup_{m,n} A_{m,n}(K)$. (We extend functions $f \in A_{m,n}(K)$ by zero outside $(K^{\circ})^m \times (K^{\circ\circ})^n$). Let x be one variable, and let $\tau(x)$ be a term of $\mathcal{L}_{\mathcal{A}(K)}$. There is a finite set $S \subset K_{alg}^{\circ}$ and a finite cover of K_{alg}° by K-annuli $\mathcal{U}_i$ such that for each i there is rational function $R_i \in K(x)$ and a strong unit $E_i \in \mathcal{O}_K^{\dagger}(\mathcal{U}_i)$ with*

$$\tau|_{\mathcal{U}_i \setminus S} = R_i \cdot E_i^{\sigma}|_{\mathcal{U}_i \setminus S}$$

i.e. τ and $R \cdot E_i$ define the same function on $\mathcal{U}_i \setminus S$. Observe that K_{alg} also has analytic $\mathcal{A}(K)$-structure, τ is also a term of $\mathcal{L}_{\mathcal{A}(K_{alg})}$ and hence defines a function $K_{alg}^{\circ} \to K_{alg}$.

Let Hen be the collection of all Henselian valued fields of characteristic zero (hence mixed characteristic and equicharacteristic zero fields are included).

For K in Hen, write K° for the valuation ring, Γ_K for the value group, $\mathrm{ord} : K^\times \to \Gamma_K$ for the (additively written) valuation, $K^{\circ\circ}$ for the maximal ideal of K°, and $\widetilde{K}$ for the residue field.

For any integer $n > 0$, write

$$rv_n : K^\times \to K^\times/1 + nK^{\circ\circ}$$

for the natural group morphism, with $nK^{\circ\circ} = \{nm \mid m \in K^{\circ\circ}\}$, and extend it to a map $rv_n : K \to (K^\times/1+nK^{\circ\circ})\cup\{0\}$ by sending zero to zero. Write RV_n, or $RV_n(K)$, for $(K^\times/1+nK^{\circ\circ})\cup\{0\}$ for integers $n > 0$. Write also ord for the natural maps $\mathrm{ord} : K^\times/1 + nK^{\circ\circ} \to \Gamma_K$. We sometimes abbreviate rv_1 to rv and RV_1 to RV. Note that in equicharacteristic zero the RV_n all are the same as RV_1.

The sorts RV and RV_n are called auxiliary. The valued field sort is the main sort. There are no other sorts. We write Val for the valued field sort.

Let $\mathcal{L}_{\text{Hen}}$ be the language of rings $(+, -, \cdot, 0, 1)$ for the valued field sort, together with function symbols rv_n for all $n > 0$, and the inclusion language on the auxiliary sorts. Let $\mathcal{T}_{\text{Hen}}$ be the theory of all fields in Hen in the language $\mathcal{L}_{\text{Hen}}$.

Let $\mathcal{A} = \{A_{m,n}\}$ be a separated Weierstrass system. Define $\mathcal{L}_{\text{Hen},\mathcal{A}}$ as the language $\mathcal{L}_{\text{Hen}}$ together with function symbols for all the elements of $\bigcup_{m,n} A_{m,n}$, with the field inverse $(\cdot)^{-1}$ on the valued field sort extended by $0^{-1} = 0$, and together with the induced language on the sorts RV_n. The analytic function symbols are interpreted as zero outside their natural domains of products of the valuation ring and the maximal ideal. On their natural domains, they are interpreted via an analytic $\mathcal{A}$-structure.

Let $\mathcal{T}_{\text{Hen},\mathcal{A}}$ be the $\mathcal{L}_{\text{Hen},\mathcal{A}}$-theory of all Henselian valued fields in Hen with analytic $\mathcal{A}$-structure.

(Likewise, one can give definitions of analytic languages and analytic theories arising from strictly convergent Weierstrass systems and strictly convergent analytic structures.)

Theorem (b-minimality, $\mathrm{Char}(K) = 0$**)** *Let $\mathcal{A} = \{A_{m,n}\}$ be a separated Weierstrass system. The theory $\mathcal{T}_{\mathrm{Hen},\mathcal{A}}$ eliminates valued field quantifiers, is b-minimal with centers and preserves all balls. Moreover, $\mathcal{T}_{\mathrm{Hen},\mathcal{A}}$ has the Jacobian property.*

Define $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}}$ as the language $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}$ together with all the functions $h_{m,n}$.

Theorem (Term structure, Char**(K)=0)** *Let K be a $\mathcal{T}_{\mathrm{Hen},\mathcal{A}}$-model. Let $X \subset K^n$ be definable and let $f : X \to K$ be a $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}(A)$-definable function for some set of parameters A. Then there exists a $\mathcal{L}_{\mathrm{Hen},\mathcal{A}}(A)$-definable function $g : X \to S$ with S auxiliary such that*

$$f(x) = t(x, g(x))$$

for each $x \in X$ and where t is a $\mathcal{L}^*_{\mathrm{Hen},\mathcal{A}}(A)$-term.