On the Mordell-Lang conjecture for curves over function fields of positive characteristic

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The generalized Mordell-Lang conjecture in char. 0

The following theorem is the result of the work of Faltings, Serre, Hindry and McQuillan:

Theorem (generalized Mordell-Lang conjecture)

Let A be an semi-abelian variety over $\mathbb{C}$. Let $X \hookrightarrow A$ be a closed subvariety with finite stabilizer. Let $\Gamma \subseteq A(\mathbb{C})$ be a group such that $\Gamma \otimes \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space.
Then $X \cap \Gamma$ is not Zariski dense in X.

Special cases

The following are consequences of GML.
Let C be a curve of genus $\geqslant 2$ defined over a number field K.

Theorem (Mordell conjecture; Faltings th. [1984])

$C(K)$ is finite.

Theorem (Manin-Mumford conjecture; Raynaud's th. [1983])

Let $P \in C(\overline{K})$. The set

$$\{Q \in C(\overline{K}) | \exists n \in \mathbb{N}^* \text{ such that } nQ - nP \text{ is principal}\}$$

is finite.

Positive characteristic

It is natural to wonder if the analog of GML is true in positive characteristic. The following is known.

Let K be an algebraically closed field of char. $p > 0$. Let A be a semi-abelian variety over K and let $\Gamma \subseteq A(K)$ be a group st $\Gamma \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p$-module of finite rank. Let $X \hookrightarrow A$ be a closed K-subvariety of A with finite stabiliser.

Theorem (Hrushovski [1996])

If $X \cap \Gamma$ is Zariski dense in X then there exists a variety X_0' over $\overline{\mathbb{F}}_p$ and a purely inseparable bijective morphism $\phi : X_0' \otimes_{\overline{\mathbb{F}}_p} K \to X$.

Around GML in positive characteristic

The only known proof of GML in positive characteristic is based on model-theoretic methods. In particular, Hrushovski uses his work with Zilber on Zariski geometries. It is not clear (to me) whether this is crucial or not.

Here is what can be proven by algebraic methods:

- the case where A is ordinary (Abramovich-Voloch [1996], Pillay-Ziegler [2003]);
- the case where X is a curve and K is of transcendence degree 1 (Grauert-Samuel [1966], Voloch [1991], Buium-Voloch [1996])

The proof of Buium-Voloch

The proof of Buium and Voloch for curves (Compositio Math. **103** (1996)) is particularly transparent and avoids the issue of ordinarity completly. It is also reminiscent of some aspects of Hrushovski's proof. It might thus point to an algebraic understanding of Hrushovski's proof of GML in positive characteristic.

In the rest of the lecture, we shall give an overview of their proof.

Reductions I

Let L be the function field of a smooth curve over $k := \overline{\mathbb{F}}_p$. Let C be a curve over L and suppose that C has no model over L^p. We shall prove that

the set $C(L)$ is finite. (*)

Let us suppose that C is embedded in $B := \mathrm{Jac}(C)$ via an L-rational point. The statement (*) implies GML for $X = C$, $A = B$, $\Gamma = B(L)$ and $K = \overline{L}$.

Reductions II

To the curve C is associated the Kodaira-Spencer map

$$\mathrm{KS} : \mathrm{Der}(L/k) \to H^1(C, \mathrm{T}C)$$

arising as a boundary map from the short exact sequence

$$0 \to \mathrm{T}C_L \to \mathrm{T}C_k \to \mathrm{Der}(L/k) \to 0$$

Proposition (Voloch)

C has a model over L^p iff $\mathrm{KS} = 0$.

Hence our hypothesis implies that $\mathrm{KS} \neq 0$.

Integral models

Let now U be a smooth curve over k with function field L.
We may suppose wlog that there exists a smooth and proper model $\pi : \mathcal{C} \to U$ of C over U (resp. $\mathcal{B} \to U$ of B) and that KS extends to a nowhere vanishing homomorphism

$$\mathrm{KS} : \mathrm{T}U_k \to \mathrm{R}^1\pi_*\mathrm{T}\mathcal{C}_U.$$

Now take any closed point

$$u : \mathrm{Spec\ k} \to \mathrm{U}$$

and let

$$t : \mathrm{Spec\ k}[\epsilon]/\epsilon^2 \to \mathrm{U}$$

be a tangent vector at u.

Weil restrictions I

Let

$$\mathcal{C}_\epsilon := W_{k[\epsilon]|k}(t^*\mathcal{C})$$

be the Weil restriction of $t^*\mathcal{C}$ to k and let

$$\mathcal{B}_\epsilon := W_{k[\epsilon]|k}(t^*\mathcal{B})$$

be the Weil restriction of $t^*\mathcal{B}$ to k.

Weil restrictions II

The properties of Weil restrictions imply the following:

- we have natural isomorphisms

$$\mathcal{C}_\epsilon(k) \simeq t^*\mathcal{C}(k[\epsilon]/\epsilon^2) \text{ and } \mathcal{B}_\epsilon(k) \simeq t^*\mathcal{B}(k[\epsilon]/\epsilon^2);$$

- $\mathcal{C}_\epsilon$ is a $\mathrm{T}\mathcal{C}_u$-torsor and $\mathcal{B}_\epsilon$ is a $\mathrm{T}\mathcal{B}_u$-torsor;
- the class of $\mathcal{C}_\epsilon$ as a torsor is $\mathrm{KS}_u(t) \in H^1(\mathcal{C}_u, \mathrm{T}\mathcal{C}_u)$ and is thus $\neq 0$;
- $\mathcal{B}_\epsilon$ is a commutative k-group scheme;

Weil restrictions III

- there is a commutative diagram of schemes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{T}\mathcal{B}_u \arrow[r] & \mathcal{B}_\epsilon \arrow[r] & \mathcal{B}_u \arrow[r] & 0 \\ & & \mathcal{C}_\epsilon \arrow[u] \arrow[r] & \mathcal{C}_u \arrow[u] & \end{tikzcd}$$

where the vertical arrows are closed immersions.

Exploiting $\mathrm{KS} \neq 0$, I

Lemma (Buium)

The scheme $\mathcal{C}_\epsilon$ is affine.

Sketch of proof. Let

$$0 \to \mathcal{O} \to E \to \mathrm{T}\mathcal{C}_u^\vee \to 0$$

be the dual of the extension corresponding to

$$\mathrm{KS}_u(t) \in H^1(\mathcal{C}_u, \mathrm{T}\mathcal{C}_u) = \mathrm{Ext}^1(\mathcal{O}, \mathrm{T}\mathcal{C}_u).$$

This extension is non-split by construction. %

Exploiting $\mathrm{KS} \neq 0$, II

End of sketch of proof. Since $\mathrm{T}\mathcal{C}_u^\vee$ is ample, this implies (Gieseker, Martin-Deschamps [1983]) that E is ample. Furthermore, we have

$$\mathcal{C}_\epsilon \simeq \mathrm{Proj}(E) \backslash \mathrm{Proj}(\mathrm{T}\mathcal{C}_u^\vee)$$

and $\mathrm{Proj}(\mathrm{T}\mathcal{C}_u^\vee)$ is a divisor of $\mathcal{O}_E(1)$ on $\mathrm{Proj}(E)$. Hence $\mathcal{C}_\epsilon$ is the complement of an ample divisor and is thus affine.

The intersection $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$, I

Note that the scheme

$$p\mathcal{B}_\epsilon \hookrightarrow \mathcal{B}_\epsilon$$

is an an abelian variety.

The scheme $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$ is a closed affine subscheme of $p\mathcal{B}_\epsilon$ and is thus finite.

Can we estimate its degree ?

The degree of $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$, I

Let

$$0 \to \mathcal{O} \to E \to \mathrm{T}\mathcal{C}_u^\vee \to 0$$

be the dual of the extension corresponding to $\mathcal{C}_\epsilon$ and let

$$0 \to \mathcal{O} \to F \to \mathrm{T}\mathcal{B}_u^\vee \to 0$$

be the dual of the extension corresponding to $\mathcal{B}_\epsilon$. The natural surjection $\mathrm{T}\mathcal{B}_u^\vee|_{\mathcal{C}_u} \to \mathrm{T}\mathcal{C}_u^\vee$ extends to a surjection

$$F|_{\mathcal{C}_u} \to E \to 0$$

The degree of $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$, II

We thus obtain a diagram of immersions

$$\begin{tikzcd} p\mathcal{B}_\epsilon \arrow[r] & \mathrm{Proj}(F) \\ & \mathrm{Proj}(E) \arrow[u] \\ & \mathcal{C}_\epsilon \arrow[u] \end{tikzcd}$$

The degree of $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$, III

Let $\pi : \mathrm{Proj}(F) \to \mathcal{B}_u$ be the natural map. Let Θ be a Theta divisor on $\mathcal{B}_u$. Let $\mathcal{L} := \pi^*\mathcal{O}(\Theta)^{\otimes 3} \otimes \mathcal{O}_E(1)$. The line bundle $\mathcal{L}$ is very ample on $\mathrm{Proj}(F)$.

Lemma

- $\deg_{\mathcal{L}}(p\mathcal{B}_\epsilon) \leqslant p^g \cdot 3^g \cdot g!$
- $\deg_{\mathcal{L}}(\mathrm{Proj}(E)) = 8g - 2$

The degree of $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$, IV

Hence, by Bezout's theorem,

$$\boxed{\deg(p\mathcal{B}_\epsilon \cap \mathcal{C}_\epsilon) \leqslant \Big(8g - 2\Big)\Big(p^g \cdot 3^g \cdot g!\Big)}$$

In particular, $\deg(p\mathcal{B}_\epsilon \cap \mathcal{C}_\epsilon)$ *is bounded independently of* u.

Proof of the Mordell conjecture for C

The proof is by contradiction. Suppose that $C(L)$ is infinite.

Since $B(L)$ is finitely generated (Mordell-Weil), we may suppose wlog that $C(L) \cap pB(L)$ is infinite. Let $P_1, \dots, P_r \in C(L) \cap pB(L)$ be pairwise distinct points such that

$$r > \Big(8g - 2\Big)\Big(p^g \cdot 3^g \cdot g!\Big).$$

We may choose u st $P_{1,u}, \dots, P_{r,u}$ are also pairwise distinct.

In that case, $P_{1,t}, \dots, P_{r,t}$ are also pairwise distinct and they lie in $\mathcal{C}_\epsilon \cap p\mathcal{B}_\epsilon$. This contradicts the estimate for $\deg(p\mathcal{B}_\epsilon \cap \mathcal{C}_\epsilon)$.

Further speculations

• Is it true that GML in char. $p > 0$ holds with the weaker condition that $\Gamma \otimes \mathbb{Q}$ (rather than $\Gamma \otimes \mathbb{Z}_p$) is finitely generated ?

• Suppose that X is any smooth proper variety over a finitely and separably generated field L of char. $p > 0$. Suppose also that the canonical bundle of X is ample and that the Kodaira-Spencer class of X is of maximal rank.

Is it true that $X(L)$ is not Zariski dense in X ?