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Effective Error Estimators for Low Order Elements —from Diffusion to Incompressible Flow Problems

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The Model Diffusion Problem and Finite Elements

The governing Poisson equation is:

$$-\nabla^2 u = f \quad \text{in} \quad \Omega,$$
$$u = g \quad \text{on} \quad \partial\Omega,$$

where u is the unknown function. Its weak formulation is,

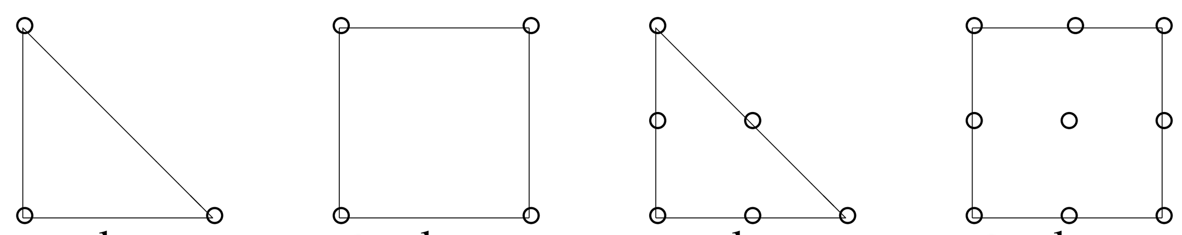
$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega v f, \quad \forall v \in H_0^1. \tag{1}$$

The finite element discretization is to find $u_h \in X_E^h \subset H_E^1$, such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega v_h f, \quad \forall v_h \in X_0^h \subset H_0^1, \tag{2}$$

where X_E^h and X_0^h are finite dimensional spaces.

The (bi-)linear and (bi-)quadratic elements are:



P_1 element Q_1 element P_2 element Q_2 element

Error Estimation Based on Solving Local Problems

The error ($e = u - u_h$) satisfies the following equation (from (2)),

$$\int_\Omega \nabla e \cdot \nabla v = \int_\Omega v f - \int_\Omega \nabla u_h \cdot \nabla v, \quad \forall v \in H_0^1. \tag{3}$$

Integrate by parts for (3),

$$\sum_{T \in T_h} (\nabla e, \nabla v)_T = \sum_{T \in T_h} \left[(f + \nabla^2 u_h, v)_T - \frac{1}{2} \sum_{E \in \partial T} \left\langle \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right], v \right\rangle_E \right]. \tag{4}$$

Then the localized error equation is,

$$(\nabla e_T, \nabla v)_T = (R_T, v)_T - \sum_{E \in \partial T} \langle R_E, v \rangle_E\,, \tag{5}$$

where $R_T = f + \nabla^2 u_h$ and $R_E = \frac{1}{2}\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]$. Note that, e_T in (5) is stronger than e in (4).

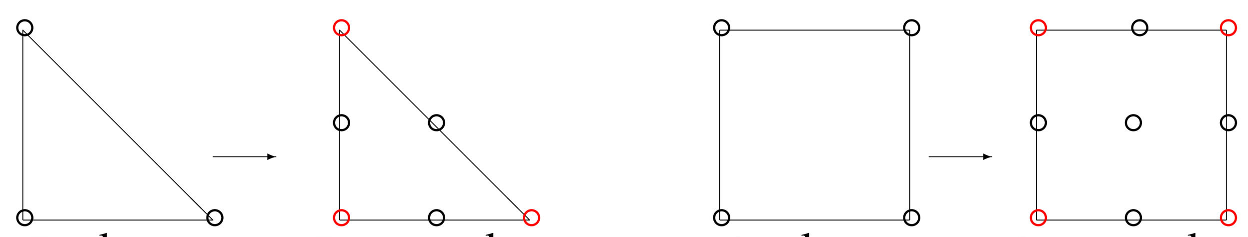
The local problem error estimation strategy is: choose a suitable finite element space $\mathbb{Q}_T$, and then find $e_h \in \mathbb{Q}_T$, such that

$$(\nabla e_h, \nabla v_h)_T = (R_T, v_h)_T - \sum_{E \in \partial T} \langle R_E, v_h \rangle_E\,, \ \forall v_h \in \mathbb{Q}_T. \tag{6}$$

Note that $\mathbb{Q}_T$ should satisfy two requirements:

- $\mathbb{Q}_T$ must be "larger" than the original approximation space;
- $\mathbb{Q}_T$ should make the problem (6) solvable—that is reasonable boundary conditions are required.

Estimators for (Bi-)linear Elements

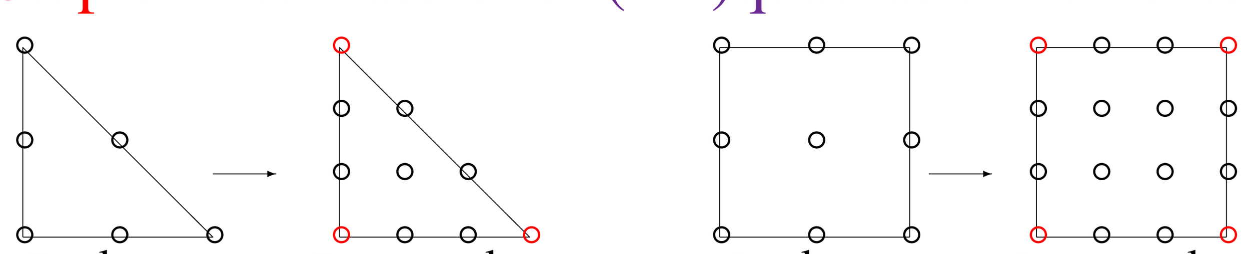


P_1 element P_2 error element Q_1 element Q_2 error element

The red circles imply these basis function nodes are removed. In other words, zero boundary values are applied at these points.

The P_2 and the Q_2 estimators can provide very accurate estimation for the exact error e and their full analysis can be found in many textbooks.

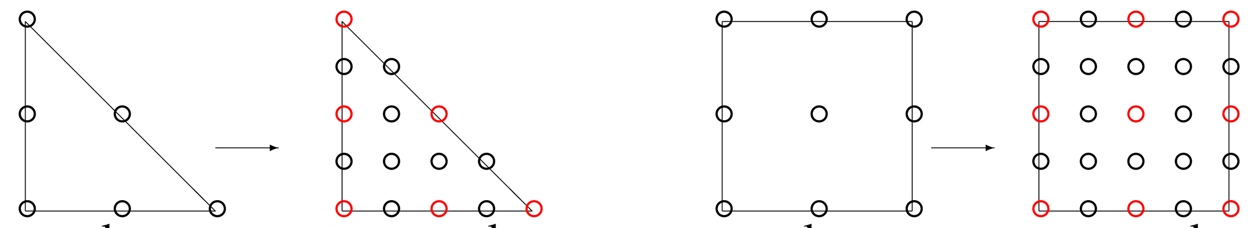
"Stupid" Estimators for (Bi-)quadratic Elements



P_2 element P_3 error element Q_2 element Q_3 error element

These estimators are simply generalized from the estimators for linear elements. With standard analysis techniques, they can mathematically be proven to be equivalent to the exact error. However, in practical computing, they do provide ineffective evaluation for the errors (see [2]).

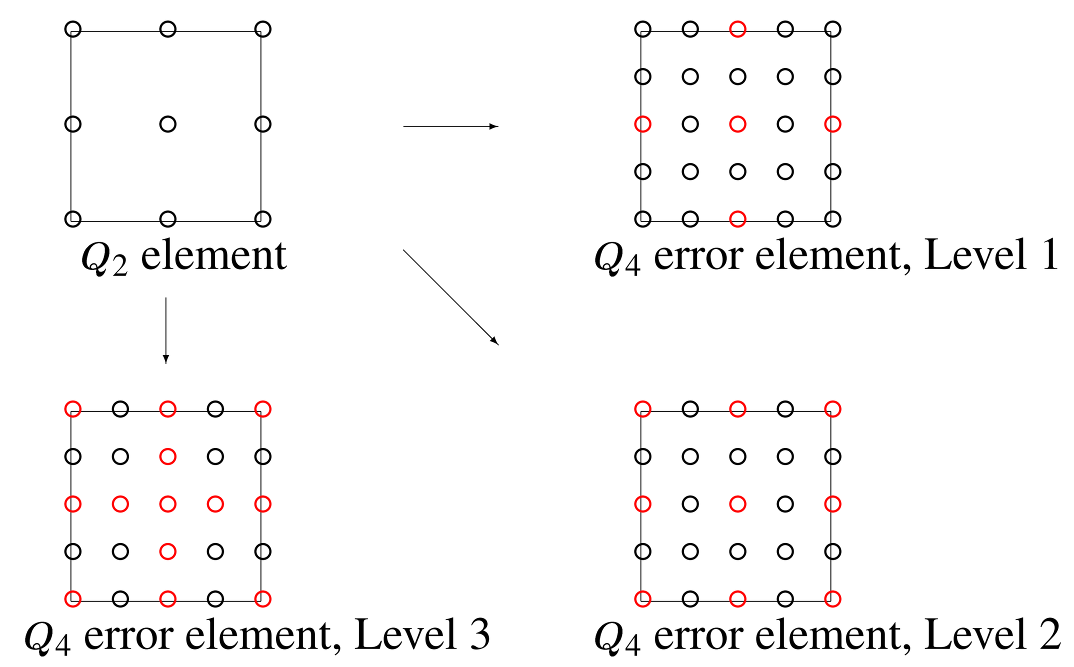
"Good" Estimators for (Bi-)quadratic Elements



P_2 element P_4 error element Q_2 element Q_4 error element

The P_4 estimator can provide a tight bound for the P_2 element, but the Q_4 estimator is still not effective.

"Perfect" Estimator for Bi-quadratic Elements



In order to find the best estimator for the Q_2 element, three levels of reduction of the Q_4 element have been tested. The Level 3 is the "perfect" choice: it is very accurate and relatively cheap (only 12 degrees of freedom).

Estimators for Mixed Approximations, Stokes Problems

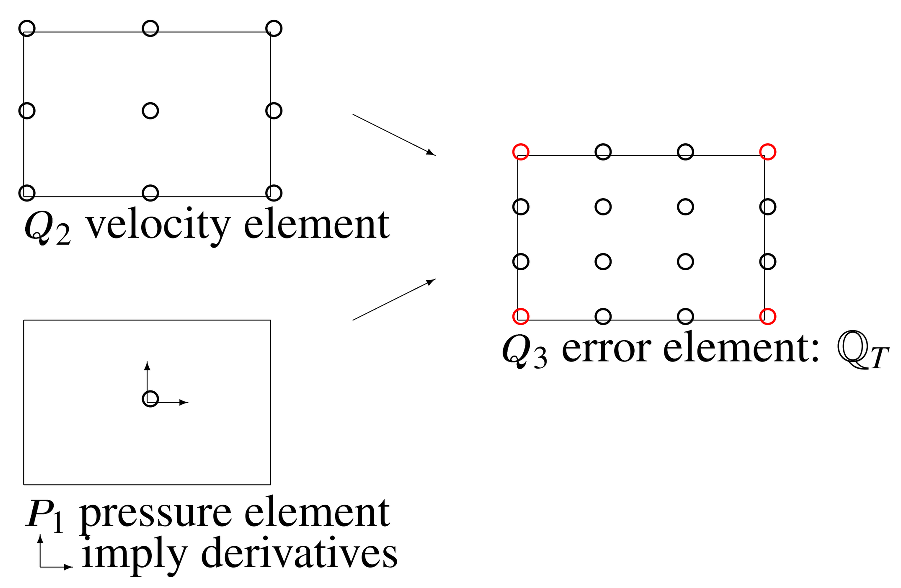
For the steady-state Stokes equations,

$$-\nabla^2 \vec{u} + \nabla p = 0 \quad \text{in} \quad \Omega,$$
$$\nabla \cdot \vec{u} = 0 \quad \text{in} \quad \Omega,$$
$$\vec{u} = \vec{g} \quad \text{on} \quad \partial\Omega_D,$$
$$\frac{\partial \vec{u}}{\partial n} - \vec{n}p = \vec{0} \quad \text{on} \quad \partial\Omega_N,$$

our local Poisson problem estimation is: compute $\eta_{P,T}^2 = |\vec{e}_{P,T}|_{1,T}^2 + \|\nabla \cdot \vec{u}_h\|_{0,T}^2$, where $\vec{e}_{P,T} \in \mathbb{Q}_T$ satisfies,

$$(\nabla \vec{e}_{P,T}, \nabla \vec{v})_T = (\vec{R}_T, \vec{v})_T - \sum_{E \in \partial T} \langle \vec{R}_E, \vec{v} \rangle_E, \quad \forall \vec{v} \in \mathbb{Q}_T.$$

For the classical mixed $Q_2 - P_1$ element, an effective estimator is



In addition, the Q_3 estimator is also effective for the $Q_2 - Q_1$ (Taylor-Hood) method. This Q_3 estimator for mixed approximations is analyzed in [1].

Conclusion

For the diffusion problem, the Q_4 (with reduction Level 4) and the P_4 estimators are effective for (bi-)quadratic elements and the Q_3 estimator is effective for the $Q_2 - P_1$ and the $Q_2 - Q_1$ mixed approximations. These new estimators are encoded in version 3.1 of the MATLAB package IFISS [3].

References

[1] Q. Liao, D. Silvester, A simple yet effective a posteriori estimator for classical mixed approximation of Stokes equations, Applied Numerical Mathematics, to appear. doi:10.1016/j.apnum.2010.05.003.

[2] Q. Liao, D. Silvester, A posteriori error estimation for low order elements, Talk in First Manchester SIAM Student Chapter Conference, Website, http://www.maths.manchester.ac.uk/~siam/contents/Liao_siam_talk.pdf (26th April 2010).

[3] D. Silvester, H. Elman, A. Ramage, Incompressible Flow & Iterative Solver Software (IFISS), http://www.manchester.ac.uk/ifiss/.