

Unravelling the String Dual of the

Gaussian Matrix Model

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Outline of the talk

◆ The Background

✧ What we know about Matrix Models.

◆ Worldsheets from Feynman Graphs

✧ A worldsheet construction and integer Strebel Differentials.

◆ Belyi Maps

✧ Interpreting matrix integrals as a sum over special holomorphic maps.

◆ The Skeleton of Gauge-String Duality

✧ An AdS/CFT picture for correlation functions.

◆ Final Remarks

Based on

Work in Progress

Worksheet Construction:

❖ [R. Gopakumar](#) (2004-06).

❖ [S. Razamat](#) (arXiv: 0803.2681, 0911.0658).

Matrix Integrals and Belyi Maps:

❖ [R. de Mello Koch and S. Ramgoolam](#) (arXiv: 1002.1634).

1 The Background

The Background

- ◆ The **AdS/CFT correspondence** has changed the **nature** as well as reshaped the **focus** of string theory.
- ◆ After more than a decade the connection between gauge theory and gravity (string theory) still appears as **miraculous** and **magical** as ever.
- ◆ Physicists are not supposed to believe in magic or miracles - our task is to **demystify!**
- ◆ We need to lay bare the **nuts and bolts** of this correspondence and understand what makes it tick **without invoking string dualities**.
- ◆ That would enable us to see how **general** the correspondence is and understand more examples and apply it to **newer settings**.
- ◆ Need to start with a simple enough toy model which is both **tractable** as well as exhibits the necessary **complexity**.

The Background

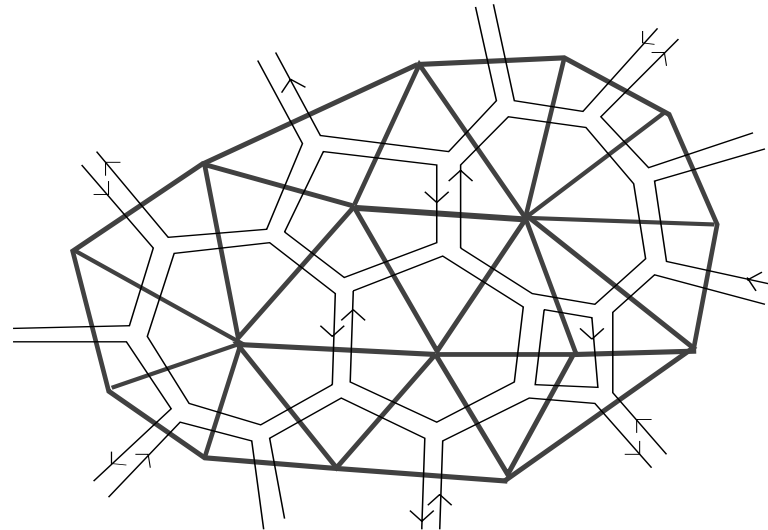
- ◆ Study the one hermitian matrix model

$$Z[t_k] = \int [dM]_{N \times N} e^{-N \text{Tr} M^2 + \sum_k t_k N \text{Tr} M^{2k}}$$

- ◆ To be thought of as the **generating function** of correlators $\langle \prod_i \text{Tr} M^{2k_i} \rangle$ in the gaussian ("free") matrix integral.
- ◆ **How** do the Feynman-'t Hooft Diagrams of this theory (**in the 't Hooft limit**) generate a closed string dual?
- ◆ Can hope to understand with **precision** (in an AdS/CFT like formulation) since this would **exhibit the structural bare bones of the gauge-string duality**.
- ◆ And provide a starting point to **deciphering the dual to free gauge theory**.
- ◆ Moreover, Gaussian matrix integrals arise as **consistent subsectors** within gauge theories (e.g. Supersymmetric Wilson Loops in $\mathcal{N} = 4$ Yang-Mills theory).

The Background *continued*

- ◆ In '80-90's, this model was studied in the "double scaling limit".
- ◆ Amplitudes dominated by Feynman diagrams with many vertices - dual graph has many regular polygonal faces - dense "triangulations" of a Riemann surface.



- ◆ Intuition: The matrix integral in this limit approximates the continuum sum over the moduli space of Riemann Surfaces.
- ◆ Dual to non-critical string theories. Actually minimal Topological string theories of topological 2d gravity with its set of physical operators \mathcal{O}_k .

The Background *continued*

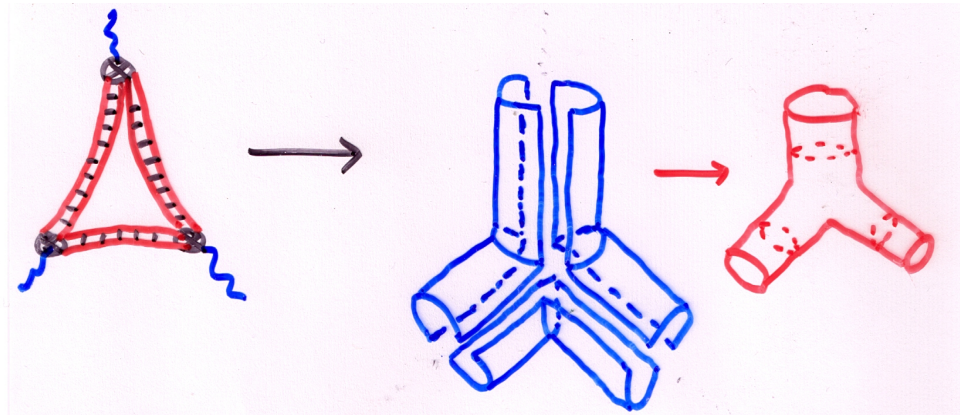
- ◆ Here we are interested in the **conventional 't Hooft limit** (N large but finite, t_k finite).
- ◆ **Earlier Intuition**: "Non-universal" theory describing a **discretely triangulated** worldsheet. No continuum string theory description.
- ◆ At odds with the seeming universality of gauge-string correspondence.
- ◆ In fact, 'tHooft dual believed to be B-model topological string theory on deformed conifold (**Dijkgraaf-Vafa**).
- ◆ What is wrong with the triangulation intuition?

Replace view of discrete worldsheet by the notion of amplitudes *localised* to a discrete set of points on the moduli space of Riemann surfaces.

2 Worldsheets from Feynman Graphs

Worksheets from Feynman Graphs

- ◆ A **general** approach to understanding gauge-string duality is to **associate world-sheets to individual Feynman diagrams** in a very natural way. (R.G.)
- ◆ **Glue** together semi-infinite cylinders along the **edges of the dual Feynman graph** extending from vertices of the original Feynman graph.



- ◆ **All inequivalent Riemann Surfaces can be generated this way.** They are parametrised by the (**Strebel**) lengths of the edges of the dual graph.
- ◆ Strebel differential $\phi(z)dz^2 \rightarrow$ An **almost flat** worldsheet metric ("**Strebel Gauge**") $ds^2 = |\phi(z)|dzd\bar{z}$. Vertices of the dual graph are zeroes of $\phi(z)$.
- ◆ Proposal is to identify Strebel lengths with (inverse) **Schwinger proper times**.

Worksheets from Feynman Graphs *continued*

- ◆ For the case of **matrix integrals** (where there is no notion of proper time) a modification proposed by Razamat.
- ◆ Identify Strebel length (of the edge of the dual graph) with the (integer) **number of contractions** along the original edge. Equivalently, face dual to the vertex with $Tr M^k$ has length k .
- ◆ Exactly the same lengths (recall regular polygons) as associated in **triangulation approach**. But interpretation is **different**.
- ◆ The associated Riemann surfaces are **continuous** (formed piecewise cylindrically) and **not discrete**.
- ◆ But integer values of Strebel Lengths pick out a set of **discrete points** (dense) **on the moduli space** of Riemann surfaces.
- ◆ **Why should a continuum string amplitude pick out a set of discrete points on moduli space?**

3 Belyi Maps

Belyi Maps

- ◆ Interesting observation of de Mello Koch and Ramgoolam on Belyi maps.
- ◆ Can interpret the contributions to the correlators $\langle \prod_i^n \text{Tr} M^{2k_i} \rangle$ as a sum over **particular holomorphic maps** $X : \Sigma_{g,n} \rightarrow \mathbb{P}^1$.
- ◆ Basically, the **different Wick contractions** of the Gaussian model can be organised in terms of **three** sets of elements of the permutation group S_{2k} where $2k = \sum_i 2k_i$.
- ◆ The first one $\beta \in (2k_1)(2k_2) \dots (2k_n)$ captures the **cyclic information at each vertex** of the n -point function.
- ◆ The second one $\alpha \in [2^k]$ consists only of two cycles (transpositions) and captures the various **possible Wick contractions** (edge information).
- ◆ The power of N for a given Wick contraction (number of faces) is associated with the **number of disjoint cycles** C_γ in the permutation $\alpha \cdot \beta \equiv \gamma^{-1}$.

Belyi Maps *continued*

- ◆ These permutations can be naturally associated with **branching structure** (ramification profile) of **holomorphic maps** from Riemann surfaces to Riemann surfaces (**Riemann**).
- ◆ Thus $N^{n-k} \langle \prod_i^n \text{Tr} M^{2k_i} \rangle_{conn} = \sum_{g=0} \sum_{\alpha \in [2^k]} N^{2-2g}$.
- ◆ Where $(2 - 2g) = C_\gamma + C_\beta - k = 2k(2 - 2 \times 0) - B$.
- ◆ With $B = (2k - C_\gamma) + (2k - C_\beta) + (2k - C_\alpha)$.
- ◆ This is the **Riemann-Hurwitz formula** for holomorphic maps from Σ_g to \mathbb{P}^1 where $2k$ is the degree of the map and B is the **branching number**.
- ◆ In this case the holomorphic map is branched over **three** points (e.g. $0, 1, \infty$) with ramification profiles given by the permutations α, β, γ (with $\alpha \cdot \beta \cdot \gamma = 1$).
- ◆ Thus matrix correlators can be interpreted as **counting** Holomorphic maps from $\Sigma_{g,n} \rightarrow \mathbb{P}^1$ which happen to be **branched over three points** (with "simple" branching over one of the points). (R. dMK and S.R)

Belyi Maps *continued*

- ◆ These are precisely the class of maps known to mathematicians as (clean) **Belyi maps**.
- ◆ Significant because **not every Riemann Surface admits a Belyi map** - only a discrete set of points on moduli space do so.
- ◆ These are the **arithmetic** riemann surfaces - those defined over the algebraic numbers $\bar{\mathbb{Q}}$ (Belyi).
- ◆ Related to Grothendieck's program (*Dessins L'Enfants*) to understand the representations of the **absolute Galois group** - $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.
- ◆ Let us now put together all this into a coherent picture of how the **string worldsheet** as well as **target space** are **built up from the large N 'tHooft diagrams** of the matrix integral.

4 The Skeleton of Gauge-String Duality

The Skeleton

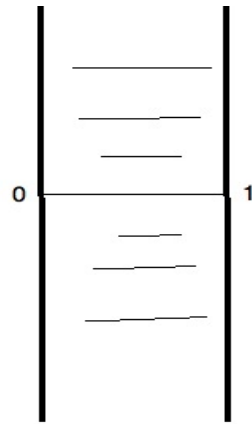
- ◆ How can a well defined continuum string theory be defined only on a discrete set of points on moduli space?
- ◆ A simple example: Consider the **A-model topological string theory** with T^2 as its target.
- ◆ The sigma model path integral gets contributions only from **holomorphic maps** from the worldsheet to the target.
- ◆ At genus one, for fixed complex structure τ_{ST} of the target T^2 , such maps exist only for **special values of the worldsheet** τ_{WS} .
- ◆ τ_{WS} must be related to τ_{ST} by special $GL(2, \mathbb{Z})$ transformations. Picks out a **discrete set of points** in the fundamental domain of τ_{WS} .
- ◆ Therefore the string amplitude gets its contribution from a sum of **delta functions on the moduli space**. Belyi maps suggest something similar in this case.

The Skeleton

- ◆ Can use a very **explicit** description of Belyi maps (**Mulase and Penkava**).
- ◆ Shows that **Belyi maps are admissible** precisely when the Strebel lengths are integers!
- ◆ Thus the **same discrete set of points** on moduli space are picked out as in the **world-sheet construction**.
- ◆ Natural to view the string dual for the matrix model as getting **localised contributions** from discrete points in moduli space corresponding to **integer Strebel lengths**.
- ◆ This "discretisation" of moduli space can **nevertheless capture the topological information** of intersection numbers on moduli space (and more). (**Chekhov, Makeenko**)
- ◆ **Topological A-model String theory provides the means for such a localisation on moduli space through holomorphic maps** - Belyi maps, in this case.

The Skeleton *continued*

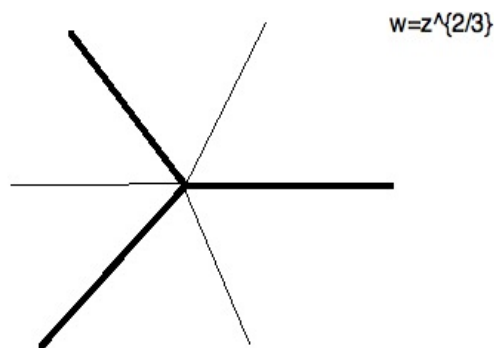
- ◆ Worldsheet riemann surface is built out of gluing **vertical strips** in the complex plane each of **width** equal to a unit Strebel Length ($0 < \text{Re}(z) < 1$).
- ◆ Essentially these are the **fattened 'tHooft lines** of the original graph which arise in the **Wick contractions** between any two vertices (insertions of $Tr M^{2k_i}$).



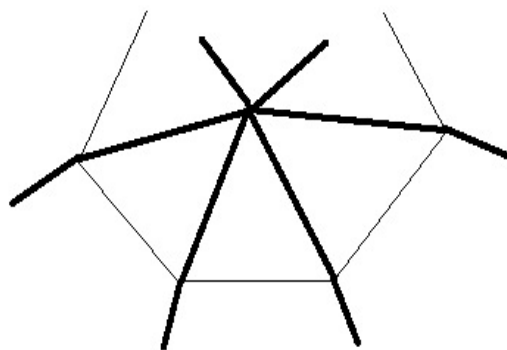
- ◆ The horizontal interval $[0, 1]$ is therefore an **edge of the dual graph**.
- ◆ The **vertices of the original graph** are at $z = \pm i\infty$.
- ◆ The Strebel differential is simply dz^2 in each such strip.

The Skeleton *continued*

- ◆ At each **dual vertex** (i.e. face of original graph), e.g. $z = 0$, the gluing is through $w = z^{\frac{2}{p}}$.



- ◆ At the centre of each **dual face** (i.e. vertex of the original graph), e.g. $z = i\infty$, the gluing is through $u \propto e^{\frac{2\pi iz}{k}}$.

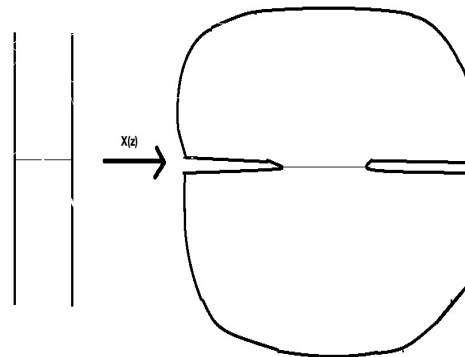


The Skeleton *continued*

- ◆ The various Feynman diagrams for the n -pt. function are supposed to be in 1-1 correspondence with *clean* Belyi maps (of degree $2k$) from this glued up worldsheet.
- ◆ We can now explicitly see this: holomorphic Belyi map on each strip

$$X(z) = \sin^2 \frac{\pi z}{2}.$$

Each strip of width one is mapped once onto a \mathbb{P}^1 .



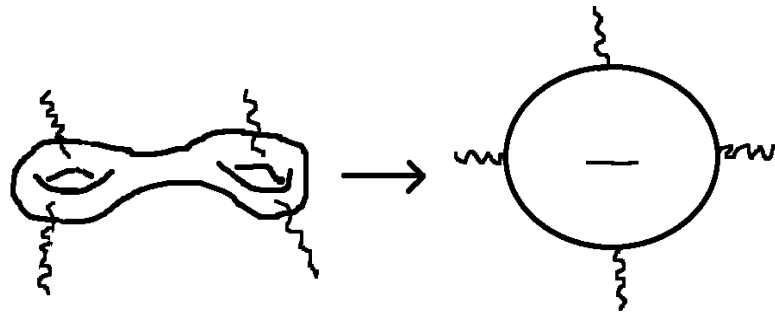
- ◆ Essentially the **vertices of the dual graph** ($z = 0, 1$ on the strip) are mapped to 0 and 1 and the **original vertices** ($z = \infty$) to ∞ , the branch points in the target space.

The Skeleton *continued*

- ◆ k separate strips in any Feynman diagram \rightarrow cover of \mathbb{P}^1 k times.
- ◆ What is the interpretation of this Belyi map?
- ◆ The differential $dz^2 = \frac{dX^2}{\pi^2 X(1-X)} = \frac{1}{\pi^2} \left(\frac{dX}{y}\right)^2 \equiv \omega^2$ where $y^2 = X(1-X)$.
- ◆ $\omega = \frac{1}{\pi} \frac{dX}{y}$ is the canonical **abelian differential** on \mathbb{P}^1 which is a **double cover** of the X plane. Hence the net map is of degree $2k$.
- ◆ We thus see that the **target space** \mathbb{P}^1 is nothing other than that of the **complexified eigenvalues** (single cut) of the Gaussian model!
- ◆ The **spatial slices of the worldsheet** (the interval $[0, 1]$ on each strip) mapped onto the **eigenvalue cut** $[0, 1]$ (the "emergent space" of the target).

The Skeleton *continued*

- ◆ External **Vertex operator insertions** $\mathcal{O}_{k_i} \leftrightarrow Tr M^{2k_i}$ at infinity ("UV") have a specific branching number k_i .
- ◆ Different Feynman graphs \rightarrow sum over holomorphic maps with **fixed branching data at infinity**: $(k_1) \dots (k_n)$. degree of map = $\sum k_i = k$.
- ◆ Leads to additional branching at 0 and 1 ("IR"), the two endpoints of the holographically emergent "space".
- ◆ Basically the strings coming in from infinity split and join at these specific points (cf. **Gross-Mende**, **Eynard-Orantin**).
- ◆ These holomorphic maps *are* **stringy Witten diagrams** in the target space \mathbb{P}^1 .



The Skeleton *continued*

- ◆ The full target space ought to be a **non-compact CY 3-fold** for consistency.
- ◆ A topological A-model mirror description of Dijkgraaf-Vafa duality.
- ◆ A \mathbb{P}^1 resolution of $xy + (e^u - 1)(e^v - 1) = 0$?
- ◆ However, free Gaussian likely to be somewhat singular.
- ◆ Can argue the dictionary $\lambda = e^{-t}$ between coupling (of quartic term) and size of \mathbb{P}^1 . Thus $\lambda = 0 \rightarrow t = \infty$!
- ◆ Need to understand how the **full CY target space** geometry emerges from the matrix integral. A bit like understanding the S^5 of the $AdS_5 \times S^5$.

To Conclude...To Continue

- ◆ Matrix models give a **hands on picture of gauge-string duality** with some unusual features.
- ◆ A dual A-model topological string with **amplitudes localised at special points** on the moduli space.
- ◆ Amplitudes count holomorphic **maps to \mathbb{P}^1 with three branchpoints**.
- ◆ The Feynman diagrams exhibit this string theory in a (singular) **"Strebel gauge" for the metric**.
- ◆ In this gauge easy to reconstruct (part of) the target space - **Riemann surface associated with the matrix model eigenvalues**.
- ◆ The holomorphic (Belyi) maps describe the **stringy Witten diagrams** in the target space \mathbb{P}^1 which **plays the role of the AdS**.
- ◆ Can hope to draw lessons for the **string duals of free (super) Yang-Mills** which one expects to be **quasi-topological**.

The end