Geometric Structures on Manifolds I: Ehresmann structures

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Geometry and Arithmetic of Lattices
London Mathematical Society — EPSRC Durham Symposium
University of Durham
5 July 2011

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  - and then by discrete groups which don't act properly.

Geometry through symmetry

In his 1872 *Erlangen Program*, Felix Klein proposed that a *geometry* is the study of properties of an abstract space X which are invariant under a transitive group G of transformations of X.

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- More exotic geometries: conformal geometries, indefinite metrics, complex, quaternionic structures, symplectic, contact structures, incidence geometries on flag manifolds, ...

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- (Ehresmann 1936): Geometric manifold M modeled on X.

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- *Locally homogeneous Riemannian geometries,* modeled on $X = G/H$, H compact.
- (Thurston 1976): 3-manifolds canonically decompose into *locally homogeneous Riemannian* pieces (8 types). (proved by Perelman)

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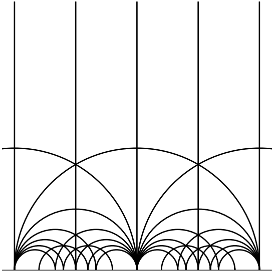

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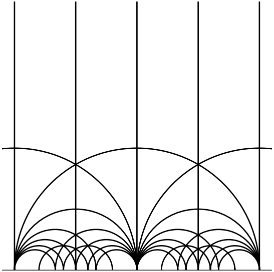

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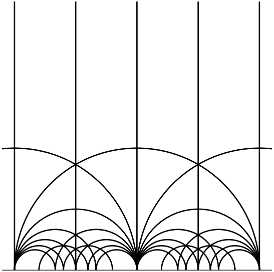
  - *Example:* The 2-torus admits a *moduli space* of Euclidean structures.



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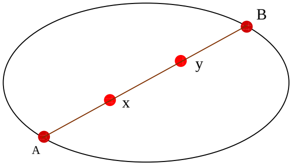

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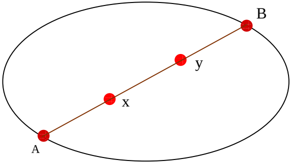

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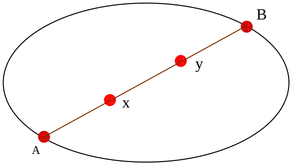

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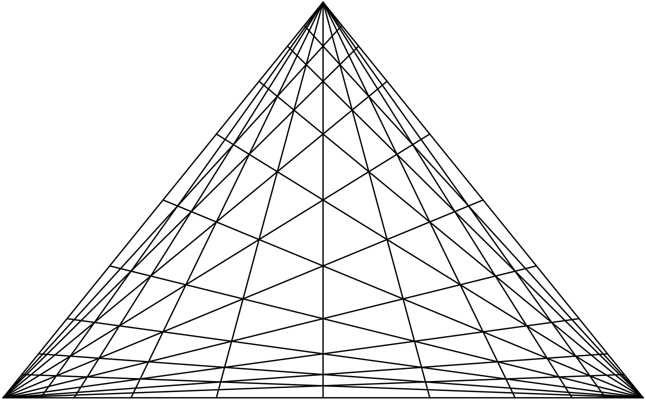
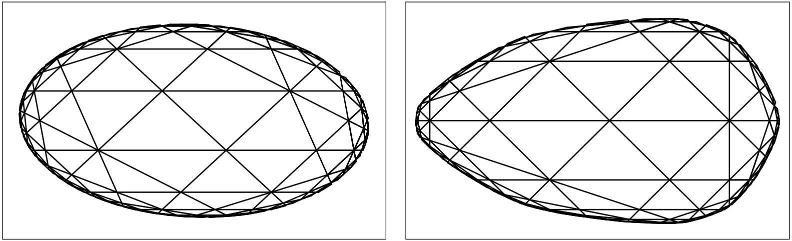
  - Hyperbolic structures *are* convex $\mathbb{R}\mathrm{P}^n$-structures.

Example: Projective tiling of $\mathbb{R}P^2$ by equilateral 60^o-triangles



This tesselation of the open triangular region is equivalent to the tiling of the Euclidean plane by equilateral triangles.

Example: A projective deformation of a tiling of the hyperbolic plane by (60^o,60^o,45^o)-triangles.



Both domains are tiled by triangles, invariant under a Coxeter group $\Gamma(3,3,4)$. First domain bounded by a conic (hyperbolic geometry), second domain bounded by $C^{1+\alpha}$-convex curve where $0 < \alpha < 1$. Second domain invariant under Zariski dense surface group in $\mathrm{SL}(3,\mathbb{R})$.

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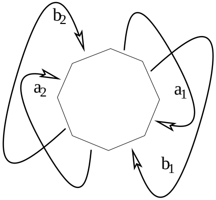

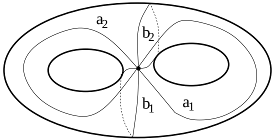

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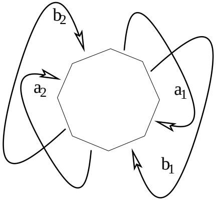
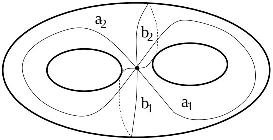
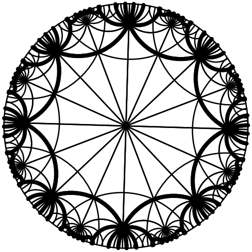
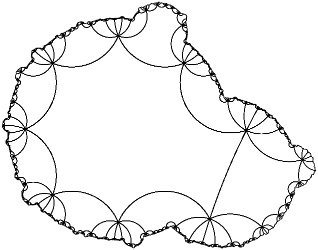




- Realize these identifications isometrically for a regular 45^o-octagon.



Example: $\mathbb{C}\mathrm{P}^1$-structures



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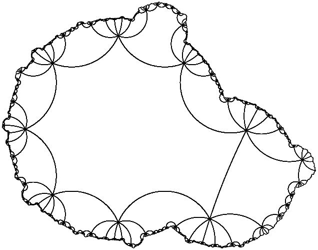

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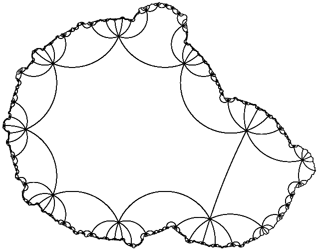

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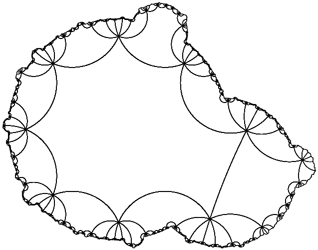

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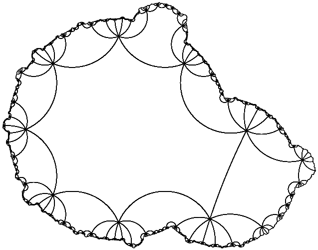

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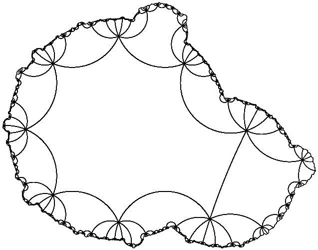
  - Hyperbolic structures on surfaces deform as $\mathbb{C}\mathrm{P}^1$-structures, through "bending" or "grafting" constructions (Thurston)



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- *Mapping class group*

$$\mathrm{Mod}(\Sigma) := \pi_0\big(\mathrm{Diff}(\Sigma)\big)$$

acts on $\mathfrak{D}_{(G,X)}(\Sigma)$.

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  enjoys the natural structure of an *affine algebraic variety*
  - Invariant under $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$.

Representation varieties

- Let $\pi = \langle X_1, \ldots, X_n \rangle$ be finitely generated and $G \subset \mathrm{GL}(N, \mathbb{R})$ a linear algebraic group.
- The set $\mathrm{Hom}(\pi, G)$ of homomorphisms

$$\pi \longrightarrow G$$

enjoys the natural structure of an *affine algebraic variety*

  - Invariant under $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$.
  - Action of $\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ on

$$\mathrm{Hom}(\pi, G)/G \ := \ \mathrm{Hom}(\pi, G)/(\{1\} \times \mathrm{Inn}(G))$$

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  - Discrete cocompact embeddings $\pi \hookrightarrow G$ form open set (Weil 1960).

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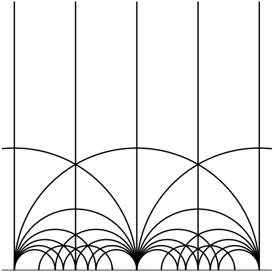

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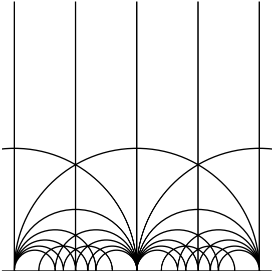

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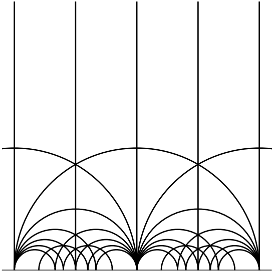

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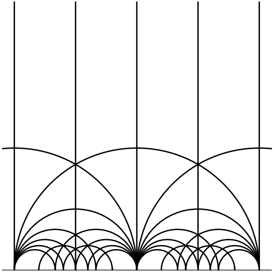

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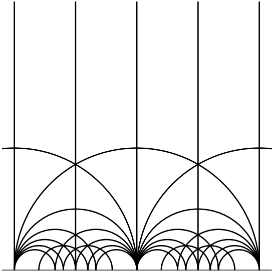

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  - Quotient $\mathfrak{F}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)$ identifies the *Riemann moduli space* $\mathfrak{T}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)$ of curves of genus g.

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- (Gallo-Kapovich-Marden) Image of hol consists of representations $\pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ which lift to absolutely irreducible unbounded representations in $\mathrm{SL}(2,\mathbb{C})$.

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- Deformation space $\mathbb{R}P^1(S^1)$ is non-Hausdorff noncompact 1-manifold

$$\Big(\widetilde{\mathsf{SL}(2,\mathbb{R})} \setminus \{1\}\Big)/\mathsf{Inn}$$

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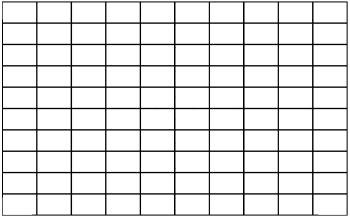
- Translation conjugated to *affine transformation*:

$$(x, y) \xrightarrow{\tau} (x + u, y + v)$$

$$(x, y) \xrightarrow{f \circ \tau \circ f^{-1}} (x - 2yv + (v^2 + u), y + v).$$

Polynomial deformations of lattices



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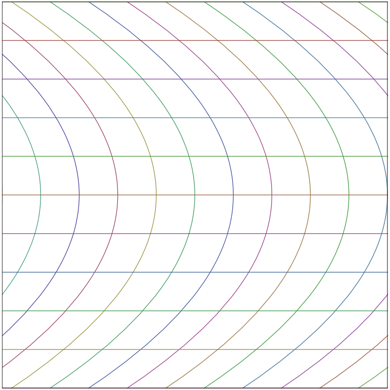

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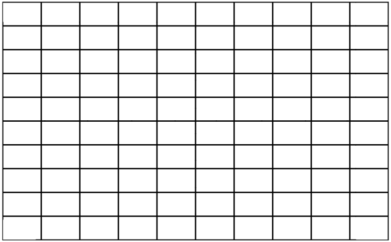

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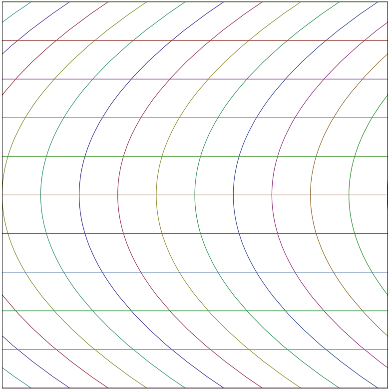

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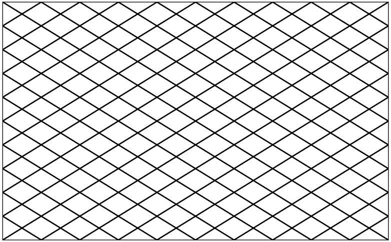

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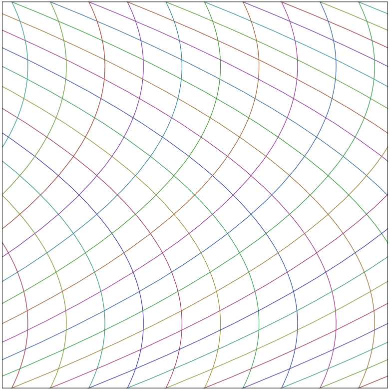

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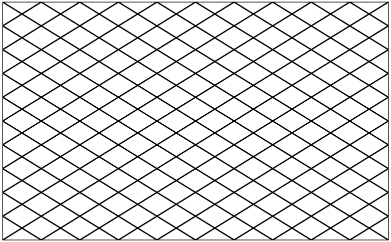

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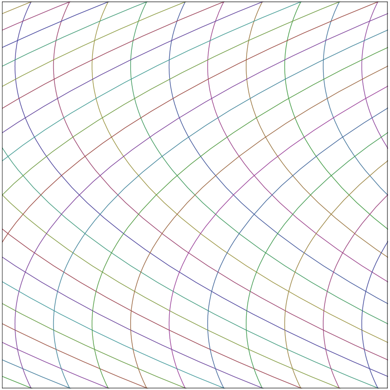

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- Contrast with the moduli space of Euclidean structures — the quotient of $\mathrm{H}^2 \times \mathbb{R}_+$ by $\mathrm{PGL}(2,\mathbb{Z})$ acting properly discretely.

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- For fixed topology Σ and geometry (G, X) define a deformation space $\mathfrak{D}_{(G,X)}(\Sigma)$ of *marked* (G, X)-structures on Σ.
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