Weakly commensurable arithmetic groups and locally symmetric spaces

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Outline

1. Weak commensurability
   - Definition and motivations
   - Basic results
   - Arithmetic Groups
   - Remarks on nonarithmetic case
2. Length-commensurable locally symmetric spaces
   - Links between length-commensurability and weak commensurability
   - Main results
   - Applications to isospectral locally symmetric spaces
3. Proofs
   - "Special" elements in Zariski-dense subgroups

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SURVEY:

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Definition

Let G_1 and G_2 be two semi-simple groups defined over a field F (of characteristic zero).

- Semi-simple $g_i \in G_i(F)$ $(i = 1, 2)$ are weakly commensurable if there exist maximal F-tori $T_i \subset G_i$ such that $g_i \in T_i(F)$ and for some $\chi_i \in X(T_i)$ (defined over $\overline{F}$) we have

$$\chi_1(g_1) = \chi_2(g_2) \neq 1.$$

- (Zariski-dense) subgroups $\Gamma_i \subset G_i(F)$ are weakly commensurable if every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

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If $T \subset \mathrm{GL}_n$ is an F-torus, then given $g \in T(F)$ and $\chi \in X(T)$ we have

$$\chi(g) = \lambda_1^{a_1} \cdots \lambda_n^{a_n}$$

where $\lambda_1, \ldots, \lambda_n$ are *eigenvalues* of g and $a_1, \ldots, a_n \in \mathbb{Z}$.

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RECALL: subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of a group $\mathcal{G}$ are *commensurable* if

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Γ_1 and Γ_2 are *commensurable up to an F-isomorphism* between G_1 and G_2 if there exists an F-isomorphism

$$\sigma\colon G_1 \to G_2$$

such that $\sigma(\Gamma_1)$ and Γ_2 are *commensurable in usual sense.*

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GENERAL FRAMEWORK: *Characterization of linear groups in terms of spectra of its elements.*

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COMPLEX REPRESENTATIONS OF FINITE GROUPS:

Let Γ be a finite group,

$$\rho_i \colon \Gamma \to GL_{n_i}(\mathbb{C}) \quad (i = 1, 2)$$

be representations. Then

$$\rho_1 \simeq \rho_2 \quad \Leftrightarrow \quad \chi_{\rho_1}(g) = \chi_{\rho_2}(g) \;\; \forall g \in \Gamma,$$

where $\chi_{\rho_i}(g) = \operatorname{tr} \rho_i(g) = \sum \lambda_j \quad (\lambda_1, \ldots, \lambda_{n_i}$ eigenvalues of $\rho_i(g))$

Algebraic perspective

- Data afforded by weak commensurability is much more *convoluted* than data afforded by character of a group representation:

  when computing
  $$\chi(g) = \lambda_1^{a_1} \cdots \lambda_n^{a_n}$$
  one can use *arbitrary* integer weights $a_1, \ldots, a_n$. So weak commensurability appears to be difficult to analyze.

- EXAMPLE. Let $\Gamma \subset SL_n(\mathbb{C})$ be a *neat* Zariski-dense subgroup. For $d > 0$, let
  $$\Gamma^{(d)} = \langle \gamma^d \mid \gamma \in \Gamma \rangle.$$
  Then any $\Gamma^{(d)} \subset \Delta \subset \Gamma$ is weakly commensurable to Γ.

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Geometric perspective

Let M be a Riemannian manifold.

$L(M)$ - (weak) length spectrum (collection of lengths of closed geodesics w/o multiplicities)

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We will demonstrate this for Riemann surfaces - for now.

Geometric perspective

- Let $G = SL_2$. Corresponding symmetric space:

$$SO_2(\mathbb{R})\backslash SL_2(\mathbb{R}) = \mathbb{H} \quad \text{(upper half-plane)}$$

- Any Riemann (compact) surface of genus > 1 is of the form

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup (with torsion-free image in $PSL_2(\mathbb{R})$).

- Any closed geodesic c in M corresponds to a semi-simple $\gamma \in \Gamma$, i.e. $c = c_\gamma$, and has *length*

$$\ell(c_\gamma) = (1/n_\gamma) \cdot \log t_\gamma$$

where t_γ is the eigenvalue of $\pm\gamma$ which is > 1,
n_γ is an integer $\geqslant 1$.

NOTE that $\pm\gamma$ is conjugate to $\begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$.

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Geometric perspective

If $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ are length-commensurable then:

- for *any* nontrivial semi-simple $\gamma_1 \in \Gamma_1$ *there exists* a nontrivial semi-simple $\gamma_2 \in \Gamma_2$ such that

$$n_1 \cdot \log t_{\gamma_1} = n_2 \cdot \log t_{\gamma_2}$$

for some integers $n_1, n_2 \geqslant 1$, and vice versa.

So, $$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$$

where χ_i is the character of the maximal $\mathbb{R}$-torus $T_i \subset \mathrm{SL}_2$ corresponding to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{n_i}$.

THUS, Γ_1 and Γ_2 are weakly commensurable.

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The first result shows that weak commensurability “almost” retains information about the type of the ambient algebraic group.

Theorem 1. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. If there exist finitely generated Zariski-dense subgroups $\Gamma_i \subset G_i(F)$ $(i = 1, 2)$ that are weakly commensurable then either G_1 and G_2 have the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$.*

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NOTE that groups of types B_n and C_n can indeed contain Zariski-dense weakly commensurable subgroups - more later.

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Theorem 2. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ $(i = 1, 2)$ be finitely generated Zariski-dense subgroups. If Γ_1 and Γ_2 are weakly commensurable then $K_{\Gamma_1} = K_{\Gamma_2}$.*

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Notion of arithmeticity

For a $\mathbb{Q}$-defined algebraic group $G \subset \mathrm{GL}_n$, we set

$$G(\mathbb{Z}) = G \cap GL_n(\mathbb{Z}).$$

The subgroups of $G(F)$ (where $F/\mathbb{Q}$) *commensurable* with $G(\mathbb{Z})$, are called arithmetic.

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Replace $\mathbb{Z}$ with $\mathbb{Z}[1/2]$ (= ring of S-integers $\mathbb{Z}_S \subset \mathbb{Q}$ for $S = \{v_\infty, v_2\}$).
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More generally, given a number field K and a (finite) $S \subset V^K$ containing V_∞^K (archimedean places), one defined the ring of S-integers

$$\mathcal{O}_K(S) = \{a \in K^\times \mid v(a) \geqslant 0 \text{ for all } v \in V^K \setminus S\} \cup \{0\}.$$

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E.g.: What is an arithmetic subgroup of $G(\mathbb{R})$ where

$$G = \mathrm{SO}_3(f) \quad \text{and} \quad f = x^2 + e \cdot y^2 - \pi \cdot z^2?$$

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In our example, we can consider rational quadratic forms that are $\mathbb{R}$-equivalent to f, e.g.:

$$f_1 = x^2 + y^2 - 3z^2 \text{ or } f_2 = x^2 + 2y^2 - 7z^2.$$

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One can also consider $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ and $f_3 = x^2 + y^2 - \sqrt{2}z^2$. Then

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One can further replace integers by S-integers, etc.

Definition of arithmeticity

Definition. Let G be an absolutely almost simple algebraic group over a field F, char $F = 0$, and $\pi \colon G \to \overline{G}$ be isogeny onto adjoint group.

1. a number field K with a *fixed* embedding $K \hookrightarrow F$;
2. a finite set $S \subset V^K$ containing V^K_∞;
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We do NOT fix an F-isomorphism ${}_F\mathcal{G} \simeq G$ in n° 3, and by varying it we obtain a class of groups invariant under F-automorphisms.

Proposition. *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a field F,* char $F = 0$*, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathcal{G}_i, K_i, S_i)$-arithmetic group $(i = 1, 2)$.*

Then Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if

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- Γ_3 is NOT commensurable with either Γ_1 or Γ_2 b/c they have different fields of definition: $\mathbb{Q}(\sqrt{2})$ for Γ_3, and $\mathbb{Q}$ for Γ_1 and Γ_2.

Theorem 3. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero.*

If Zariski-dense $(\mathcal{G}_i, K_i, S_i)$-arithmetic $\Gamma_i \subset G_i(F)$ are weakly commensurable for $i = 1, 2$, then $K_1 = K_2$ and $S_1 = S_2$.

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The forms $\mathcal{G}_1$ and $\mathcal{G}_2$ may NOT be K-isomorphic in general, but we have the following.

Theorem 4. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, of the same type different from A_n, D_{2n+1} with $n > 1$, and E_6, and let $\Gamma_i \subset G_i(F)$ be a $(\mathcal{G}_i, K, S)$-arithmetic subgroup.*

If Γ_1 and Γ_2 are weakly commensurable then $\mathcal{G}_1 \simeq \mathcal{G}_2$ over K, and hence Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$.

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[1] - groups of type $\neq D_{2n}$; [2] - groups of type D_{2n} other than D_4;
Skip Garibaldi - type D_4 and alternative proof for all D_{2n}.

Theorem 5. (Garibaldi-R.) *Let G_1 and G_2 be connected absolutely almost simple groups of types B_n and C_n $(n \geqslant 3)$ respectively, defined over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup.*

Then Γ_1 and Γ_2 are weakly commensurable if and only if

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Theorem 6. *Let G_1 and G_2 be two connected absolutely almost simple groups defined over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense (K, S)-arithmetic subgroup.*

Then the set of Zariski-dense (K, S)-arithmetic subgroups $\Gamma_2 \subset G_2(F)$ which are weakly commensurable to Γ_1, is a union of finitely many commensurability classes.

Theorem 7. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup for $i = 1, 2$.*

If Γ_1 and Γ_2 are weakly commensurable then $\mathrm{rk}_K\, \mathcal{G}_1 = \mathrm{rk}_K\, \mathcal{G}_2$; in particular, if $\mathcal{G}_1$ is K-isotropic then so is $\mathcal{G}_2$.

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Theorem 8. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense lattice for $i = 1, 2$. Assume that Γ_1 is a (K, S)-arithmetic subgroup of $G_1(F)$.*

If Γ_1 and Γ_2 are weakly commensurable, then Γ_2 is a (K, S)-arithmetic subgroup of $G_2(F)$.

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    - Definition and motivations
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Two aspects:

1. Given a Zariski-dense subgroup $\Gamma_1 \subset G_1(F)$ with $K_{\Gamma_1} =: K$, determine possible K-groups $\mathcal{G}_2$ for which there exists a Zariski-dense subgroup $\Gamma_2 \subset \mathcal{G}_2(K)$ which is weakly commensurable to Γ_1;
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Item 1° is closely related to the following classical question:

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(Our results solve this problem for a number field K.)

(∗) *Let* D_1 *and* D_2 *be quaternion division algebras over a field* K $(\mathrm{char}\, K \neq 2)$. *Assume that* D_1 *and* D_2 *have same maximal subfields.*

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(.. well, one usually considers $\mathbb{Q}[\Gamma^{(2)}]$ where $\Gamma^{(2)} \subset \Gamma$ is generated by squares ...)

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Then

$$Z(D_1) = Z(D_2) =: K,$$

and for any semi-simple $\gamma_1 \in \Gamma_1$ there exists a semi-simple $\gamma_2 \in \Gamma_2$ s. t.

$$\gamma_1^m \text{ and } \gamma_2^n \text{ are conjugate in } SL_2(\mathbb{R}) \text{ for some } m, n \geqslant 1.$$

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Thus, length-commensurability of M_1 and M_2 implies that D_1 *and* D_2 *have the same isomorphism classes of étale subalgebras that intersect* Γ_1 *and* Γ_2*, respectively.*

On the other hand,

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$(*)$ has affirmative answer over number fields $\Rightarrow$

$L(M_1) = L(M_2)$ for arithmetically defined Riemann surfaces M_1 & M_2 implies that M_1 and M_2 are commensurable (A. Reid).

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$(*)$ can have negative answer over "large" fields (Rost, Wadsworth, Schacher ...), but remains widely open over finitely generated fields.

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Definition. Let D be a finite-dimensional central division algebra $/K$. The genus of D is

$$\mathbf{gen}(D) = \{\, [D'] \in \mathrm{Br}(K) \mid D' \text{ division algebra with same maximal subfields as } D\,\}.$$

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Theorem 10. (Chernousov + R^2) *Let K be a field of characteristic $\neq 2$. If K satisfies the following property*

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Theorem 11. (C + R^2) *Let K be a finitely generated field, and let D be a central division algebra $/K$ of degree n which is prime to* char *K. Then $\mathbf{gen}(D)$ is finite.*

Conjecture. *Let G_1, G_2 be absolutely simple algebraic groups over a field F,* char $F = 0$*, let $\Gamma_1 \subset G_1(F)$ be a finitely generated Zariski-dense subgroup.*

Set $K = K_{\Gamma_1}$.

Then there exist a finite collection $\mathcal{G}_2^{(1)}, \ldots, \mathcal{G}_2^{(r)}$ of F/K-forms of G_2 such that

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Question: *When can one take $r = 1$?*

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Notations

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  $\mathcal{G} = G(\mathbb{R})$
- $\mathcal{K}$ a maximal compact subgroup of $\mathcal{G}$;
  $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ associated symmetric space, $\quad \mathrm{rk}\,\mathfrak{X} = \mathrm{rk}_{\mathbb{R}}\, G$
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- $\mathcal{K}$ a maximal compact subgroup of $\mathcal{G}$; $\mathfrak{X} = \mathcal{K} \backslash \mathcal{G}$ associated symmetric space, $\quad \mathrm{rk}\, \mathfrak{X} = \mathrm{rk}_{\mathbb{R}}\, G$
- Γ a discrete torsion-free subgroup of $\mathcal{G}$, $\ \mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$
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Given $G_1, G_2, \quad \Gamma_i \subset \mathcal{G}_i := G_i(\mathbb{R})$ etc. as above, we will denote the corresponding *locally symmetric spaces* by $\mathfrak{X}_{\Gamma_i}$.

Two Riemannian manifolds M_1 and M_2 are:

- commensurable if they have a common finite-sheeted cover;
- length-commensurable if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$, where $L(M_i)$ is the set of lengths of all closed geodesics in M_i.

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Fact. *Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are of finite volume.*

If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable then (under minor technical assumptions) *Γ_1 and Γ_2 are weakly commensurable.*

The proof relies:

- in rank one case - on the result of Gel'fond and Schneider (1934): *if α and β are algebraic numbers $\neq 0, 1$ then $\dfrac{\log \alpha}{\log \beta}$ is either rational or transcendental.*

- in higher rank case - on the following

  Conjecture (Shanuel) *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree of the field generated by*

$$z_1, \ldots, z_n;\ e^{z_1}, \ldots, e^{z_n}$$

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So, our results for higher rank spaces are *conditional*.

Outline

Theorem 12. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be locally symmetric spaces of finite volume. If they are length-commensurable then*

- *either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n;*
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Theorem 13. *Let $\mathfrak{X}_{\Gamma_1}$ be an arithmetically defined locally symmetric space. The set of arithmetically defined locally symmetric spaces $\mathfrak{X}_{\Gamma_2}$ which are length-commensurable to $\mathfrak{X}_{\Gamma_1}$, is a union of finitely many commensurability classes. It consists of a single commensurability class if G_1 and G_2 have the same type different from A_n, D_{2n+1} with $n > 1$ and E_6.*

Corollary.

1. *Let d be even or $\equiv 3 \pmod 4$, and let M_1 and M_2 be arithmetic quotients of the d-dimensional real hyperbolic space.*

   If M_1 and M_2 are not commensurable, then (after a possible interchange of M_1 and M_2) there exists $\lambda_1 \in L(M_1)$ such that for any $\lambda_2 \in L(M_2)$, the ratio λ_1/λ_2 is transcendental over $\mathbb{Q}$

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2. *For any $d \equiv 1 \pmod 4$ there exist length-commensurable, but not commensurable, arithmetic quotients of the real hyperbolic d-space.*

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Theorem 14. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be locally symmetric spaces of finite volume which are length-commensurable. Assume that one of the spaces is arithmetically defined. Then*

1. *the other space is also arithmetically defined;*
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RECALL that for any *lattice* Γ, compactness of $\mathfrak{X}_\Gamma$ is equivalent to the existence of nontrivial unipotents in Γ. So, one can ask: *Suppose two lattices are weakly commensurable. Does the existence of nontrivial unipotents in one of them implies their existence in the other?* This question makes sense for arbitrary Zariski-dense subgroups.

Outline

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Theorem 15. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be isospectral compact locally symmetric spaces. If Γ_1 is arithmetic then Γ_2 is also arithmetic.*

Theorem 16. *Assume that* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are isospectral compact locally symmetric spaces, and at least one of the subgroups* Γ_1 *or* Γ_2 *is arithmetic. Then* $G_1 = G_2 =: G$*. Moreover, unless* G *is type* A_n, D_{2n+1} $(n > 1)$ *or* E_6*, the spaces* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are commensurable.*

Theorem 16. *Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral compact locally symmetric spaces, and at least one of the subgroups Γ_1 or Γ_2 is arithmetic. Then $G_1 = G_2 =: G$. Moreover, unless G is type A_n, D_{2n+1} $(n > 1)$ or E_6, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.*

It would be interesting to determine if Theorem 16 remains valid without any assumptions of arithmeticity.

Outline

Proofs rely on the existence of "special" elements in Zariski-dense subgroups.

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Question 2: *Let G be a reductive algebraic group over a field K (of characteristic zero), and let $\Gamma \subset G(K)$ be a Zariski-dense subgroup.*
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Elements of this kind will be called generic (this notion will be specialized further later on)

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Then Γ is dense in $\mathcal{G}$, but for any

$$\gamma = \left(\sqrt{2}m(\mathrm{mod}\,\mathbb{Z})\,,\,\sqrt{2}n(\mathrm{mod}\,\mathbb{Z})\right) \in \Gamma$$

we have $\overline{\langle\gamma\rangle} \subset \{(a(\mathrm{mod}\,\mathbb{Z})\,,\,b(\mathrm{mod}\,\mathbb{Z}))\mid na - mb \equiv 0(\mathrm{mod}\,\mathbb{Z})\}$,
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Example 2: Let $G = \mathbb{C}^\times \times \mathbb{C}^\times$, and let $\varepsilon \in \mathbb{C}^\times$ be NOT a root of unity.
Then $\Gamma = \langle\varepsilon\rangle \times \langle\varepsilon\rangle$ is Zariski-dense in G, but for any $\gamma = (\varepsilon^m, \varepsilon^n) \in \Gamma$,
we have $\overline{\langle\gamma\rangle} \subset \{(x,y) \in G \mid x^n = y^m\} \neq G$.

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Thus, for any $\gamma \in T \cap \Gamma$, we have $\gamma^n \in T'$ or T'', and therefore $T \neq \overline{\langle \gamma \rangle}$.

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Lemma 1. *If T is irreducible over K then for any $\gamma \in T(K)$ of infinite order, $\overline{\langle \gamma \rangle} = T$.*

Thus, a regular semi-simple $\gamma \in \Gamma \subset G(K)$ is "generic" if $T = C_G(\gamma)^\circ$ is K-irreducible.

Let T be a K-torus.

- $X(T)$ - group of characters of T
- K_T - minimal splitting field of T
- $\mathcal{G}_T = \mathrm{Gal}(K_T/K)$
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Explicit construction can be implemented for other classical types.

Additional problem: embed resulting generic tori into a given group.

GENERAL CASE:

Fact (Voskresenskii) *There exists a purely transcendental extension $\mathcal{K} = K(x_1, \ldots, x_r)$ and a $\mathcal{K}$-defined maximal torus $\mathcal{T} \subset G$ such that*

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For K a number field, one can construct such generic tori with prescribed local behavior at finitely many places.

Then, if Γ is S-arithmetic, one can find generic tori containing $\gamma \in \Gamma$ of infinite order.

Generic tori constructed by this method may not contain elements $\gamma \in \Gamma$ of infinite order if Γ is not S-arithmetic.

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Theorem 17. *Let G be a connected semi-simple real algebraic group. Then any Zariski-dense subsemigroup $\Gamma \subset G(\mathbb{R})$ contain a regular $\mathbb{R}$-regular γ such that $\langle\gamma\rangle$ is Zariski-dense in $T = C_G(\gamma)^\circ$.*

Theorem 18. *Let G be a semi-simple algebraic group over a field K of characteristic zero, and let $\Gamma \subset G(K)$ be a Zariski-dense subgroup. Then there exists a regular semi-simple $\gamma \in \Gamma$ such that $\langle \gamma \rangle$ is Zariski-dense in $T = C_G(\gamma)^{\circ}$.*

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We want to construct a regular semi-simple $\gamma \in \Gamma$ of infinite order such that $T = C_G(\gamma)^\circ$ is generic over K.

Proposition. *Let K be a finitely generated field, and $R \subset K$ be a finitely generated ring. There exists an infinite set of primes Π such that for each $p \in \Pi$ there exists an embedding $\varepsilon \colon K \hookrightarrow \mathbb{Q}_p$ such that $\varepsilon_p(R) \subset \mathbb{Z}_p$.*

- Pick a maximal K-torus $T_0 \subset G$ and fix a conjugacy class C in $W(G, T_0)$.
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Using Galois cohomology, we find an open $\Omega_p(C) \subset G(\mathbb{Q}_p)$ satisfying

- $\Omega_p(C)$ consists of regular semi-simple elements and intersects every open subgroup of $G(\mathbb{Q}_p)$;
- for $\omega \in \Omega_p(C)$ and $T_\omega = C_G(\omega)^\circ$, we have

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Pick r primes $p_1, \ldots, p_r \in \Pi$, and consider $\Omega_{p_i}(C_i) \subset G(\mathbb{Q}_{p_i})$.

One shows that

$$\Omega := \bigcap_{i=1}^{r} (\Gamma \cap \Omega_{p_i}(C_i)) \neq \emptyset,$$

and any $\gamma \in \Omega$ is generic. $\square$

Some other applications of p-adic embeddings:

- (Platonov) *Let $\pi\colon \tilde{G} \to G$ be a nontrivial isogeny of semi-simple groups over a finitely generated field K. Then $\pi(\tilde{G}(K)) \neq G(K)$.*
- (R.) Let Γ be a group with bounded generation, i.e.

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Assume that *any subgroup of finite index $\Gamma_1 \subset \Gamma$ has finite abelianization* $\Gamma_1^{ab} = \Gamma_1/[\Gamma_1, \Gamma_1]$. Then there are only *finitely many inequivalent irreducible representations $\rho\colon \Gamma \to GL_n(\mathbb{C})$.*

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