Data assimilation in slow-fast systems using stochastic subgridscale forecast models

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Stochastic parametrizations for data assimilation

GENERAL QUESTION: Can reduced stochastic climate models be beneficial for forecasting and prediction?

Our setting here: Data assimilation with Ensemble Kalman filters

Under what circumstances and why can stochastic reduced models be beneficial as forecast models in an ensemble Kalman filter setting? Can we achieve

- computational gain
- better skill?

Stochastic homogenization (*Khasminsky '66, Kurtz '73, Papanicolaou '76*) has been recently taken up in the context of climate models (works by *Crommelin, Franzke, Majda, Timofejev, Vanden-Eijnden*).

IDEA: Consider $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$dx = \frac{1}{\varepsilon} f_0(x,y)\,dt + f_1(x,y)\,dt$$
$$dy = \frac{1}{\varepsilon^2} g_0(x,y)\,dt + \frac{1}{\varepsilon}\sigma(x,y)\,dW_t$$

(For purely deterministic dynamics see *Melbourne and Stuart, Nonlinearity 2011*)

Assume the fast y-process is ergodic, and the average of f_0 over this measure is zero; then the statistics of the slow x-dynamics can be approximated in the limit $\varepsilon \to 0$ by

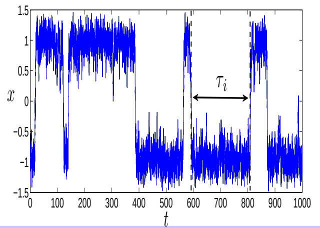
$$dX = F(X)\,dt + \Sigma(X)\,dB_t$$

Toy Model

We study the skew product system of a chaotically forced bistable system (*Givon et al., Nonlinearity* **17** *(2004)*)

$$\frac{dx}{dt} = x - x^3 + \frac{4}{90\varepsilon} y_2$$

$$\frac{dy_1}{dt} = \frac{10}{\varepsilon^2}(y_2 - y_1) \qquad \frac{dy_2}{dt} = \frac{1}{\varepsilon^2}(28y_1 - y_2 - y_1 y_3) \qquad \frac{dy_3}{dt} = \frac{1}{\varepsilon^2}\left(y_1 y_2 - \frac{8}{3} y_3\right)$$



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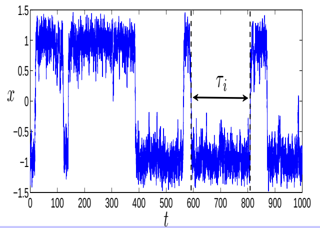

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Homogenization yields

$$dx = (x - x^3)dt + \sigma dW$$

where

$$\frac{\sigma^2}{2} = -\left(\frac{4}{90}\right)^2 \int_0^\infty y_2(t) \lim_{T\to\infty} \frac{1}{T} \int_0^T y_2(t+s)\, ds\, dt$$

has to be numerically estimated.

Parameter estimation (*Siegert et al.* 1998)

Assume that the slow dynamics of the deterministic system is modelled (on a coarse time scale) by a Langevin equation

$$dx = d(x)\, dt + \sigma(x)\, dW_t$$

Estimate drift and diffusion from a long trajectory; partition phase space into bins $[X, X + \Delta X]$, sample at coarse sampling time $h \gg dt$

$$D(X) = \frac{1}{h}\langle (x^{n+1} - x^n)\rangle\Big|_{x^n \in (X+\Delta X)} \xrightarrow[h \to 0]{} \mathrm{d}(X)$$
$$S(X) = \frac{1}{h}\langle (x^{n+1} - x^n)^2\rangle\Big|_{x^n \in (X+\Delta X)} \xrightarrow[h \to 0]{} \sigma^2(X)$$

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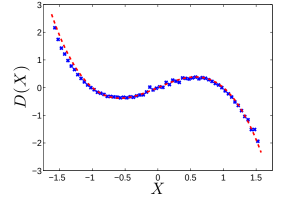

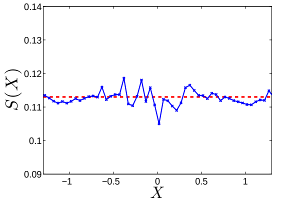

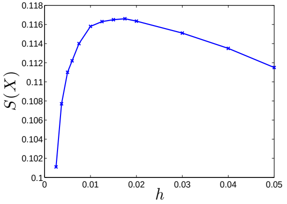

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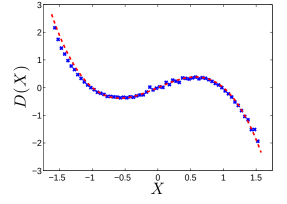

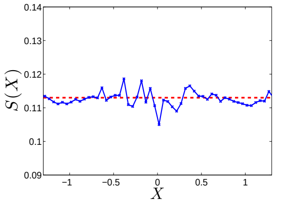

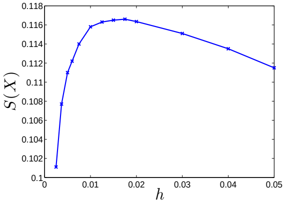

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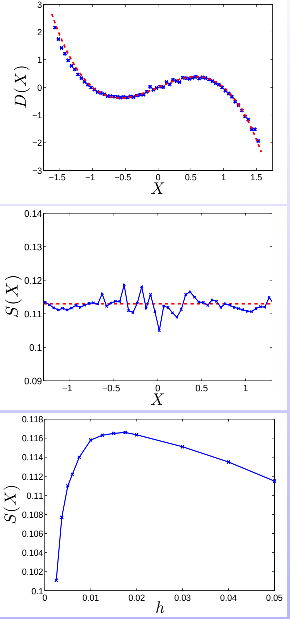

$$dx = d(x)\,dt + \sigma(x)\,dW_t$$

How to choose the subsampling time h?

- If h is chosen too small, diffusion coefficient does not exist
- If h is chosen too large,
  - $S(X) \approx X^2(1-X^2)^2\,h^2 + \sigma^2 \xrightarrow[h\to 0]{} \mathrm{d}^2(X)$
  - $D(X) \approx \frac{1}{h}\int (x - X)\hat{\rho}(x)\,dx = -\frac{X}{h}$

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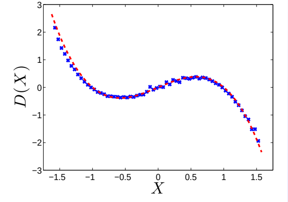

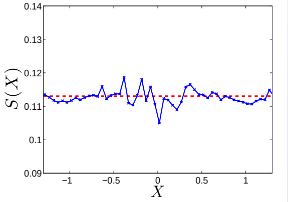

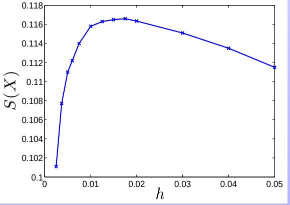

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Rule of thumb: Choose $h \approx 3T_f$, where T_f is the characteristic time of the fast dynamics

$$\sigma^2 = 0.113$$

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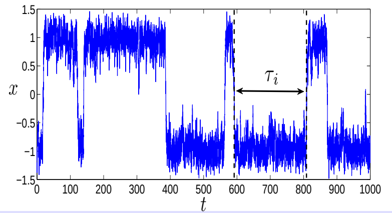
Characteristic time scales



- Autocorrelation time τ_{corr}
  - $C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)x(\tau+s)\,ds$: $\quad (\tau_{\text{corr}} = 208)$

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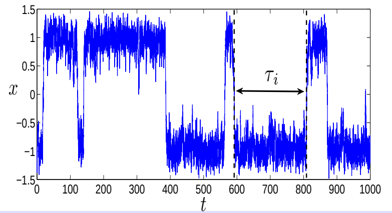

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- Mean sojourn time $\bar{\tau}$ (Mean exit time $\tau_e = \bar{\tau}/2$)
  - average over individual τ_i: $(\bar{\tau} = 218)$
  - assume Poisson process $P_c(\tau_i) = 1 - \exp\left(-\frac{\tau_i}{\bar{\tau}}\right)$: $(\bar{\tau} = 214)$
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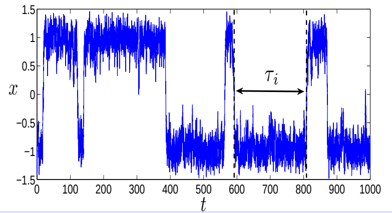
Characteristic time scales



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- Mean transit time $\bar{\tau}_t$
  - average over individual $\tau_{t,i}$: $(\bar{\tau}_t = 5.9)$
  - homogenized model: $(\bar{\tau} = 5.66)$

Accuracy and sensitivity of homogenized model

How good are climate models to reproduce the statistics?

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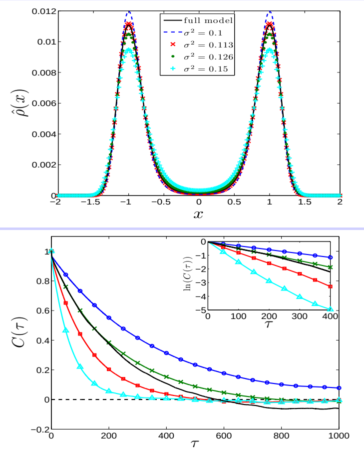
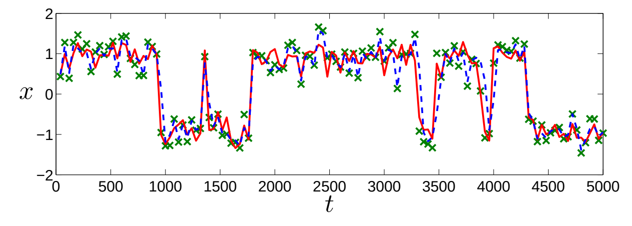


The climate model is not sensitive to uncertainties in $\varepsilon < 0.05$ but very sensitive to changes in drift and diffusion coefficients:

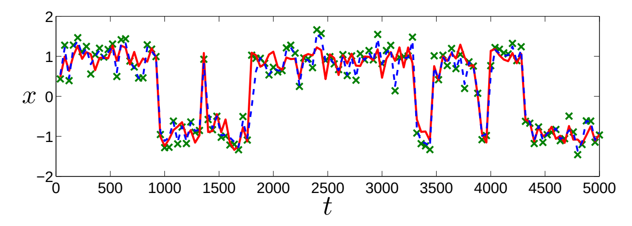
	full model	climate model ($\sigma^2 = 0.1$)	climate model ($\sigma^2 = 0.113$)	climate model ($\sigma^2 = 0.126$)	climate model ($\sigma^2 = 0.15$)
$\tau_{\rm corr}$	208.3	353.9	221.7	129.0	70.5
τ_e	108.6	205.7	117.8	75.6	40.8
τ_t	5.9	5.86	5.66	5.48	5.17
$\lambda_{\rm max}^{-1}$	0.0103	n.a.	n.a.	n.a	n.a
λ_{LS}^{-1}	233.7	588.2	398.6	206.4	108.6

Numerical results

Use the climate model as a forecast model in an Ensemble Transform Kalman filter (ETKF) setting (*Tippet et al. 2003*). Only the slow variable x is observed.

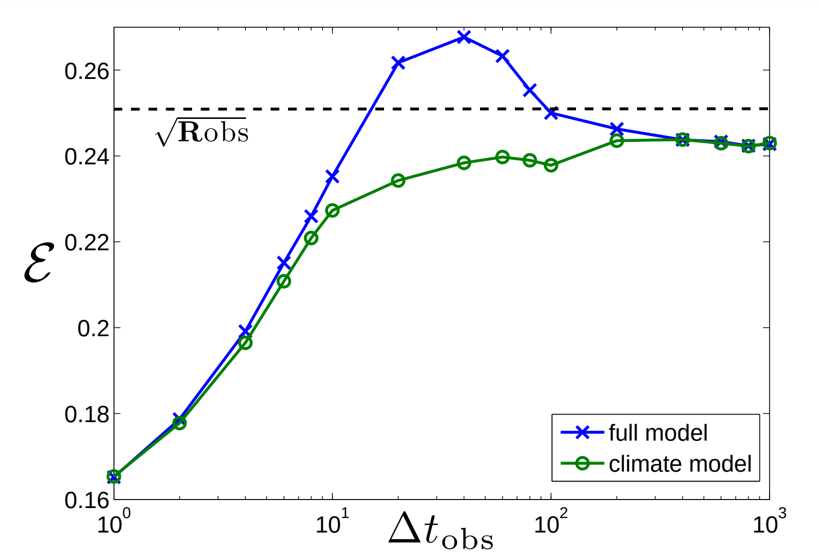


Full deterministic model



Reduced climate model ($\sigma^2 = 0.126 > 0.113$)

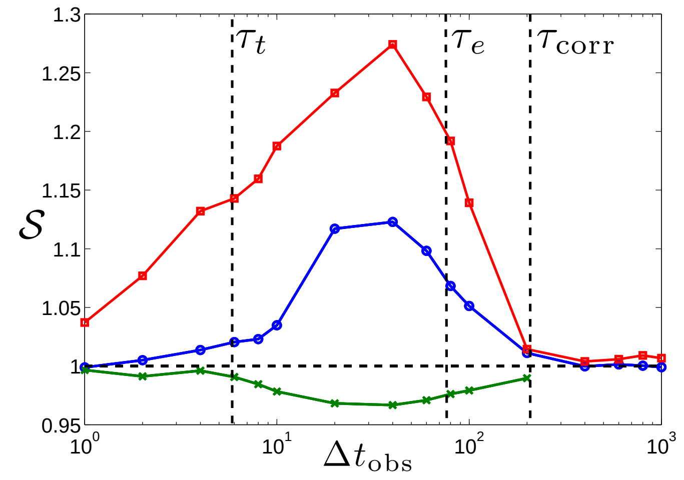
Numerical results



Numerical results

We define the *skill* $\mathcal{S} = \frac{\mathcal{E}_{\text{full}}}{\mathcal{E}_{\text{climate}}}$, $\mathcal{S} > 1$ is good!

Blue - all analyses, Green - metastable states, Red - transitions



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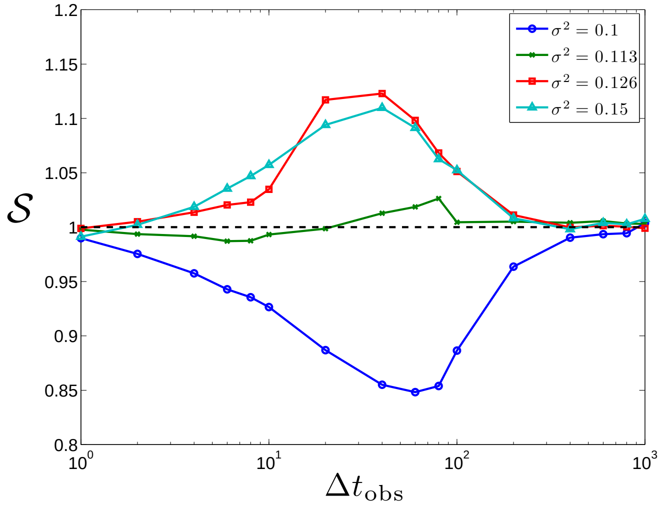

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The numerical results suggests that stochastic climate models are

- beneficial for observation time intervals $\Delta t_{\mathrm{obs}} \in (\tau_t, \tau_{\mathrm{corr}})$
- good at capturing the transitions between slow metastable states
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So why, if the climate model fails to accurately reproduce the statistics of the full model, does it perform better?

Numerical results

Small ensemble sizes $\longrightarrow$ underestimation of $\mathbf{P}_f$ $\nearrow$ larger ensembles

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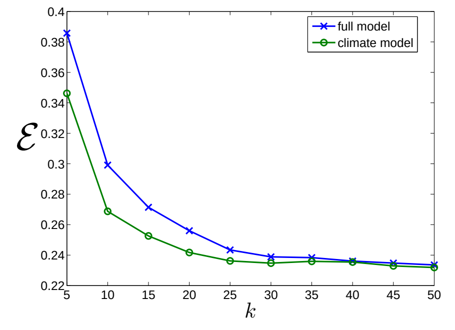
Increasing ensemble size k:



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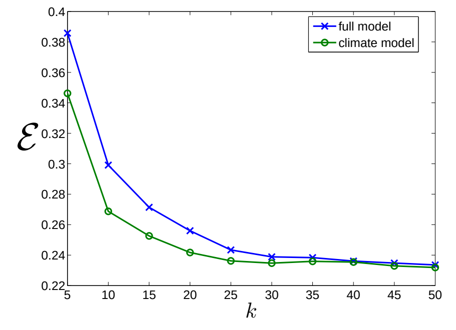
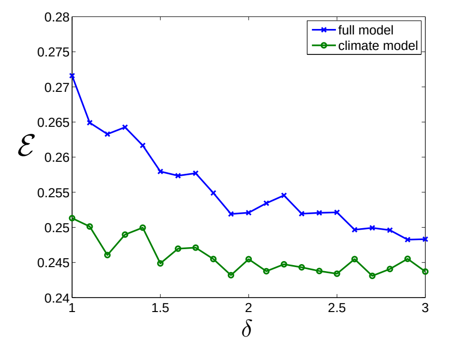


Increasing covariance inflation δ:



Reliability and Talagrand diagrams

- sort the forecast ensemble $\mathbf{X}_f = [x_{f,1}, x_{f,2}, ..., x_{f,k}]$ and create bins $(-\infty, x_{f,1}]$, $(x_{f,1}, x_{f,2}]$, ... , $(x_{f,k}, \infty)$ at each forecast step
- increment whichever bin the actual truth falls into at each forecast step

Convex histogram: underestimating ensemble
Concave histogram: overestimating ensemble
Flat histogram: reliable ensemble for which each ensemble member has equal probability of being nearest to the truth

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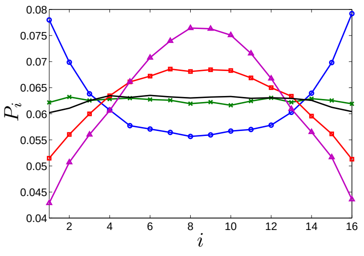

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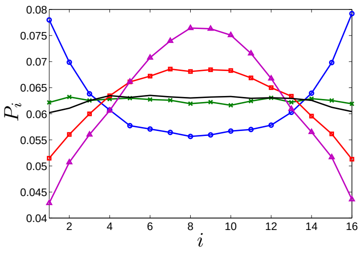
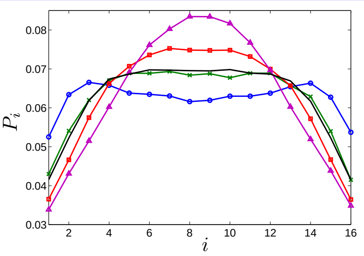
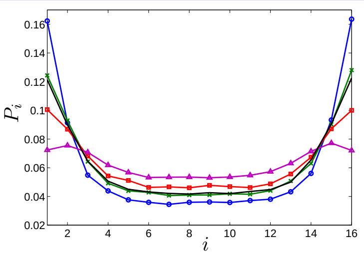




Metastable states



Transitions

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We would like to

- explore the usefulness of climate models in more realistic settings
- study the effectiveness of stochastic climate models in other data assimilation schemes