Motivic Superstring Amplitudes

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Polylogarithms as a Bridge between
Number Theory and Particle Physics
London Mathematical Society – EPSRC Durham Symposium
July, 8 – 12, 2013

Outline: Superstring amplitudes

- Superstring amplitudes: generalized Euler integrals,
  multiple Gaussian hypergeometric functions

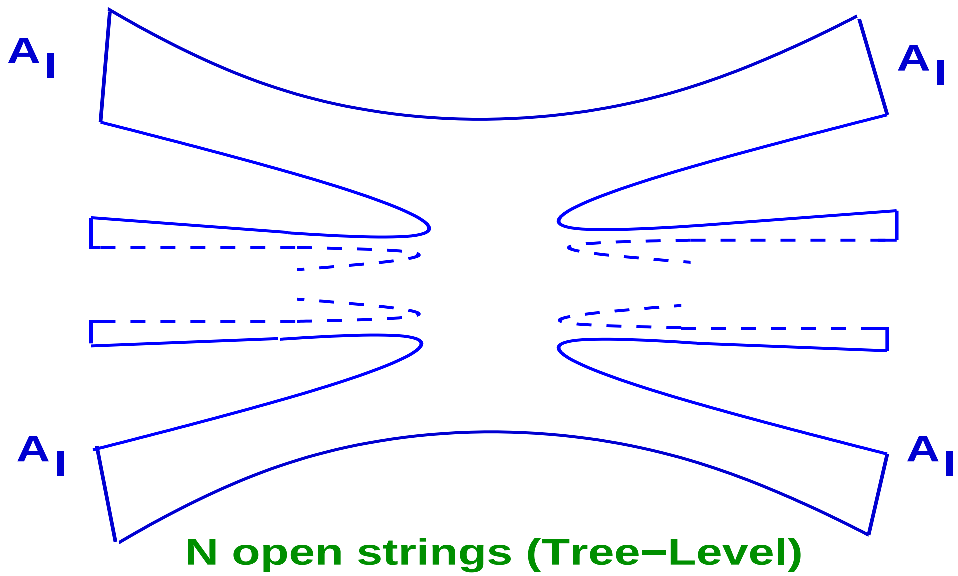
- Structure of α'–expansions is encoded by
  decompositon of motivic multi zeta values

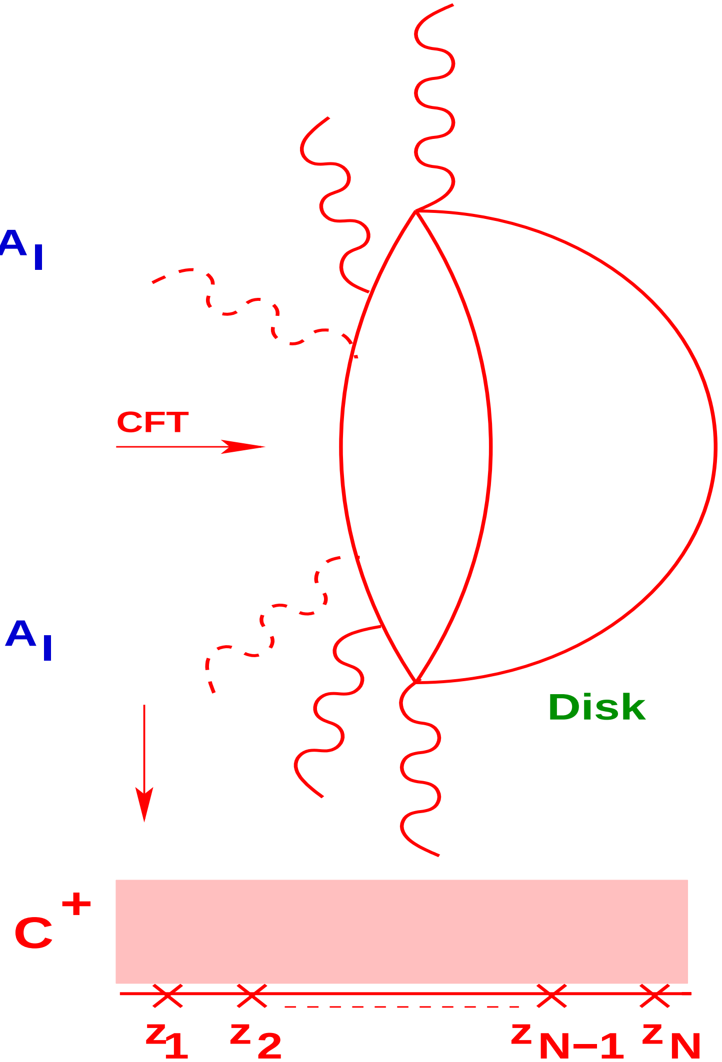
- coproduct maps the α'–expansion onto a
  non–commutative Hopf algebra

- Graded Lie algebra structure of the α'–expansion:
  interesting algebras related to Grothendiecks Galois theory

- Multiple inverse Mellin transformations:
  string amplitudes treated as distributions

Disk scattering of open strings





$$\mathcal{A}(1, 2, \ldots, N; \alpha')$$

Disk scattering of open strings

Compact and short expression

in terms of a *minimal basis* of $(N-3)!$ *building blocks*:

$$\boxed{\mathcal{A}(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\ F^{\sigma}_{(1,\ldots,N)}(\alpha')}$$

Mafra, Schlotterer, St.St., `arXiv:1106.2645` and `arXiv:1106.2646`

St.St., Taylor, `arXiv:1204.3848`

A_{YM}	Yang–Mills subamplitudes with colour ordering $1,2,\ldots,N$
$F^{\sigma}(\alpha')$	generalized Euler integral, multiple Gaussian hypergeometric functions

Scattering of N open strings

Consider (generalized) Selberg integrals:

$$\boxed{F\left[\{n_{ij}^I\}\right] = \int\limits_{z_i<z_{i+1}} \left(\prod_{j=2}^{N-2} dz_j\right) \prod_{1\leq i<j\leq N-1} |z_i - z_j|^{s_{ij}} \ (z_j - z_i)^{n_{ij}^I}}$$

Integration over $N-3$ ordered points

$$s_{ij} = \alpha'(k_i + k_j)^2 = 2\alpha' k_i k_j,\ n_{ij}^I \in \mathbf{Z}$$

With parameterization:

$$\begin{aligned} z_1 &= 0 \quad , \quad z_{N-1} = 1 \ , \\ z_k &= \prod_{l=k-1}^{N-3} x_l \quad , \quad k = 2, \ldots, N-2 \end{aligned}$$

Generalized Euler integrals:

$$B_N\,[n] = \left(\prod_{i=1}^{N-3} \int_0^1 dx_i\right) \prod_{j=1}^{N-3} x_j^{a_j}\ (1-x_j)^{b_j} \prod_{j<k}^{N-3} \left(1 - x_j x_{j+1} \ldots x_k\right)^{c_{jk}}$$

Scattering of N open strings

- $\frac{1}{2}N(N-3)$ Laurent polynomials = number of kinematic invariants s_{ij}
- Integrals on the moduli space of Riemann spheres with N marked points $\mathcal{M}_{0,N}$
- There is a basis of $(N-3)!$ independent integrals $F\left[\{n^I_{ij}\}\right]$

Examples:

$$B_4[n] \;=\; \int_0^1 dx\ x^{s_{12}+n_{12}}\ (1-x)^{s_{23}+n_{23}} = \frac{\Gamma(s_{12}+n_{12}+1)\ \Gamma(s_{23}+n_{23}+1)}{\Gamma(s_{12}+s_{23}+n_{12}+n_{23}+2)}$$

$$\begin{aligned}
B_5[n] \;&=\; \int_0^1 dx_1 \int_0^1 dx_2\ x_1^{s_{12}+n_1}\ x_2^{s_{45}+n_2}\ \ (1-x_1)^{s_{23}+n_{11}}\ (1-x_2)^{s_{34}+n_{22}}\ (1-x_1x_2)^{s_{24}+n_{12}} \\
&=\; \frac{\Gamma(1+s_{12}+n_1)\Gamma(1+s_{45}+n_2)\Gamma(1+s_{23}+n_{11})\Gamma(1+s_{34}+n_{22})}{\Gamma(2+s_{12}+s_{23}+n_1+n_{11})\ \Gamma(2+s_{34}+s_{45}+n_2+n_{22})} \\
&\times\ \ {}_3F_2\left[\begin{matrix} 1+s_{12}+n_1,\,1+s_{45}+n_2,\,-s_{24}-n_{12} \\ 2+s_{12}+s_{23}+n_1+n_{11},\,2+s_{34}+s_{45}+n_2+n_{22}\end{matrix}\right]
\end{aligned}$$

α'–expansion $\Longleftrightarrow$ multiple Euler–Zagier sums

E.g. N=7:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \; \frac{x^{s_2}\ (1-x)^{s_3}\ y^{t_2}\ (1-y)^{s_4}\ z^{t_6}\ (1-z)^{s_5}\ w^{s_7}\ (1-w)^{s_6}}{(1-xy)\ (1-wz)\ (1-yz)}\ (1-wxyz)^{s_1-t_1+t_4-t_7}$$

$$\times\ (1-xy)^{-s_3-s_4+t_3}\ (1-wz)^{-s_5-s_6+t_5}\ (1-yz)^{-s_4-s_5+t_4}\ (1-wyz)^{s_5+t_1-t_4-t_5}\ (1-xyz)^{s_4-t_3-t_4+t_7}$$

$$= \mathcal{I}_0 \ + \ \mathcal{I}_{1a}\ (s_1+s_2+s_3+s_4+s_5+s_6+s_7) \ + \ \mathcal{I}_{1b}\ (t_1+t_2+t_3+t_4+t_5+t_6+t_7) + \mathcal{O}(\alpha'^2)$$

with Multiple Euler–Zagier sums $\mathcal{I}_0,\ \mathcal{I}_{1a},\ \mathcal{I}_{1b}$:

$$\boxed{\begin{aligned}
\mathcal{I}_0 &= \int \frac{1}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1,n_2=0\\ n_3=1}}^{\infty} \frac{1}{n_3\ (1+n_1)\ (n_1+n_2+1)\ (n_2+n_3)} = \frac{27}{4}\zeta(4)\\
\mathcal{I}_{1a} &= \int \frac{\ln w}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1,n_3=1\\ n_2=0}}^{\infty} \frac{1}{n_1\ n_3^2\ (n_1+n_2)\ (n_2+n_3)} = \frac{7}{2}\zeta(5)-4\zeta(2)\zeta(3)\\
\mathcal{I}_{1b} &= \int \frac{\ln y}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1,n_2=1\\ n_3=0}}^{\infty} \frac{1}{n_1\ n_2\ (n_2+n_3)\ (n_1+n_3)^2} = -\frac{9}{2}\zeta(5)+\zeta(2)\zeta(3)
\end{aligned}}$$

Generalized Selberg integrals

$z_1 = 0,\ z_{N-1} = 1$

$$\boxed{\int_{z_i<z_{i+1}} \left(\prod_{j=2}^{N-2} dz_j\right) \prod_{1\le i<j\le N-1} |z_i - z_j|^{s_{ij}}\ (z_j - z_i)^{n_{ij}}}$$

Terasoma & Brown: The coefficients of the Taylor expansion of the Selberg integrals w.r.t. the variables s_{ij} can be expressed as linear combinations of MZVs over $\mathbf{Q}$.

However:

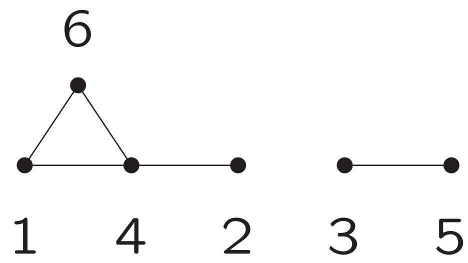
$$\int_{z_i<z_{i+1}} \left(\prod_{j=2}^{5} dz_j\right) \prod_{1\le i<j\le 6} |z_{ij}|^{s_{ij}} \frac{1}{z_{14}z_{16}z_{24}z_{35}z_{46}} = 2\ \zeta(2) + 2\ \zeta(3) + \cdots$$

$$\int_{z_i<z_{i+1}} \left(\prod_{j=2}^{5} dz_j\right) \prod_{1\le i<j\le 6} |z_{ij}|^{s_{ij}} \frac{1}{z_{13}z_{16}z_{24}z_{35}z_{46}} = \frac{27}{4}\ \zeta(4) + \cdots$$

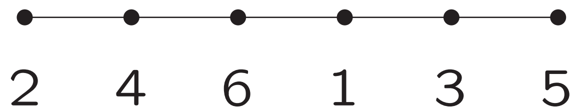
Selberg integrals

Graphs:

$$\frac{1}{z_{14} z_{16} z_{24} z_{35} z_{46}} =$$



$$\frac{1}{z_{13} z_{16} z_{24} z_{35} z_{46}} =$$



$\hookrightarrow$ mixing criterion from graphs

St.St., Taylor, `arXiv:1204.3848`
Mafra, Schlotterer, St.St., `arXiv:1106.2646`

α'–expansion and multiple polylogarithms

Multiple polylogarithms appear to be suitable language to **systemize** our task

Consider (regular) integral: $z_1 = 0,\ z_{N-1} = 1,\ z_N = \infty$

$$I_{\{a_k\}} = \prod_{k=2}^{N-2} \int_0^{z_{k+1}} \frac{dz_k}{z_k - a_k} \underbrace{\prod_{i<j} |z_{ij}|^{s_{ij}}}_{\text{expand w.r.t. } \alpha'}, \qquad a_k \in \{0, z_{k+1}, z_{k+2}, \ldots, z_{N-2}, 1\}$$

$$= \prod_{k=2}^{N-2} \int_0^{z_{k+1}} \frac{dz_k}{z_k - a_k} \prod_{i<j} \left(\sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} \frac{(\ln |z_{ij}|)^{n_{ij}}}{n_{ij}!} \right)$$

$$I = \frac{1}{2}\, s_{23}^2 \int_0^1 \frac{dz_3}{z_3 - 1} \int_0^{z_3} \frac{dz_2}{z_2} \, [\ln(z_3 - z_2)]^2$$

$$= \frac{1}{2}\, s_{23}^2 \int_0^1 \frac{dz_3}{z_3 - 1} \int_0^{z_3} \frac{dz_2}{z_2} \left\{ [\ln(z_3)]^2 + 2\ \ln z_3 \ln\left(1 - \frac{z_2}{z_3}\right) + \left[\ln\left(1 - \frac{z_2}{z_3}\right)\right]^2 \right\}$$

Performing the integration using multiple polylogarithms

Multiple polylogarithms: $\boxed{G(a_1, a_2, \ldots, a_n; z) \quad := \quad \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t)}$

Multiple polylogarithms constitute a graded Lie algebra with shuffle product:

$$G(a_1, \ldots, a_r; z)\ G(a_{r+1}, \ldots, a_{r+s}; z) = \sum_{\sigma \in \Sigma(r,s)} G(a_{\sigma(1)}, \ldots, a_{\sigma(r+s)}; z)$$

$$G(\underbrace{0, 0, \ldots, 0}_{w}; z) \;=\; \frac{1}{w!} (\ln\ z)^w \qquad G(\underbrace{1, 1 \ldots, 1}_{w}; z) \;=\; \frac{1}{w!} \ln^w (1-z)$$

$$G(\underbrace{a, a, \ldots, a}_{w}; z) \;=\; \frac{1}{w!} \ln \left(1 - \frac{z}{a}\right)^w$$

$$I \;=\; \int_0^1 \frac{dz_3}{z_3 - 1} \int_0^{z_3} \frac{dz_2}{z_2} \left\{ G(0,0; z_3) + G(0; z_3)\, G(z_3; z_2) + G(z_3, z_3; z_2) \right\}$$

$$=\; \int_0^1 \frac{dz_3}{z_3 - 1} \left\{ G(0,0; z_3) G(0; z_3) + G(0; z_3)\ G(0, z_3; z_3) + G(0, z_3, z_3; z_3) \right\} = \frac{\zeta_2^2}{5}$$

Performing the integration using multiple polylogarithms

Problem, *e.g.:*

$$I = \int_0^1 \frac{dz_3}{z_3 - 1} \int_0^{z_3} \frac{dz_2}{z_2} G(z_3; z_2)\, G(1; z_2)$$

$$= \int_0^1 \frac{dz_3}{z_3 - 1} \left\{ G(0, z_3, 1; z_3) + G(0, 1, z_3; z_3) \right\}$$

$\hookrightarrow$ How to write a multiple polylogarithm of the form

$$G(\{0, a_1, a_2, \ldots, z, \ldots, a_n\}_w; z)$$

in terms of objects without z in their labels ?

Proceed as follows:

- use Hopf-algebra structure of polylogarithms
- decompose polyogs step by step using coproduct
- express the result in the appropriate basis

Performing the integration using multiple polylogarithms

Example: $G(0,z,1;z) = G(0,0,1;z) - G(1,0,1;z) - G(1;z)\,\zeta_2$

$$G(0,1,z;z) = -2G(0,0,1;z) + G(0,1,1;z) + G(1,0,1;z) + G(1;z)\,\zeta_2$$

$$\boxed{\begin{aligned} G(a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_n;z) &= G(a_{i-1},a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n;z) \\ &- G(a_{i+1},a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n;z) \\ &- \int_0^z \frac{dt}{t-a_{i-1}} G(a_1,\ldots,\hat{a}_{i-1},t,a_{i+1},\ldots,a_n;t) \\ &+ \int_0^z \frac{dt}{t-a_{i+1}} G(a_1,\ldots,a_{i-1},t,\hat{a}_{i+1},\ldots,a_n;t) \\ &+ \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_{i-1},t,a_{i+1},\ldots,a_n;t) \end{aligned}}$$

Broedel, Schlotterer, St.St., `arXiv:1304.7267 [hep-th]`

Scattering of N open strings

Recall, we had an ordering:

$$\int_{z_i<z_{i+1}} \left(\prod_{j=2}^{N-2} dz_j\right) \ldots$$

$\hookrightarrow$ an other set of $(N-3)!$ different orderings !

$\hookrightarrow$ basis of $(N-3)! \times (N-3)!$ hypergeometric functions

$\hookrightarrow$ put them into a matrix F

F = period matrix of the moduli space of Riemann spheres with N marked points

$\mathrm{rk}(F) = (N-3)!$ (Goncharov, private communication)

Superstring amplitude : $\boxed{\mathcal{A} \;=\; F\; A}$ Mafra, Schlotterer, St.St.

$\mathcal{A} \;=\;$ $(N-3)!$ dimensional vector encoding the **string basis**

$F \;=\;$ $(N-3)! \times (N-3)!$ matrix encoding α'–expansion

$A \;=\;$ $(N-3)!$ dimensional vector encoding the **YM-basis**

E.g. $N=5$*:* $$F = \begin{pmatrix} F_1 & F_2 \\ \tilde{F}_2 & \tilde{F}_1 \end{pmatrix} \quad , \quad \tilde{F}_1 = F_1|_{2\leftrightarrow 3}\,, \quad \tilde{F}_2 = F_2|_{2\leftrightarrow 3}$$

$$\boxed{\begin{aligned} F_1 &= s_{12}\; s_{34} \int_0^1 dx \int_0^1 dy\; x^{s_{45}}\; y^{s_{12}-1}\; (1-x)^{s_{34}-1}\; (1-y)^{s_{23}}\; (1-xy)^{s_{24}} \\ &= 1 + \zeta(2)\; (s_1 s_3 - s_3 s_4 - s_1 s_5) \\ &- \;\zeta(3)\; (s_1^2 s_3 + 2 s_1 s_2 s_3 + s_1 s_3^2 - s_3^2 s_4 - s_3 s_4^2 - s_1^2 s_5 - s_1 s_5^2) + \mathcal{O}(\alpha'^4)\ , \\ \\ F_2 &= s_{13}\; s_{24} \int_0^1 dx \int_0^1 dy\; x^{s_{45}}\; y^{s_{12}}\; (1-x)^{s_{34}}\; (1-y)^{s_{23}}\; (1-xy)^{s_{24}-1} \\ &= \zeta(2)\; s_{13}\; s_{24} - \zeta(3)\; s_{13}\; s_{24}\; (s_1 + s_2 + s_3 + s_4 + s_5) + \mathcal{O}(\alpha'^4)\ . \end{aligned}}$$

with: $s_{ij} = \alpha'(k_i + k_j)^2,\; s_i \equiv s_{i,i+1} = \alpha'\;(k_i + k_{i+1})^2\ ,\; i+5 \equiv i$

$$\boxed{\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l}} \quad , \quad n_l \in \mathbf{N}^+ \, , \; n_r \geq 2 \, .$$

E.g.: Riemann zeta–function: $\zeta_n = \sum_{k>0} \frac{1}{k^n} \quad , \quad \zeta_{2n} = \frac{2^{2n-1}\ \pi^{2n}}{(2n)!}\ |B_{2n}|$

- $w = \sum_{l=1}^{r} n_l$ is called the transcendentality degree or weight
- r depth of MZV

$$\zeta_{n_1,\ldots,n_r} = (-1)^r\ G(\underbrace{0,\ldots,0}_{n_r-1},1\ldots,\underbrace{0,\ldots,0}_{n_1-1},1;1)$$

$$\boxed{\zeta_{n_1,\ldots,n_r} = (-1)^r\ I_\gamma(0;\rho(n_1\ldots n_r);1) \quad , \quad \rho(n_1,\ldots,n_r) = 10^{n_1-1}\ldots 10^{n_r-1}}$$

$$I_\gamma(a_0;a_1,\ldots,a_n;a_{n+1}) = \int_{\Delta_{n,\gamma}} \frac{dt_1}{t_1-a_1} \wedge \ldots \wedge \frac{dt_n}{t_n-a_n}$$

(Commutative) graded $\mathbf{Q}$–algebra:

$$\boxed{\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k \quad , \quad \dim_{\mathbf{Q}}(\mathcal{Z}_N) = d_N} \, ,$$

with: $d_N = d_{N-2} + d_{N-3}$, $\quad d_0 = 1,\ d_1 = 0,\ d_2 = 1, \ldots$ (Zagier)

w	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5 $\zeta_2\ \zeta_3$	ζ_3^2 ζ_2^3	ζ_7 $\zeta_2\ \zeta_5$ $\zeta_2^2\ \zeta_3$	$\zeta_{3,5}$ $\zeta_3\ \zeta_5$ $\zeta_2\ \zeta_3^2$ ζ_2^4	ζ_9 ζ_3^3 $\zeta_2\ \zeta_7$ $\zeta_2^2\ \zeta_5$ $\zeta_2^3\ \zeta_3$	$\zeta_{3,7}$ $\zeta_3\ \zeta_7$ ζ_5^2 $\zeta_2\ \zeta_{3,5}$ $\zeta_2\ \zeta_3\ \zeta_5$ $\zeta_2^2\ \zeta_3^2$ ζ_2^5	$\zeta_{3,3,5}$ $\quad$ $\zeta_2\ \zeta_3^3$ $\zeta_{3,5}\ \zeta_3$ $\quad$ $\zeta_2\ \zeta_9$ ζ_{11} $\quad$ $\zeta_2^2\ \zeta_7$ $\zeta_3^2\ \zeta_5$ $\quad$ $\zeta_2^3\ \zeta_5$ $\zeta_2^4\ \zeta_3$	$\zeta_{1,1,4,6}$ $\quad$ $\zeta_2\ \zeta_{3,7}$ $\zeta_{3,9}$ $\quad$ $\zeta_2^2\ \zeta_{3,5}$ $\zeta_3\ \zeta_9$ $\quad$ $\zeta_2\ \zeta_5^2$ $\zeta_5\ \zeta_7$ $\quad$ $\zeta_2\ \zeta_3\ \zeta_7$ ζ_3^4 $\quad$ $\zeta_2^2\ \zeta_3\ \zeta_5$ $\zeta_2^3\ \zeta_3^2$ ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9	12

<u>*E.g.*</u> weight 12 : $\quad \zeta_{5,7} = \frac{14}{9}\ \zeta_{3,9} + \frac{28}{3}\ \zeta_5\ \zeta_7 - \frac{776224}{1576575}\ \zeta_2^6$

Blümlein, Broadhurst, Vermaseren

Structure of open string amplitude

$$F = P\ Q\ :\exp\left\{\sum_{n\geq 1}\zeta_{2n+1}\ M_{2n+1}\right\}:$$

$$\begin{aligned}
P &= (N-3)! \times (N-3)! \text{ matrix encoding } \alpha'\text{-expansions with } \boldsymbol{\zeta_2^k} \\
M_{2n+1} &= (N-3)! \times (N-3)! \text{ matrix encoding } \alpha'\text{-expansions with } \boldsymbol{\zeta_{2n+1}} \\
Q &= (N-3)! \times (N-3)! \text{ matrix encoding } \alpha'\text{-expansions with } \boldsymbol{\zeta_{n_1,\ldots,n_r}}
\end{aligned}$$

with:

$$\begin{aligned}
P &= 1+\sum_{n\geq 1}\zeta_2^n\ P_{2n} := 1+\sum_{n\geq 1}\zeta_2^n\ F|_{\zeta_{2n}} \\
M_{2n+1} &= F|_{\zeta_{2n+1}}
\end{aligned}$$

Structure of open string amplitude

and:

$$
\begin{aligned}
Q \;=\;& 1+\frac{1}{5}\,\zeta_{3,5}\,[M_5,M_3]+\left\{\frac{3}{14}\,\zeta_5^2+\frac{1}{14}\,\zeta_{3,7}\right\}\,[M_7,M_3]\\
&+\left\{9\,\zeta_2\,\zeta_9+\frac{6}{25}\,\zeta_2^2\,\zeta_7-\frac{4}{35}\,\zeta_2^3\,\zeta_5+\frac{1}{5}\,\zeta_{3,3,5}\right\}\,[M_3,[M_5,M_3]]\\
&+\left\{\frac{2}{9}\,\zeta_5\,\zeta_7+\frac{1}{27}\,\zeta_{3,9}\right\}\,[M_9,M_3]\\
&+\frac{48}{691}\left\{\frac{18}{35}\,\zeta_2^3\,\zeta_3^2+\frac{1}{5}\,\zeta_2^2\,\zeta_3\,\zeta_5-10\,\zeta_2\,\zeta_3\,\zeta_7-\frac{7}{2}\,\zeta_2\,\zeta_5^2-\frac{3}{5}\,\zeta_2^2\,\zeta_{3,5}-3\,\zeta_2\,\zeta_{3,7}\right.\\
&\left.-\frac{1}{12}\,\zeta_3^4-\frac{467}{108}\,\zeta_5\,\zeta_7+\frac{799}{72}\,\zeta_3\,\zeta_9+\frac{2665}{648}\,\zeta_{3,9}+\zeta_{1,1,4,6}\right\}\,\{\,[M_9,M_3]-3\,[M_7,M_5]\,\}+\ldots
\end{aligned}
$$

This form is bolstered by the algebraic structure of **motivic MZVs**

Schlotterer, Stieberger, `arXiv:1205.1516 [hep-th]`

Structure of α'–expansions

Illustrate the structure of the matrices P and M, *e.g.:* P_2 and M_3:

$$P_2 = \begin{pmatrix} -s_3 s_4 + s_1\ (s_3 - s_5) & s_{13}\ s_{24} \\ s_1\ s_3 & (s_1 + s_2)\ (s_2 + s_3) - s_4 s_5 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

with:

$$\begin{aligned} m_{11} &= s_3\ [\ -s_1\ (s_1 + 2s_2 + s_3) + s_3 s_4 + s_4^2\ ] + s_1 s_5\ (s_1 + s_5)\ , \\ m_{12} &= -s_{13}\ s_{24}\ (s_1 + s_2 + s_3 + s_4 + s_5)\ , \\ m_{21} &= s_1\ s_3\ [\ s_1 + s_2 + s_3 - 2\ (s_4 + s_5)\ ]\ , \\ m_{22} &= (s_2 + s_3)\ [\ (s_1 + s_2)(s_1 + s_3) - 2\ s_1 s_4\ ] - [\ 2s_1 s_3 - s_4^2 + 2s_2\ (s_3 + s_4)\ ] s_5 + s_4 s_5^2\ , \end{aligned}$$

and $s_i \equiv \alpha'(k_i + k_{i+1})^2$ subject to cyclic identification $k_{i+N} \equiv k_i$.

Structure of α'–expansions

More material to download from:

$\boxed{\text{http://mzv.mpp.mpg.de}}$

Broedel, Schlotterer, St.St., arXiv:1304.7267 [hep-th]

Structure of α'–expansions

Open string α'–expansions:

E.g. weight 8:

$$\boxed{\begin{aligned} F\,\big|_{\zeta_3\,\zeta_5} &= M_5\;M_3 \\ F\,\big|_{\zeta_{3,5}} &= \frac{1}{5}\;[M_5,M_3] \\ &\vdots \end{aligned}}$$

E.g. weight 10:

$$\boxed{\begin{aligned} F\,\big|_{\zeta_3\,\zeta_7} &= M_7\;M_3 \\ F\,\big|_{\zeta_{3,7}} &= \frac{1}{14}\;[M_7,M_3] \\ F\,\big|_{\zeta_5^2} &= \left(\frac{1}{2}\;M_5\;M_5+\frac{3}{14}\;[M_7,M_3]\;\right) \\ &\vdots \end{aligned}}$$

Remarks

- There is an **intrinsic form** in Q given by the commutator terms:

$$\zeta_{n_1,\ldots,n_r} \quad [M_{n_2},[M_{n_3},\ldots,[M_{n_r},M_{n_1}]]\ldots]$$

$\hookrightarrow$ Minimal depth representation with Euler sums

- Q may be written as an **exponential**:

$$Q = \exp\left\{\tfrac{1}{5}\,\zeta_{3,5}\,\,[M_5,M_3] + \left(\tfrac{3}{14}\,\zeta_5^2 + \tfrac{1}{14}\,\zeta_{3,7}\right)\,\,[M_7,M_3] + \ldots\right\}$$

- Forms of Q and M are **basis dependent** !

E.g.: $\zeta_{3,5} = -\frac{5}{2}\,\zeta_{2,6} + 5\,\zeta_3\,\zeta_5 - \frac{21}{25}\,\zeta_2^4$ changes form of Q.

- Consider **Hoffman basis**: *Every MZV is a* $\mathbf{Q}$*–linear combination of*

$$\{\ \zeta_{n_1,\ldots,n_r}\ |\ n_1,\ldots,n_r \in \{2,3\}\ \},\ \textit{e.g.:}\ \begin{pmatrix}\zeta_{3,2}\\ \zeta_{2,3}\end{pmatrix} = \begin{pmatrix}\frac{9}{2} & -2\\ -\frac{11}{2} & 3\end{pmatrix}\begin{pmatrix}\zeta_5\\ \zeta_2\,\zeta_3\end{pmatrix}$$

$\hookrightarrow$ Coproduct structure will remove all these ambiguities !

Aspects of motivic MZVs

Recall: $\zeta_{1,3,7} = -\zeta_{3,3,5} + \frac{703}{4}\,\zeta_{11} + \frac{1}{2}\,\zeta_5\,(\zeta_3)^2 - \frac{173}{2}\,\zeta_9\zeta_2 - \frac{47}{5}\,\zeta_7\,(\zeta_2)^2 - \frac{52}{35}\,\zeta_5\,(\zeta_2)^3 - \frac{6}{35}\,\zeta_3\,(\zeta_2)^4$

$\hookrightarrow$ How to determine coefficients $\in \mathbf{Q}$?

Task:

To explicitly describe the structure of the algebra $\mathcal{Z}$
MZVs are replaced by their symbols (or motivic MZVs)

Goncharov, Brown, ...

$$\boxed{\begin{array}{ccc} \zeta^m & \longrightarrow & \zeta \\ \mathcal{H} & \longrightarrow & \mathcal{Z} \end{array}}$$

This defines a commutative graded **Hopf algebra**: $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$

Aspects of motivic MZVs

For for $a_0, \ldots, a_{n+1} \in \{0,1\}$ define **motivic MZVs**:

$$\boxed{\zeta^m(n_1,\ldots,n_r) \;=\; (-1)^r \; I^m(0;\rho(n_1,\ldots,n_r);1)}$$

with period map: $per : \mathcal{H} \longrightarrow \mathbf{R}$

<u>Non–canonical isomorphism:</u> $\boxed{\mathcal{H} \simeq \mathcal{H}\big/\zeta_2^m\mathcal{H} \;\otimes_{\mathbf{Q}}\; \mathbf{Q}[\zeta_2^m]}$

<u>*Note:*</u> $\mathcal{H}_k \to \mathcal{Z}_k$ is surjective, i.e.: $\dim_{\mathbf{Q}}(\mathcal{Z}_k) \leq \dim_{\mathbf{Q}}(\mathcal{H}_k) = d_k$

Deligne, Goncharov, Brown

Work of F. Brown: *On the decomposition of motivic MZVs*:

To explicitly describe the structure of $\mathcal{H}$ introduce the (trivial) algebra–comodule:

introduce elements f_i: $$\boxed{\mathcal{U} = \mathbf{Q}\langle f_3, f_5, \ldots \rangle \ \otimes_{\mathbf{Q}} \ \mathbf{Q}[f_2]}$$

There exists a morphism ϕ of graded algebra–comodules:

$$\boxed{\phi : \ \mathcal{H} \longrightarrow \mathcal{U}}$$

normalized by: $\phi(\zeta_n^m) = f_n \quad , \quad n \geq 2$.

Note:

$$\boxed{\begin{array}{ccccc} \zeta & \longleftarrow & \zeta^m & \longrightarrow & \phi(\zeta^m) \\ \mathcal{Z} & \longleftarrow & \mathcal{H} & \longrightarrow & \mathcal{U} \end{array}}$$

It is believed: $\mathcal{Z}_n \simeq \mathcal{U}_n$ over $\mathbf{Q}$

Aspects of motivic MZVs

Map ϕ sends every motivic MZV $\xi \in \mathcal{H}_{N+1}$ to a non–commutative polynomial in the f_i's:

$$\boxed{\phi(\xi) = \sum_{3 \leq 2r+1 \leq N} f_{2r+1}\ \xi_{2r+1} \in \mathcal{U}_{N+1}}$$

expansion w.r.t. basis $\{f_{2r+1}\}$

coefficients $\xi_{2r+1} \in \mathcal{U}_{N-2r}$

computed from coproduct

Example: weight 10

$$B_{10} = \{\ \zeta^m_{3,7},\ \zeta^m_3\ \zeta^m_7,\ (\zeta^m_5)^2,\ \zeta^m_{3,5}\ \zeta^m_2,\ \zeta^m_3\ \zeta^m_5\ \zeta^m_2,\ (\zeta^m_3)^2\ (\zeta^m_2)^2,\ (\zeta^m_2)^5\ \}$$

Compute $\phi(B_{10})$:

$$\boxed{\begin{aligned}
\phi\left(\zeta^m_{3,7}\right) &= -14\ f_7 f_3 - 6\ f_5 f_5, \quad \phi\left(\zeta^m_3 \zeta^m_7\right) = f_3 \sqcup\!\sqcup f_7\ , \\
\phi\left((\zeta^m_5)^2\right) &= f_5 \sqcup\!\sqcup f_5, \quad \phi\left(\zeta^m_{3,5}\zeta^m_2\right) = -5\ f_5 f_3 f_2\ , \\
\phi\left(\zeta^m_3 \zeta^m_5 \zeta^m_2\right) &= f_3 \sqcup\!\sqcup f_5 f_2, \quad \phi\left((\zeta^m_3)^2 (\zeta^m_2)^2\right) = f_3 \sqcup\!\sqcup f_3 f_2^2 \\
\phi\left((\zeta^m_2)^5\right) &= f_2^5
\end{aligned}}$$

Decomposition of motivic MZVs

$\Longrightarrow$ Inverting the map ϕ gives the decomposition of $\zeta^m(n_1,\ldots,n_r)$ w.r.t. to a basis B_n of motivic MZVs.

Moving to motivic MZVs allows to compute the decomposition directly:

$$\boxed{\begin{aligned}\xi_{10} &= a_0\ (\zeta_2^m)^5 + a_1\ (\zeta_2^m)^2\ (\zeta_3^m)^2 + a_2\ \zeta_2^m\ \zeta_3^m\ \zeta_5^m + a_3\ (\zeta_5^m)^2\\ &+\ a_4\ \zeta_2^m\ \zeta_{3,5}^m + a_5\ \zeta_3^m\ \zeta_7^m + a_6\ \zeta_{3,7}^m\end{aligned}}$$

Operators acting on $\phi(\xi_{10})$:

$$\boxed{\begin{aligned}a_1 &= \tfrac{1}{2}\ c_2^2\ \partial_3^2,\ \ a_2 = c_2\ \partial_5\partial_3,\ \ a_3 = \tfrac{1}{2}\ \partial_5^2 + \tfrac{3}{14}\ [\partial_7,\partial_3]\\ a_4 &= \tfrac{1}{5}\ c_2\ [\partial_5,\partial_3],\ \ a_5 = \partial_7\partial_3,\ \ a_6 = \tfrac{1}{14}\ [\partial_7,\partial_3]\end{aligned}}$$

Decomposition of motivic MZVs and α'–expansion

Example: weight 11

$$\phi(\zeta^m_{3,3,5}) = -\frac{5}{2}\ f_5(f_3 \sqcup\!\sqcup f_3) + \frac{4}{7}\ f_5 f_2^3 - \frac{6}{5}\ f_7 f_2^2 - 45\ f_9 f_2\ ,$$

$$\vdots$$

Exact match in the coefficient and commutator structure

by identifying the

motivic derivation operators $\boldsymbol{\partial}$ and the **matrix operators $\boldsymbol{M}$**

and the **coefficient operator $\boldsymbol{c_2}$** with the **matrix operators$\boldsymbol{P_2}$**

$$\boxed{\begin{aligned} \partial_{2n+1} &\simeq M_{2n+1}\ , \\ c_2^k &\simeq P_{2k}\quad ,\qquad k \geq 1\ . \end{aligned}}$$

$\hookrightarrow$ Motivic multi zeta values encapsulate α'–expansion

Schlotterer, Stieberger, `arXiv:1205.1516 [hep-th]`

Motivic structure of the open superstring amplitude

E.g: $\phi\left(\ \zeta_3^m \zeta_5^m\ M_5 M_3 + \frac{1}{5}\ \zeta_{3,5}^m\ [M_5, M_3]\ \right) = f_3 f_5\ M_5 M_3 + f_5 f_3\ M_3 M_5$

$$\boxed{\phi(F^m) = \left(\sum_{k=1}^{\infty} f_2^k\ P_{2k}\right) \left(\sum_{p=0}^{\infty}\ \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} f_{i_1} f_{i_2} \ldots f_{i_p}\quad M_{i_p} \ldots M_{i_2} M_{i_1}\right)}$$

$\hookrightarrow$ Motivic period matrix F^m takes simple structures in terms of Hopf algebra !

Schlotterer, Stieberger, arXiv:1205.1516 [hep-th]

Remark: $\phi(F^m) = \left(\sum_{k=1}^{\infty} f_2^k\ P_{2k}\right)\ \left(1 - \sum_{k=1}^{\infty} f_{2k+1}\ M_{2k+1}\right)^{-1}$

Proof in Broedel, Schlotterer, St.St., Terasoma, to appear

Lie algebra structure of the α'–expansion

Recall: Commutator structure in Q: $[M_{n_2},[M_{n_3},\ldots,[M_{n_r},M_{n_1}]]$

Graded Lie algebra over $\mathbf{Q}$

$$\mathcal{L}=\bigoplus_{r\geq 1}\mathcal{L}_r\ ,$$

which is generated by the symbols M_{2n+1} of degree $2n+1$, with the Lie bracket $(M_i,M_j)\mapsto [M_i,M_j]$.

The algebra $\mathcal{L}$ is generated by the following elements:

$$M_3,\ M_5,\ M_7,\ [M_5,M_3],\ M_9,\ [M_7,M_3],$$

$$M_{11},\ [M_3,[M_5,M_3]],\ [M_9,M_3],\ [M_7,M_5],\ldots$$

Linearly independent elements in $\mathcal{L}_m$ and primitive MZVs

m	$\dim(\mathcal{L}_m)$	linearly independent elements at α'^m	irreducible MZVs
1	0	—	—
2	0	—	—
3	1	M_3	ζ_3
4	0	—	—
5	1	M_5	ζ_5
6	0	—	—
7	1	M_7	ζ_7
8	1	$[M_5, M_3]$	$\zeta_{3,5}$
9	1	M_9	ζ_9
10	1	$[M_7, M_3]$	$\zeta_{3,7}$
11	2	M_{11}, $[M_3, [M_5, M_3]]$	ζ_{11}, $\zeta_{3,3,5}$
12	2	$[M_9, M_3]$, $[M_7, M_5]$	$\zeta_{3,9}$, $\zeta_{1,1,4,6}$

m	δ_m	linearly independent elements at α'^m	irreducible MZVs
11	2	$M_{11},\ [M_3,[M_5,M_3]]$	$\zeta_{11},\ \zeta_{3,3,5}$
12	2	$[M_9,M_3],\ [M_7,M_5]$	$\zeta_{3,9}\ ,\zeta_{1,1,4,6}$
13	3	$M_{13},\ [M_3,[M_7,M_3]],\ [M_5,[M_5,M_3]]$	$\zeta_{13},\ \zeta_{3,3,7},\ \zeta_{3,5,5}$
14	3	$[M_{11},M_3],\ [M_9,M_5],\ [M_3,[M_3,[M_5,M_3]]]$	$\zeta_{3,11},\ \zeta_{5,9},\ \zeta_{3,3,3,5}$
15	4	$M_{15},$ $[M_3,[M_9,M_3]],\ [M_5,[M_7,M_3]],\ [M_7,[M_5,M_3]]$	ζ_{15} $\zeta_{5,3,7},\ \zeta_{3,3,9},\ \zeta_{1,1,3,4,6}$
16	5	$[M_{13},M_3],\ [M_{11},M_5],\ [M_9,M_7],$ $[M_3,[M_5,[M_5,M_3]]],\ [M_3,[M_3,[M_7,M_3]]]$	$\zeta_{3,13},\ \zeta_{5,11},\ \zeta_{1,1,6,8},$ $\zeta_{3,3,3,7},\ \zeta_{3,3,5,5}$
17	7	$M_{17},$ $[M_3,[M_3,[M_3,[M_5,M_3]]]],\ [M_7,[M_7,M_3]],\ [M_5,[M_7,M_5]],$ $[M_3,[M_{11},M_3]],\ [M_9,[M_5,M_3]],\ [M_5,[M_9,M_3]]$	$\zeta_{17},$ $\zeta_{3,3,3,3,5},\ \zeta_{1,1,3,6,6},\ \zeta_{5,5,7},$ $\zeta_{3,3,11},\ \zeta_{5,3,9},\ \zeta_{3,5,9}$
18	8	$[M_{15},M_3],\ [M_{13},M_5],\ [M_{11},M_7],$ $[M_5,[M_5,[M_5,M_3]]],\ [M_3,[M_3,[M_7,M_5]]],$ $[M_5,[M_3,[M_7,M_3]]],\ [M_3,[M_3,[M_9,M_3]]],\ [M_3,[M_5,[M_7,M_3]]]$	$\zeta_{3,15},\ \zeta_{5,13},\ \zeta_{1,1,6,10},$ $\zeta_{3,5,5,5},\ \zeta_{5,3,3,7},$ $\zeta_{3,3,3,9},\ \zeta_{3,5,3,7},\ \zeta_{1,1,3,3,4,6}$
19	11	$M_{19},$ $[M_3,[M_{13},M_3]],\ [M_7,[M_9,M_3]],\ [M_9,[M_7,M_3]],$ $[M_5,[M_{11},M_3]],\ [M_{11},[M_5,M_3]],\ [M_5,[M_9,M_5]],$ $[M_7,[M_7,M_5]],$ $[M_3,[M_3,[M_5,[M_5,M_3]]]],\ [M_5,[M_3,[M_3,[M_5,M_3]]]],$ $[M_3,[M_3,[M_3,[M_7,M_3]]]]$	$\zeta_{19},$ $\zeta_{3,3,13},\ \zeta_{7,3,9},\ \zeta_{1,1,3,6,8},$ $\zeta_{5,3,11},\ \zeta_{3,5,11},\ \zeta_{5,5,9},$ $\zeta_{1,1,5,6,6},$ $\zeta_{3,3,5,3,5},\ \zeta_{3,3,3,5,5},$ $\zeta_{3,3,3,3,7}$
20	13	$[M_{17},M_3],\ [M_{15},M_5],\ [M_{13},M_7],\ [M_{11},M_9],$ $[M_3,[M_3,[M_3,[M_3,[M_5,M_3]]]]],$ $[M_5,[M_5,[M_7,M_3]]],\ [M_3,[M_5,[M_7,M_5]]],\ [M_3,[M_3,[M_9,M_5]]],$ $[M_3,[M_7,[M_7,M_3]]],\ [M_3,[M_5,[M_9,M_3]]],\ [M_3,[M_3,[M_{11},M_3]]],$ $[M_5,[M_3,[M_7,M_5]]],\ [M_5,[M_3,[M_9,M_3]]]$	$\zeta_{7,13},\ \zeta_{5,15},\ \zeta_{3,17},\ \zeta_{1,1,8,10},$ $\zeta_{3,3,3,3,3,5},$ $\zeta_{5,5,3,7},\ \zeta_{3,5,5,7},\ \zeta_{5,3,3,9},$ $\zeta_{3,3,7,7},\ \zeta_{3,5,3,9},\ \zeta_{3,3,3,11},$ $\zeta_{1,1,3,3,4,8},\ \zeta_{1,1,5,3,4,6}$

$\delta_m := \dim(\mathcal{L}_m)$

A similar algebra $\boldsymbol{\mathcal{F}}$ has been studied by **Ihara** to relate the **Galois Lie algebra** $\mathcal{G}$ of the Galois group G to the more tractable object $\mathcal{F}$.

The **free graded Lie algebra $\boldsymbol{\mathcal{F}}$** over $\mathbf{Q}$ is freely generated by the symbols $\boldsymbol{\tau_{2n+1}}$ of degree $2n+1$, with the dimension dim$(\mathcal{F}_m)$.

Consider graded space of irreducible (primitive) MZVs:

$$\frac{\mathcal{Z}_{>0}}{\mathcal{Z}_{>0}\ \mathcal{Z}_{>0}}\ , \qquad \text{with:}\ \ \mathcal{Z}_{>0} = \bigoplus_{w>0} \mathcal{Z}_w$$

Goncharov:
$$\boxed{\frac{\mathcal{Z}_{>0}}{\mathcal{Z}_{>0}\ \mathcal{Z}_{>0}} \simeq \langle \zeta_2 \rangle_{\mathbf{Q}} \oplus \mathcal{F}}$$

<u>Observation:</u> $\dim(\mathcal{L}_m) = \dim(\mathcal{F}_m)$ *i.e.:* $M_{2n+1} \simeq \tau_{2n+1}$

This relates linearly independent elements $\mathcal{L}_m$ in α'–expansion to primitive MZVs

Linearly independent elements in $\mathcal{L}_m$ for $N=5$

E.g. $N=5$: For any nested commutator

$$\mathcal{Q}_{(r)} = [M_{n_2},[M_{n_3},\ldots,[M_{n_r},M_{n_1}]]\ldots]\ , \qquad r\geq 2:$$

$$\boxed{\begin{array}{lll} r_1+r_2\in 2\mathbf{Z}: & [\ \mathcal{Q}_{(r_1)},\tilde{\mathcal{Q}}_{(r_2)}\ ] & =0 \\ r_1+r_2\in 2\mathbf{Z}+1: & \{\ \mathcal{Q}_{(r_1)},\tilde{\mathcal{Q}}_{(r_2)}\ \} & =0 \end{array}}$$

As a consequence, e.g. $w=18$: $[M_3,[M_5,[M_7,M_3]]]=[M_5,[M_3,[M_7,M_3]]]$

General counting formula for $\dim(\mathcal{L}_m)$

in Broedel, Schlotterer, St.St., Terasoma, to appear

m	$\dim(\mathcal{F}_m)$	$\dim(\mathcal{L}_m)$	irreducible MZVs
18	8	7	7
19	11	11	11
20	13	11	11
21	17	16	16
22	21	16	16
23	28	25	...

$\mathcal{L}$ gives rise to an other algebra, which may be related to Galois Lie group

Linearly independent elements in $\mathcal{L}_m$ for $N = 5$

$$\begin{aligned}\dim(\mathcal{L}_{2r}) &= w^8 + w^{10} + 2w^{12} + 3w^{14} + 5w^{16} + 7w^{18} + 11w^{20} \\ &+ 16w^{22} + 24w^{24} + 34w^{26} 49w^{28} + 69w^{30} + 98w^{32} + \ldots\end{aligned}$$

$$\begin{aligned}\dim(\mathcal{L}_{2r+1}) &= t^{11} + 2t^{13} + 3t^{15} + 6t^{17} + 10t^{19} + 15t^{21} + 24t^{23} \\ &+ 36t^{25} + 54t^{27} + 80t^{29} + 116t^{31} + 166t^{33} + 237t^{35} \\ &+ 334t^{37} + 466t^{39} + 646t^{41} + 889t^{43} + 1213t^{45} + \ldots\end{aligned}$$

Broedel, Schlotterer, St.St., Terasoma, to appear

String theory in Mellin space

Taylor, St.St. arXiv:1303.1532

Mellin transformation: $M_f(s) := \int_0^\infty x^{s-1} f(x)$

Inverse Mellin transformation: $f(x) = (2\pi i)^{-1} \int_{-i\infty+c}^{+i\infty+c} ds\ x^{-s}\ M_f(s)$

Consider multiple inverse Mellin transformations:

$$\frac{1}{(2\pi i)^2} \int\limits_{-i\infty+c}^{+i\infty+c} ds \int\limits_{-i\infty+c}^{+i\infty+c} du\ x^{-s}\ y^{-u}\ \frac{\Gamma(s)\ \Gamma(u)}{\Gamma(s+u)} = \delta(1-x-y)\ \theta(1-x)\ \theta(1-y)$$

Kang: Extension of Mellin transformation to distributional setup

$$\frac{1}{s} + \frac{1}{u} - \zeta(2)\ (s+u) + \ldots$$

$$\simeq\ \theta(1-x)\ \theta(1-y)\ \Big\{\ \delta(1-x) + \delta(1-y) + \zeta(2)\ \Big[x\ \delta'(1-x) + y\ \delta'(1-y)\Big]\Big\} + \ldots$$

Introduce $m = \frac{1}{2}N(N-3)$ dihedral coordinates $u_{i,j}$:

$$(2\pi i)^{-m}\ \left(\prod_{(i,j)\in P}\ \int_{-i\infty+c}^{+i\infty+c} ds_{i,j}\ u_{i,j}^{-s_{i,j}}\right)\ B_N(\{s_{k,l}\},\{n_{k,l}\})$$

$$= \left(\prod_{(i,j)\in P}\ u_{i,j}^{n_{i,j}}\ \theta(1-u_{i,j})\right)\ \delta(\{u_{k,l}\})\ .$$

Polylogarithms and MZVs from expanding delta–functions !

- Superstring amplitudes:

  generalized Euler integrals, multiple polylogarithms and MZVs

- Structure of α'–expansions is encoded by
  decompositon of motivic multi zeta values

- coproduct maps the α'–expansion onto a
  non–commutative Hopf algebra

- Graded Lie algebra structure of the α'–expansion:
  interesting algebras related to Galois Lie algebra

- Multiple inverse Mellin transformations:
  polylogarithms and MZVs from expanding delta–functions !

Appendix: Hopf algebra

A Hopf algebra is an **algebra** $\mathcal{A}$ with **multiplication** $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, i.e. $\mu(x_1 \otimes x_2) = x_1 \cdot x_2$ and associativity. At the same time it is also a **coalgebra** with **coproduct** $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and coassociativity such that the product and coproduct are compatible: $\Delta(x_1 \cdot x_2) = \Delta(x_1) \cdot \Delta(x_2)$, with $x_1, x_2 \in \mathcal{A}$.

$$I^m(x; a_1, \ldots, a_r; y) \cdot I^m(x; a_{r+1}, \ldots, a_{r+s}; y) = \sum_{\sigma \in \Sigma(r,s)} I^m(x; a_{\sigma(1)}, \ldots, a_{\sigma(r+s)}; y)$$

with $\Sigma(r,s) = \{\sigma \in \Sigma(r+s) \mid \sigma^{-1}(1) < \ldots < \sigma^{-1}(r) \cap \sigma^{-1}(r+1) < \ldots < \sigma^{-1}(r+s)\}$

$$\begin{aligned} \Delta\, I^m(a_0; a_1, \ldots, a_n; a_{n+1}) \; &= \sum_{0=i_0<i_1<\ldots<i_k<i_{k+1}=n+1} I^m(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \\ &\otimes \prod_{p=0}^{k} I^m(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \;, \end{aligned}$$

with $0 \le k \le n$ and $a_i, x, y \in F$.