A spectral parameter for scattering amplitudes in N=4 SYM

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based on work with
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Introduction

Yang-Mills theories in 4d are special and relevant.

Huge progress in last 10 years in the study of maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) – dual to superstings in $AdS_5 \times S^5$ – a superconformal qft

1. Integrability in AdS/CFT:
   - Scaling dimensions alias string spectrum from Bethe equations [......]
     $\Rightarrow$ In principle complete solution of the spectral problem available
   - Finite size problem: Y-system and TBA.
2. Scattering amplitudes in maximally susy Yang-Mills:
   - On-shell recursion relations [Britto,Cachazo,Feng,Witten;...]
     $\Rightarrow$ all tree-level amplitudes known [Drummond,Henn]
   - Generalized unitarity methods [Bern,Dixon,Dunbar,Kosower;...]
   - Many high-loop/high-multiplicity results available
   - Integrand of planar n-point and l-loop amplitude principally known [Arkani-Hamed et al]
   - Dual superconformal symmetry and Yangian invariance [Drummond,Henn,Korchemsky,Sokatchev] [Drummond,Henn,JP]
   - Generalizable to 1-loop order [Sever,Vieira] [Beisert,Henn,JP,McLoughlin]

This talk

Shortcomings of present state of the art:

1. Integrability $\hat{=}$ Yangian symmetry. How to employ it for explicit results?
2. Loop level integrals following from constructed integrands undefined due to IR divergencies!

   Regularization prescriptions:
   - Dimensional regularization/reduction $D = 4 \to D = 4 + 2\epsilon$
   - "Built in" Massive/Higgs regulator: Vev for $\mathcal{N} = 4$ scalars [Alday,Henn,Schuster,JP]

   Both break superconformal symmetry or lead to deformations [Sever,Vieira] [Beisert,Henn,JP,McLoughlin]
3. Needs formalism to produce regulated integrands.

This talk:

1. Take inspiration from quantum inverse scattering method: Introduce Yang-Baxter equation and a spectral parameter z for scattering amplitudes $\to$ deformation of amplitudes.
2. Possibility to use z as regulator and stay in $D = 4$ and maintain superconformal symmetry?

Message of this talk: Establish z and use it instead of ϵ or m!

$\mathcal{N}=4$ super Yang Mills: Most symmetric interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: A_μ
  - 6 real Scalars: Φ_I
  - 4 Gluinos: $\Psi_{\alpha\, A}$, $\bar{\psi}^A_{\dot{\alpha}}$
- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{{g_{\mathrm{YM}}}^2} \int d^4x \, \mathrm{Tr} \Big[\tfrac{1}{4} F_{\mu\nu}^2 + \tfrac{1}{2} (D_\mu \Phi_I)^2 - \tfrac{1}{4} [\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \bar{\Psi}^A_{\dot{\alpha}} \sigma_\mu^{\dot{\alpha}\beta} \mathcal{D}^\mu \Psi_{\beta\, A} - \tfrac{i}{2} \Psi_{\alpha\, A} \sigma_I^{AB} \epsilon^{\alpha\beta} [\Phi^I, \Psi_{\beta\, B}] - \tfrac{i}{2} \bar{\Psi}_{\dot{\alpha}\, A} \sigma_I^{AB} \epsilon^{\dot{\alpha}\dot{\beta}} [\Phi^I, \bar{\Psi}_{\dot{\beta}\, B}] \Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}} = 0}$: Quantum Conformal Field Theory
- Two freely tunable parameters: N & 't Hooft-coupling $\lambda = {g_{\mathrm{YM}}}^2 N$
- Shall consider 't Hooft limit: $N \to \infty$ with λ fixed: Only planar Feynman diagrams survive
  $\rightarrow$ Suppression of instanton contributions

Superconformal symmetry

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{su}(2,2|4)$

  Poincaré: $p^{\alpha\dot{\alpha}} = p_\mu \, (\sigma^\mu)^{\dot{\alpha}\beta}, \quad m_{\alpha\beta}, \quad \bar{m}_{\dot{\alpha}\dot{\beta}}$

  Conformal: $k_{\alpha\dot{\alpha}}, \quad d \qquad c$: central charge

  R-symmetry: $r^A{}_B$

  Poincaré Susy: $q^{\alpha\,A}, \bar{q}^{\dot{\alpha}}_A$

  Conformal Susy: $s_{\alpha\,A}, \bar{s}^A_{\dot{\alpha}}$

- Algebra:

$$\{q^{\alpha\,A}, \bar{q}^{\dot{\alpha}}_B\} = \delta^A_B \, p^{\alpha\dot{\alpha}} \qquad \{s_{\alpha\,B}, \bar{s}^A_{\dot{\alpha}}\} = \delta^A_B \, k_{\alpha\dot{\alpha}}$$

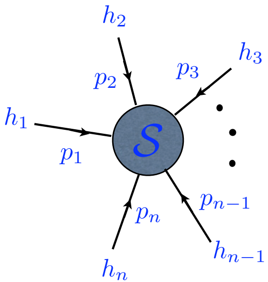
$$[p^{\alpha\dot{\alpha}}, s_{\beta\,A}] = \delta^\alpha_\beta \, \bar{q}^{\dot{\alpha}}_A \qquad [k_{\alpha\dot{\alpha}}, q^{\beta A}] = \delta^\beta_\alpha \, \bar{s}^A_{\dot{\alpha}}$$

$$[k_{\alpha\dot{\alpha}}, p^{\beta\dot{\beta}}] = \delta^\beta_\alpha \, \delta^{\dot{\beta}}_{\dot{\alpha}} \, d + \delta^{\dot{\beta}}_{\dot{\alpha}} \, m^\beta_\alpha + \delta^\beta_\alpha \bar{m}^\beta_{\dot{\beta}}$$

$$\{q^{\alpha\,A}, s_{\beta\,B}\} = m^\alpha{}_\beta \, \delta^A_B + r^A{}_B \, \delta^\alpha_\beta + \tfrac{1}{2}\delta^\alpha_\beta \, \delta^A_B \, (d + c)$$

Scattering amplitudes in $\mathcal{N} = 4$ SYM

- Consider n-particle scattering amplitude



- Planar amplitudes most conveniently expressed in color ordered formalism:

$$A_n(\{p_i, h_i, a_i\}) = \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2}\, \mathrm{tr}[t^{a_{\sigma_1}} \ldots t^{a_{\sigma_n}}]$$
$$\times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2 N)$$

$\mathcal{A}_n$: Color ordered amplitude. Color structure is stripped off.

Helicity of ith particle: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

Spinor helicity formalism

- Express momentum for massless particles via commuting spinors λ^α, $\tilde{\lambda}^{\dot{\alpha}}$:

$$p^{\alpha\dot{\alpha}} = (\sigma^\mu)^{\alpha\dot{\alpha}}\, p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} = |\tilde{\lambda}^{\dot{\alpha}}]\, \langle \lambda^\alpha|$$

$$\Leftrightarrow \quad p_\mu\, p^\mu = \det p^{\alpha\dot{\alpha}} = 0$$

- Choice of helicity determines polarization vector ε^μ of external gluon

$$h = -1 \qquad \varepsilon_-^{\alpha\dot{\alpha}} = \frac{\lambda^\alpha \tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda}\, \tilde{\mu}]} \qquad [\tilde{\lambda}\, \tilde{\mu}] := \epsilon^{\dot{\alpha}\dot{\beta}}\, \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}_{\dot{\beta}}$$

$$h = +1 \qquad \varepsilon_+^{\alpha\dot{\alpha}} = \frac{\mu^\alpha \tilde{\lambda}^{\dot{\alpha}}}{\langle \lambda\, \mu \rangle} \qquad \langle \lambda\, \mu \rangle := \epsilon_{\alpha\beta}\, \lambda^\alpha \mu^\beta$$

$\mu, \bar{\mu}$ arbitrary reference spinors.

- E.g. scalar products: $2\, p_1 \cdot p_2 = \langle \lambda_1\, \lambda_2 \rangle\, [\tilde{\lambda}_2\, \tilde{\lambda}_1] = \langle 12 \rangle\, [21]$
- Helicity assignments:

$$h(\lambda^\alpha) = -1/2 \qquad h(\tilde{\lambda}^{\dot{\alpha}}) = +1/2$$

Trees

Gluon Amplitudes and Helicity Classification

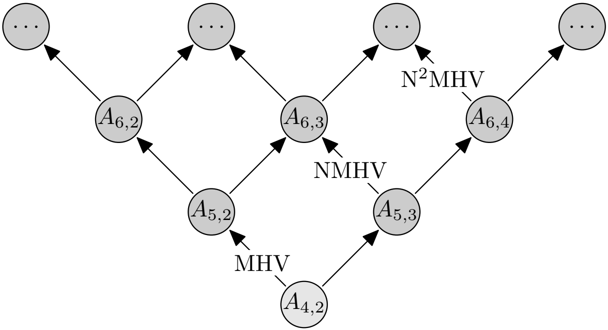
Classify gluon amplitudes by # of helicity flips

- By SUSY Ward identities: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ true to all loops
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^{(4)}(\sum_i p_i) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

[Parke, Taylor]

- Next-to-maximally helicity amplitudes (N^{k}MHV) have more involved structure!



$$A_{n,m} : g_+^{n-m} g_-^m$$

[Picture from T. McLoughlin]

On-shell superspace

- Augment λ_i^α and $\tilde{\lambda}_i^{\dot\alpha}$ by Graßmann-odd variables η_i^A $\quad A = 1,2,3,4$ [Nair]
- On-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}^{\dot\alpha}, \eta_i^A)$ with on-shell superfield:

$$\varphi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\,\epsilon_{ABCD}G^-(p)$$

- Superamplitudes: $\left\langle \varphi(\lambda_1,\tilde{\lambda}_1,\eta_1)\,\varphi(\lambda_2,\tilde{\lambda}_2,\eta_2)\ldots\varphi(\lambda_n,\tilde{\lambda}_n,\eta_n)\right\rangle$
  Packages all n-parton gluon$^\pm$-gluino$^{\pm 1/2}$-scalar amplitudes
- General form of **tree superamplitudes**:

$$\mathcal{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i\,\tilde{\lambda}_i)\,\delta^{(8)}(\sum_i \lambda_i\,\eta_i)}{\langle 12\rangle\,\langle 23\rangle\ldots\langle n1\rangle}\,\mathcal{P}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})$$

Conservation of super-momentum: $\quad \delta^{(8)}(\sum_i \lambda^\alpha\eta_i^A) = (\sum_i \lambda^\alpha\eta_i^A)^8$

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- Superamplitudes: $\left\langle \varphi(\lambda_1, \tilde{\lambda}_1, \eta_1)\,\varphi(\lambda_2, \tilde{\lambda}_2, \eta_2) \ldots \varphi(\lambda_n, \tilde{\lambda}_n, \eta_n) \right\rangle$

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- General form of **tree superamplitudes**:

$$\boxed{\mathcal{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i\,\tilde{\lambda}_i)\,\delta^{(8)}(\sum_i \lambda_i\,\eta_i)}{\langle 12\rangle\,\langle 23\rangle\ldots\langle n1\rangle}\,\mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})}$$

Conservation of super-momentum: $\quad \delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$

Superamplitudes and BCFW recursion

$$\mathcal{A}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\}) = \frac{\delta^{(4)}(\sum_i \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}})\,\delta^{(8)}(\sum_i \lambda_i^{\alpha}\eta_i^{A})}{\langle 1,2\rangle\,\langle 2,3\rangle\ldots\langle n,1\rangle}\,\mathcal{P}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})$$

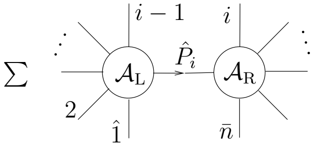
- η-expansion of $\mathcal{P}_n$ yields N^{k}MHV-classification of superamps as $h(\eta)=-1/2$

$$\mathcal{P}_n = 1 + \eta^4\,\mathcal{P}_n^{\text{NMHV}}(\lambda,\tilde{\lambda}) + \eta^8\,\mathcal{P}_n^{\text{NNMHV}}(\lambda,\tilde{\lambda}) + \ldots + \eta^{4n-8}\,\mathcal{P}_n^{\overline{\text{MHV}}}(\lambda,\tilde{\lambda})$$

- Efficient way of computing tree-level amplitudes via BCFW recursion

  [Britto,Cachazo,Feng,Witten]

$$A_n = \sum_i A_{i+1}^{h}\,\frac{1}{P_i^2}\,A_{n-i+1}^{-h}$$



  - N-point amplitudes are obtained recursively from lower-point amplitudes
  - All amplitudes are on-shell
- Reformulation of recursion relations in on-shell superspace → super BCFW

  [Bianchi,Elvang,Freedman; Brandhuber,Heslop,Travaglini; Arkani-Hamed,Cachazo,Kaplan]
- Super BCFW recursion is much simpler and can be solved analytically!

Symmetries

$\mathfrak{su}(2,2|4)$ invariance

- Superamplitude: $(i = 1, \ldots, n)$

$$\boxed{\mathcal{A}_n^{\text{tree}}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}})\, \delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle}\, \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})}$$

- Representation of $\mathfrak{su}(2,2|4)$ generators in **on-shell superspace**, e.g. [Witten]

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^\alpha\, \tilde{\lambda}_i^{\dot{\alpha}} \qquad q^{\alpha\, A} = \sum_{i=1}^{n} \lambda_i^\alpha\, \eta_i^A \qquad \Rightarrow \text{ obvious symmetries}$$

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha}\, \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \qquad s_{\alpha\, A} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha}\, \frac{\partial}{\partial \eta_i^A} \qquad \Rightarrow \text{ less obvious sym}$$

- Invariance: $\boxed{\{\, p, k, m, \bar{m}, d, r, q, \bar{q}, s, \bar{s}, c_i \,\} \circ \mathcal{A}_n^{\text{tree}}(\{\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A\}) = 0}$
- N.B.: **Local** invariance $\quad h_i\, \mathcal{A}_n = 1 \cdot \mathcal{A}_n$

Helicity operator: $\quad h_i = -\tfrac{1}{2}\, \lambda_i^\alpha\, \partial_{i\,\alpha} + \tfrac{1}{2}\, \tilde{\lambda}_i^{\dot{\alpha}}\, \partial_{i\,\dot{\alpha}} + \tfrac{1}{2}\, \eta_i^A\, \partial_{i\,A} = 1 - c_i$

Dual conformal and Yangian symmetries

- Superconformal + Dual superconformal algebra [Drummod,Henn,Korchemsky,Sokatchev]
  $= \text{Yangian } Y[\mathfrak{psu}(2,2|4)]$ algebra [Drummond, Henn, JP]

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(0)}$$ conventional superconformal symmetry

$$[J_a^{(0)}, J_b^{(1)}\} = f_{ab}{}^c\, J_c^{(1)}$$ from dual conformal symmetry

$$[J_a^{(1)}, J_b^{(1)}\} = f_{ab}{}^c\, J_c^{(2)} + g_{ab}(J^{(0)}, J^{(1)})$$

$\vdots$ and super Serre relations [Dolan,Nappi,Witten]

- Coproducts:

$$\Delta(J_a^{(0)}) = J_a^{(0)} \otimes 1 + 1 \otimes J_a^{(0)} \qquad \Delta(J_a^{(1)}) = J_a^{(1)} \otimes 1 + 1 \otimes J_a^{(1)} + f_a^{cb}\, J_b^{(0)} \otimes J_c^{(0)}$$

- Or explicitly

Local generators $J_a^{(0)} = \sum_{i=1}^{n} J_{a,i}^{(0)}$

Nonlocal generators $J_a^{(1)} = \sum_{i=1}^{n} \alpha_i\, J_{a,i}^{(0)} + f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)}\, J_{j,c}^{(0)}$

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

$$J_a^{(1)} = \sum_{i=1}^{n} \alpha_i \, J_{a,i}^{(0)} + f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)} \, J_{j,c}^{(0)}$$

- For tree-level superamplitudes $\alpha_i = 0$ (trivial evaluation representation)
- Explicit example: Bosonic invariance $\boxed{p^{(1)}_{\alpha\dot\alpha}\,\mathcal{A}_n = 0}$ with

$$\begin{aligned} p^{(1)}_{\alpha\dot\alpha} &= \tfrac{1}{2}(m + \bar m - d) \otimes p + \bar q \otimes q \\ &= \tfrac{1}{2} \sum_{i<j} (m_{i,\,\alpha}{}^{\gamma} \delta^{\dot\gamma}_{\dot\alpha} + \bar m_{i,\,\dot\alpha}{}^{\dot\gamma} \delta^{\gamma}_{\alpha} - d_i \, \delta^{\gamma}_{\alpha} \delta^{\dot\gamma}_{\dot\alpha}) \, p_{j,\,\gamma\dot\gamma} + \bar q_{i,\,\dot\alpha C} \, q^{C}_{j,\alpha} - (i \leftrightarrow j) \end{aligned}$$

- In fact $J_a^{(0)}$ and $p^{(1)}$ generate all of $Y[\mathfrak{psu}(2,2|4)]$

On-shell diagrams

Twistor representation and Graßmannian formulation

- Take Fourier transform $\lambda_i^\alpha \to \tilde{\mu}_i^\alpha$ to super-twistors $\mathcal{Z}^{\mathcal{A}} = (\tilde{\mu}^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta^A)$
- Graßmannian formulation of tree-amplitudes

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Cheung, Goncharov, Hodges, Kaplan, Postnikov, Trnka][Mason, Skinner]

$$\mathcal{A}_{n,k} = \oint_\Gamma \frac{\prod_{a=1}^{k}\prod_{i=k+1}^{n} dc_{ai}}{(1...k)(2...k{+}1)...(n...n{+}k{-}1)} \prod_{a=1}^{k} \delta^{4|4}\left(\mathcal{Z}_a^{\mathcal{A}} + \sum_{i=k+1}^{n} c_{ai}\mathcal{Z}_i^{\mathcal{A}}\right),$$

- Yields N^{k-2}MHV$_n$ amplitudes
- Γ (unspecified) set of contours for $c_{ai} \in \mathbb{C}$ which are entries of $(k \times n)$ matrix

$$C = \left(\begin{array}{c|cccc} & c_{1,k+1} & c_{1,k+2} & \cdots & c_{1,n} \\ \mathbf{1}_{k\times k} & \vdots & \vdots & \ddots & \vdots \\ & c_{k,k+1} & c_{k,k+2} & \cdots & c_{k,n} \end{array} \right)$$

with $k \times k$ sub determinants $(p \ldots p + k - 1)$

Form of Yangian generators

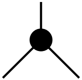
- Integrand of Graßmanian formulation of scattering amplitude is Yangian invariant up to total derivatives [Drummond, Ferro]
- Super-twistors yield natural homogenous order form of Yangian generators [Henn, Drummond, JP]

$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}$$

$$(J^{(1)})^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-1)^{|\mathcal{C}|} \left[\mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{C}}} \mathcal{Z}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{Z}_j^{\mathcal{B}}} - (i \leftrightarrow j) \right]$$

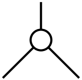
Three point amplitudes

- The "atoms": MHV_3 and $\overline{\text{MHV}}_3$ amplitudes



$$= \oint \frac{dc_1\, dc_2}{c_1 c_2}\, \delta^{(4|4)}(C_{(2,3)} \cdot \mathcal{Z}) \mathrel{\hat{=}} \frac{\delta^{(4)}(p^{\alpha\dot{\alpha}})\, \delta^{(8)}(q^{\alpha\, a})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

$$C_{(2,3)} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{pmatrix}$$



$$= \oint \frac{dc_1\, dc_2}{c_1 c_2}\, \delta^{(4|4)}(C_{(1,3)} \cdot \mathcal{Z}) \mathrel{\hat{=}} \frac{\delta^{(4)}(p^{\alpha\dot{\alpha}})\, \delta^{(4)}(\eta_1\, [23] + \text{cyclic})}{[12][23][31]}$$

$$C_{(1,3)} = \begin{pmatrix} 1 & c_1 & c_2 \end{pmatrix}$$

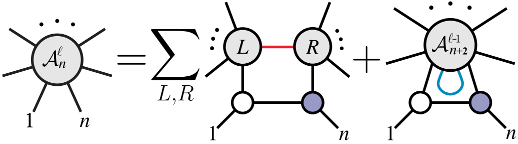
- Only exist for $\mathbb{R}^{2,2}$ signature or complex momenta:

$$\left.\begin{array}{llll} \text{MHV}_3: & \tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3 & \Rightarrow & [ij] = 0 \\ \overline{\text{MHV}}_3: & \lambda_1 \sim \lambda_2 \sim \lambda_3 & \Rightarrow & \langle ij\rangle = 0 \end{array}\right\} \Rightarrow \quad 2\, p_i \cdot p_j = \langle ij\rangle [ji] = 0$$

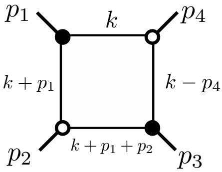
On shell diagrams

- On-shell diagram "all-loop" BCFW recursion relation:

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot,Trnka][Picture from arXiv:1212.5605]



- An example: MHV$_4$ amplitude:



$$= \int_{\mathbb{R}^{2,2}} d^4k\, \delta(k^2)\, \delta((k+p_1)^2)\, \delta((k+p_1+p_2)^2)\, \delta((k-p_4)^2)$$
$$w_\bullet(k, p_1, k+p_1)\, w_\circ(k+p_1, p_2, k+p_1+p_2)$$
$$w_\bullet(k+p_1+p_2, p_3, k-p_4)\, w_\circ(k-p_4, p_2, p_4, k)$$

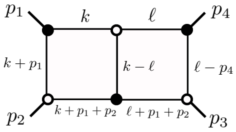
Integral completely localizes: $= \dfrac{\delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$

- We see: 1-loop on-shell = 0-loop off-shell

Off-shell loop amplitudes from on-shell diagrams

[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka]

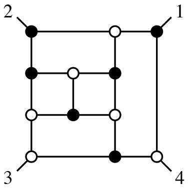
- Proceed from tree-level example: 2-loop on-shell $= \frac{1}{4}$-loop off-shell



$$= \underbrace{\int_{\mathbb{R}^{2,2}} d^4k \int_{\mathbb{R}^{2,2}} d^4l}_{\text{8-dim}} \underbrace{\delta(k^2)\ldots\delta(l^2)}_{\text{7-dim}} w_\bullet w_\circ \ldots w_\bullet w_\circ$$

- Construct full MHV$_4$ off-shell one-loop amplitude

$$\mathcal{A}^{\text{1-loop}}_{4,2} =$$



$$= \underbrace{\int d^4k_1 \ldots \int d^4k_5}_{\text{20-dim}} \underbrace{\delta(\ldots)\ldots\delta(\ldots)}_{\text{16-dim}} w_\bullet \ldots w_\circ$$

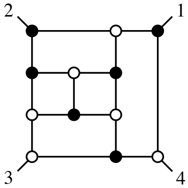
One integration remains: 5-loop on-shell $=$ 1-loop off-shell. Result:

$$= \mathcal{A}^{\text{tree-level}}_{4,2}\, st \int \frac{d^4q}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2}$$

One-loop MHV_4 from on-shell diagrams

- Famous one-loop box integral

$$\mathcal{A}_{4,2}^{\text{1-loop}} =$$



$$= \mathcal{A}_{4,2}^{\text{tree-level}} st \int \frac{d^4q}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2}$$

$$= \infty!!$$

- The box integral is IR divergent $\rightarrow$ Ad hoc dimensional (or mass) regularization

$$\mathcal{A}_{4,2}^{\text{(1-loop)}} = \mathcal{A}_{4,2}^{\text{tree-level}} \left(\frac{2}{\epsilon^2} \left(\left(\frac{s}{\mu^2} \right)^{-\epsilon} + \left(\frac{t}{\mu^2} \right)^{-\epsilon} \right) - \log^2 \left(\frac{s}{t} \right) - \frac{4\pi^2}{3} \right)$$

- On-shell diagram technique undefined in D-dimensions!
- Regulator breaks superconformal invariance!

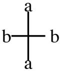
Spectral parameters for amplitudes

Quantum inverse scattering theory

- Yangian symmetry is "smoking gun" signature of integrability
- In quantum inverse scattering theory: Monodromy matrix

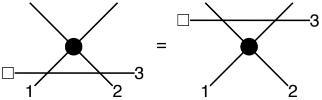
$$T(z) = \exp\left[-\frac{1}{z}\, t^a\, J_a^{(0)} + \frac{1}{z^2}\, t^a\, J_a^{(1)} + \dots\right] \qquad z\text{: spectral parameter}$$

$(t^a)_{mn}$ fundamental repr. matrices of underlying algebra (here $\mathfrak{su}(2,2|4)$)

- Intertwiner of $T(z)$: R-matrix $\quad R_{ab}(z) =$ 

$$R_{ab}(z_1 - z_2)\, T_{a_1}(z_1)\, T_{a_2}(z_2) = T_{a_1}(z_1)\, T_{a_2}(z_2)\, R_{ab}(z_1 - z_2)$$

- R-matrix satisfies Yang-Baxter equation:



$$\boxed{R_{12}(z_3)\, R_{13}(z_2)\, R_{23}(z_1) = R_{23}(z_1)\, R_{13}(z_2)\, R_{12}(z_3)} \qquad z_3 = z_2 - z_1$$

- $T_a(z) = L_{1a} L_{2a} \dots L_{Na}$ $\qquad$ with L_{ia}: Lax-operators $\hat{=}\ R_{ia}$

A spectral parameter for scattering amplitudes I

- R-matrix acting on tensor product of fundamental and super-twistor space representation $\mathcal{Z}^{\mathcal{A}}{}_i = (\tilde{\mu}_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^{A})$:

$$R_{i\,3}{}^{\mathcal{A}}{}_{\mathcal{B}}(z) = z\,\delta^{\mathcal{A}}_{\mathcal{B}} + (-)^{\mathcal{B}}\, J_i^{(0)\,\mathcal{A}}{}_{\mathcal{B}}\,, \qquad J_i^{(0)\,\mathcal{A}}{}_{\mathcal{B}} = \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}$$

- Seek solution of Yang-Baxter equation in this setup ($\begin{smallmatrix} 3 \,\hat{=}\, \text{fund.} \\ 1,\,2 \,\hat{=}\, \text{super-twistor} \end{smallmatrix}$ rep.)

$$R_{12}(z_3)\, R_{13}(z_2)\, R_{23}(z_1) = R_{23}(z_1)\, R_{13}(z_2)\, R_{12}(z_3)$$

with the kernel

$$R_{12}(z) \circ g(\mathcal{Z}_1, \mathcal{Z}_2) = \int d^{4|4}(\mathcal{Z}_3, \mathcal{Z}_4)\, \mathcal{R}(z; \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3 \mathcal{Z}_4)\, g(\mathcal{Z}_3, \mathcal{Z}_4)$$

and make the Graßmannian inspired ansatz

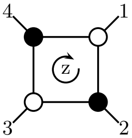
$$\mathcal{R}(z; \mathcal{Z}) = \oint \frac{dc_{13}\, dc_{14}\, dc_{23}\, dc_{24}}{c_{13}\, c_{24}\, (c_{13}c_{24} - c_{14}c_{23})}\, F(C_{(2,4)}; z)\, \delta^{(4|4)}(C_{(2,4)} \cdot \mathcal{Z})$$

A spectral parameter for scattering amplitudes II

- Yang-Baxter equation yields solution : $F(C_{(2,4)};z) = \left(\frac{c_{13}\,c_{24}}{(c_{13}c_{24}-c_{14}c_{23})}\right)^{z}$

  Assumptions:
  - Partial integrations w/o boundary terms
  - Physical helicities on all legs: $c_i \circ \mathcal{R}(z;\mathcal{Z}) = 0$
- Integrating we find the four-point R-matrix

$$\mathcal{R}_4(z) =$$



$$= \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(\frac{s}{t}\right)^{z}$$

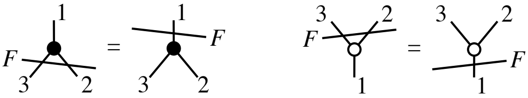
  A spectral parameter deformation of the MHV$_4$ amplitude!
- Symmetries:

$$J^{(0)\mathcal{A}}{}_{\mathcal{B}} \circ \mathcal{R}_4(z) = 0\,, \qquad J^{(1)\mathcal{A}}{}_{\mathcal{B}} \circ \mathcal{R}_4(z) = -z\sum_{i=1}^{4}(-1)^i\, J_i^{(0)\mathcal{A}}{}_{\mathcal{B}} \circ \mathcal{R}_4(z)$$

- Hence z-deformed level-one generator: with $\alpha_i = z\,(-1)^i$ leaves $\mathcal{R}_4(z)$ invariant

3-point R-matrices I

- Is there a similar deformation for the "atoms" of on-shell diagramatics?
- Postulate bootstrap equations:



e.g. $$\boxed{z_2\,(z_1+J_1)\,\mathcal{R}_\bullet(z_1,z_2)=\mathcal{R}_\bullet(z_1,z_2)\,(z_1+J_3)\,(z_2+J_2)}$$

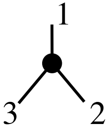
- Solution yields spectral parameter deformed 3-point vertices

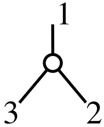
$$\mathcal{R}_\bullet(z_1,z_2)=\frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle^{1+z_3}\langle 23\rangle^{1+z_1}\langle 31\rangle^{1+z_2}}\hat{=}\oint\frac{dc_1dc_2}{c_1^{1+z_1}c_2^{1+z_2}}\delta^{4|4}(C_{(2,3)}\cdot\mathcal{Z})$$

$$\mathcal{R}_\circ(z_1,z_2)=\frac{\delta^4(P)\,\delta^4([12]\eta_3^A+\text{cyclic})}{[12]^{1+z_3}[23]^{1+z_1}[31]^{1+z_2}}\hat{=}\oint\frac{dc_1dc_2}{c_1^{1+z_1}c_2^{1+z_2}}\delta^{4|4}(C_{(1,3)}\cdot\mathcal{Z})$$

- Reminds of 3pt functions in CFT! (Rewrite $\langle 12\rangle^{1+z_3}=\langle 12\rangle^{h_1+h_2-h_3}$)

3-point R-matrices II

$\mathcal{R}_\bullet(z_1, z_2) =$ 

$\mathcal{R}_\circ(z_1, z_2) =$ 

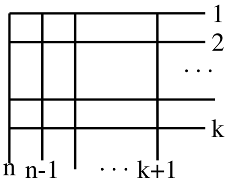
- Interpretation of spectral parameters:
  Central charges or deformed helicities of external legs

$$c_i \circ \mathcal{R}_\bullet = \frac{z_i}{2} \mathcal{R}_\bullet \,, \qquad c_i \circ \mathcal{R}_\circ = \frac{z_i}{2} \mathcal{R}_\circ \,,$$

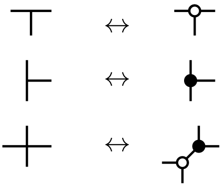
  recall $c_i = \frac{1}{2}(\lambda_i \partial_{\lambda_i} - \tilde{\lambda}_i \partial_{\tilde{\lambda}_i} - \eta_i \partial_{\eta_i}) + 1 = h_i - 1$
- Central charge conservation at each vertex: $z_1 + z_2 + z_3 = 0$
- May now build arbitrary z-deformed amplitudes via on-shell digrams $\hat{=}$ higher-point R-matrices with unphysical or physical helicities of external and internal legs.
- Connects and generalizes the on-shell diagram construction

Plabic diagrams and deformation of any scattering amplitude

- Solution to tree-level BCFW recursion:



Dictionary [Postnikov]



- Number of c's equals number of faces of the diagram above. Alternative parametrisation of the Graßmannian integrand

$$\mathcal{A}_{n,k}^{(\text{tree})} = \int \prod_{i=1}^{(n-k)k} \frac{df_i}{f_i} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} c_{ai}(f_1, \ldots, f_{(n-k)k})\mathcal{Z}_i\right)$$

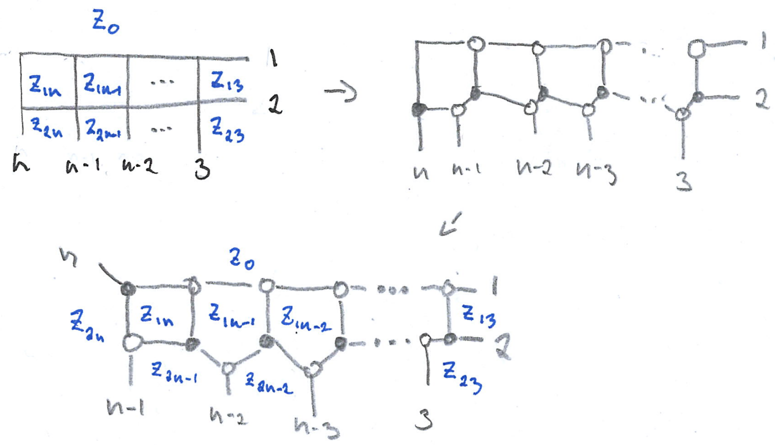
- Deformations of tree amplitudes can be easily written using face variables

$$\frac{df_i}{f_i} \longrightarrow \frac{df_i}{f_i^{1+z_i}}$$

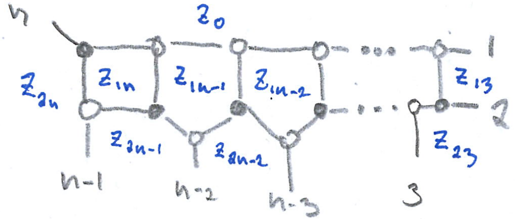
- Harmonic R-matrix $\mathcal{R}_{n,k}^{(\text{tree})}$ depends on $k\,(n-k)$ spectral face parameters

Spectral parameter deformed MHV$_n$ amplitude

- Relevant on-shell graph following plabic diagram dictionary



Spectral parameter deformed MHV$_n$ amplitude



- Deformed MHV$_n$ amplitude (setting $z_0 = 0$):

$$\mathcal{R}_{n,2} = \mathcal{A}_{n,2} \left(\frac{\langle 23\rangle}{\langle 13\rangle}\right)^{z_{13}} \prod_{i=4}^{n} \left(\frac{\langle i-1\, i\rangle\langle 1\, i-2\rangle}{\langle i-2\; i-1\rangle\langle 1\, i\rangle}\right)^{z_{1\,i}} \prod_{i=3}^{n} \left(\frac{\langle 1\, i\rangle}{\langle 1\, i-1\rangle}\right)^{z_{2\,i}}$$

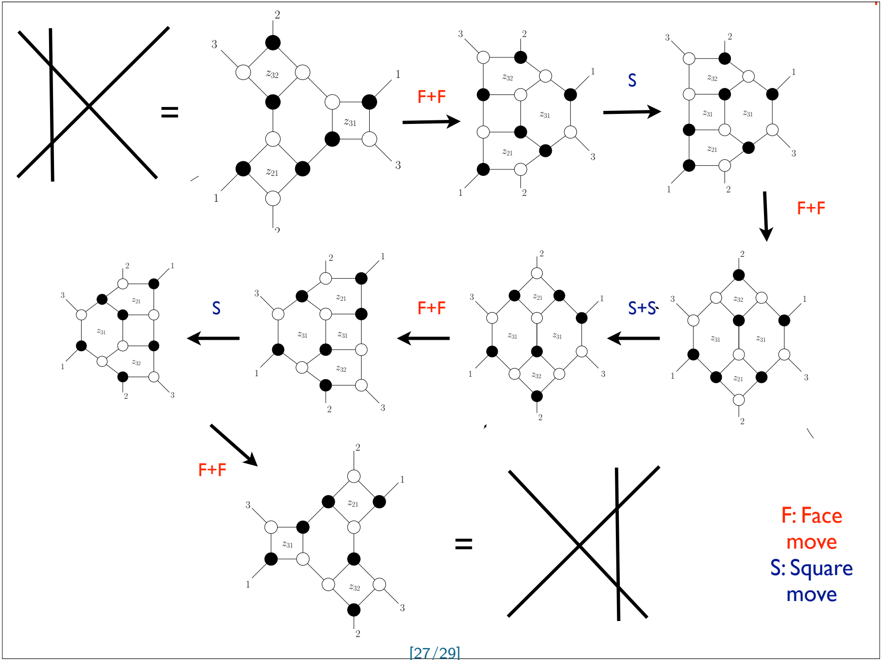
- Is superconformal and Yangian invariant $\boxed{J_a^{(1)} \circ \mathcal{R}_{n,2} = 0}$ provided

$$z_{2\,j} + z_{1\,j-1} - z_{2\,j-1} - z_{1\,j+1} = 0$$

  Reduces # of spectral parameters for $\mathcal{R}_{n,2}$ to $n-1$.

- Incidentally the same restrictions arise for "square move" of on-shell diagram to be valid for spectral deformed case

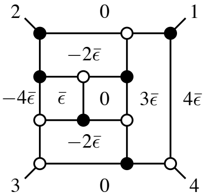
An atomistic view of the Yang-Baxter equation



Loop amplitudes and spectral regularization

- **Idea**: Use spectral parameter as regulator! Posibility to stay in 4d with a symmetry respecting regulator
- Concrete construction: BCFW recursion relation + unphysical helicities on external legs $h_i = \{4\bar{\epsilon}, 4\bar{\epsilon}, -4\bar{\epsilon}, -4\bar{\epsilon}\}$

$$\mathcal{R}_{4,2}^{\text{(1-loop)}}(\bar{\epsilon}) =$$



$$= \mathcal{A}_{4,2}^{\text{tree}}\, st \int d^4 q \frac{(\langle 34\rangle [21])^{-4\bar{\epsilon}}}{q^{2(1-\bar{\epsilon})}(q+p_1)^{2(1-\bar{\epsilon})}(q+p_1+p_2)^{2(1-\bar{\epsilon})}(q-p_4)^{2(1-\bar{\epsilon})}}$$

$$= \mathcal{A}_{4,2}^{\text{tree}} \left(\frac{\langle 12\rangle}{\langle 43\rangle}\right)^{4\bar{\epsilon}} \left[\frac{(s/t)^{2\bar{\epsilon}}}{\bar{\epsilon}^2} - \frac{1}{2}\log^2(s/t) - \frac{7}{6}\pi^2\right]$$

- Superconformal invariant: $\boxed{J_a^{(0)} \circ \mathcal{R}_{4,2}^{\text{(1-loop)}}(\bar{\epsilon}) = 0}$

Summary and Outlook

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- Introduced spectral parameters z_i for amplitudes solving Yang-Baxter and bootstrap equation in the Graßmannian language
- Mathematical interpretation of z_i: Central charges;
  Physical interpretation of z_i: Unquantized, complex helicities of particles
- Presented initial evidence for the use of z_i as a symmetry preserving regulator for the one-loop 4-point amplitude
- Presented deformed MHV$_n$ amplitude $\mathcal{R}_{n,2}$.
- Yangian invariance restricts # of spectral parameters

Outlook: Plenty of open questions

- Embarrassment of riches: What is the role of multiple spectral parameters?
- N^{k}MHV story, is there a s[ectral parameter deformed version of BCFW?
- Not all choices for z_i yield IR-finite one-loop amplitude. Non-physical helicities are needed for external legs. Organizing principle?
- Higher loops?

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