Chapter 1

Multiple Polylogarithms[1]

In this chapter we consider single- and multi-variable generalizations of the natural logarithm, called the polylogarithms and the multiple polylogarithms, respectively.

1.1 Chen's iterated integrals

We briefly recall the theory of Chen's iterated integrals in this section. It is an indispensable tool in the subsequent study of the analytic properties of multiple polylogarithms.

For $r > 1$ define inductively

$$\int_a^b f_1(t)\,dt \cdots f_r(t)\,dt = \int_a^b f_1(\tau)\,d\tau \cdots f_{r-1}(\tau)\left(\int_a^\tau f_r(t)dt\right)\,d\tau.$$

When $r = 0$ one sets the integral to be 1 as a convention. More generally, let $w_1, \dots, w_r$ be some 1-forms (repetition allowed) on a manifold M and let $\alpha : [0,1] \to M$ be a piecewise smooth path. Write $\alpha^* w_i = f_i(t)dt$ and define the iterated integral

$$\int_\alpha w_1 \cdots w_r := \int_0^1 f_1(t)\,dt \cdots f_r(t)dt.$$

Remark 1.1.1. Here our order of the 1-forms in the iteration is opposite to Chen's original order.

The following results are crucial in the application of the Chen's theory of iterated path integrals.

Lemma 1.1.2. *Let w_i $(i \geq 1)$ be $\mathbb{C}$-valued 1-forms on a manifold M.*

(i) *The value of $\int_\alpha w_1 \cdots w_r$ is independent of the parametrization of α.*

(ii) *If $\alpha, \beta : [0,1] \longrightarrow M$ are composable paths (i.e. $\alpha(1) = \beta(0)$), then*

$$\int_{\beta\alpha} w_1 \cdots w_r = \sum_{j=0}^r \int_\beta w_1 \cdots w_i \int_\alpha w_{i+1} \cdots w_r$$

where $\beta\alpha$ denotes the composition of α and β and we set $\int_\alpha \phi_1 \cdots \phi_m = 1$ if $m = 0$.

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(iii) *For every path* α,

$$\int_{\alpha^{-1}} w_1 \cdots w_r = (-1)^r \int_\alpha w_r \cdots w_1.$$

(iv) *For every path* α,

$$\int_\alpha w_1 \cdots w_r \int_\alpha w_{r+1} \cdots w_{r+s} = \sum_\sigma \int_\alpha w_{\sigma(1)} \cdots w_{\sigma(r+s)}$$

where σ *ranges over all shuffles of type* (r,s), *i.e., permutations* σ *of* $r+s$ *letters with* $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ *and* $\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$.

Proof. (i) can be derived from the theorem on [73, p. 361]. (ii) and (iii) are formulas (1.6.1) and (1.6.2) of [73] respectively. Ree [292] discovered the shuffle relation (iv) which appeared as (1.5.1) in [73]. □

Lemma 1.1.3. *If* $w_i^{(j)}$ *are closed 1-forms for* $1 \le i \le r$ *and* $1 \le j \le n$ *such that* $\sum_j w_1^{(j)} \wedge w_2^{(j)} = \sum_j w_2^{(j)} \wedge w_3^{(j)} = \cdots = \sum_j w_{r-1}^{(j)} \wedge w_r^{(j)} = 0$ *then* $\sum_j \int_\alpha w_1^{(j)} w_2^{(j)} \cdots w_r^{(j)}$ *only depends on the homotopy class of* α.

Proof. The case $j=1$ is proved on [73, p. 366]. The case $r=2$ can be found on [73, p. 368]. The general case follows from a similar argument. □

1.2 Classical polylogarithms

The Riemann zeta value $\zeta(n)$ for positive integer $n > 1$ can be regarded as a special value of the classical polylogarithm

$$Li_n(x) := \sum_{k=1}^\infty \frac{x^k}{k^n}, \qquad |x| \le 1$$

when $x = 1$. If $n = 1$ one sees easily that

$$Li_1(x) := \sum_{k=1}^\infty \frac{x^k}{k} = -\log(1-x), |x| < 1.$$

Notice that one can express $Li_n(x)$ using the following iterated integral

$$Li_n(x) = \int_0^x \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{1-t},$$

from which one obtains an analytic continuation of $Li_n(x)$ to $\mathbb{C}$ as a multi-valued function with singularities at 1. Since multi-valued functions are hard to grasp there have been several attempts to find their single-valued cousins. Lewin [234] defined the following polylogarithm of real variable $x < 1$:

$$L_n(x) = \sum_{j=0}^n \frac{(-\log|x|)^j}{j!} Li_{n-j}(x) + \frac{(-\log|x|)^{n-1}}{n!} \log|1-x|, \tag{1.1}$$

which can be extended to $x \in \mathbb{R}$ by setting $L_n(x) = (-1)^{n-1} L_n(1/x)$ for $x \ge 1$. He further conjectured that $L_n(x)$ should satisfy "clean" functional equations and proved this for those equations found by Kummer with $n<5$. Another version of single-valued polylogarithm is defined by Ramakrishnan [287, 351] for complex variable x with $|x|\le 1$

$$D_n(x)=\left\{\begin{array}{l}\mathrm{Re}\\ \mathrm{Im}\end{array}\right. \quad \sum_{j=0}^{n}\frac{(-\log|x|)^j}{j!}Li_{n-j}(x)-\frac{(-\log|x|)^n}{2\cdot n!} \quad \begin{array}{l}\text{if } n \text{ is odd;}\\ \text{if } n \text{ is even,}\end{array} \tag{1.2}$$

and then extend it to $\mathbb{C}$ by $D_n(x)=(-1)^{n-1}D_n(1/x)$ for $|x|\ge 1$. Notice that this function is real analytic. But it seems that Zagier's version of single-valued polylogarithms $\mathcal{L}_n$ in [353] behaves the most regularly. It is defined on $\mathbb{CP}^1$ for any positive integer $n\ge 2$ by

$$\mathcal{L}_n(x)=\left\{\begin{array}{l}\mathrm{Re}\\ \mathrm{Im}\end{array}\right. \quad \sum_{j=0}^{n}\frac{2^jB_j}{j!}\log^j|x|Li_{n-j}(x) \quad \begin{array}{l}\text{if } n \text{ is odd;}\\ \text{if } n \text{ is even,}\end{array} \tag{1.3}$$

where B_j is the j-th Bernoulli number and $Li_0(x)=1$. These functions are all single-valued and real-analytic by straight-forward monodromy computations. In particular $\mathcal{L}_n(0)=\mathcal{L}_n(\infty)=\mathcal{L}_{2n}(0)=0$, and $\mathcal{L}_{2n+1}(1)=\zeta(2n+1)$.

We turn now briefly to the discussion of polylogarithm functional equations. In high school students already know that $\log|x|+\log|y|=\log|xy|$. For dilogarithm one has the following fact (see [356, p. 9]): if $1/x+1/y+1/z=1$ then (see Exercise 1.1)

$$Li_2(x)+Li_2(y)+Li_2(z)=\frac{1}{2}\left[Li_2\Big(-\frac{xy}{z}\Big)+Li_2\Big(-\frac{yz}{x}\Big)+Li_2\Big(-\frac{zx}{y}\Big)\right]. \tag{1.4}$$

In fact this is equivalent to the so-called five-term relation discovered by Abel [1], which says, in terms of $\mathcal{L}_2(x)$, that for all $x,y\ne 0,1$ and $x\ne y$

$$\mathcal{L}_2(x)-\mathcal{L}_2(y)+\mathcal{L}_2\Big(\frac{y}{x}\Big)-\mathcal{L}_2\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big)+\mathcal{L}_2\Big(\frac{1-x}{1-y}\Big)=0. \tag{1.5}$$

It has a more symmetric form, see Exercise (1.2).

Similar result holds for trilogarithm $\mathcal{L}_3(x)$: for all x,y,z

$$\begin{aligned}\mathcal{L}_3(-xyz)+\bigoplus_{\mathrm{Cyc}(x,y,z)}\Big\{&\mathcal{L}_3(zx-x+1)+\mathcal{L}_3\big(\frac{zx-x+1}{zx}\big)-\mathcal{L}_3\big(\frac{zx-x+1}{z}\big)\\ &+\mathcal{L}_3\big(\frac{x(yz-z+1)}{-(zx-x+1)}\big)+\mathcal{L}_3\big(\frac{yz-z+1}{y(zx-x+1)}\big)+\mathcal{L}_3(z)-\mathcal{L}_3\big(\frac{yz-z+1}{yz(zx-x+1)}\big)\Big\}=3\mathcal{L}_3(1).\end{aligned} \tag{1.6}$$

Here $\bigoplus_{\mathrm{Cyc}(x,y,z)}f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y)$. This was discovered first by Goncharov [142] when he proved the following Zagier's Conjecture for $\zeta_F(3)$ where $\zeta_F(s)$ is the Dedekind zeta function over a number field F. Zagier proved his conjecture for $\zeta_F(2)$ in [350].

Conjecture 1.2.1. (Zagier's Conjecture) *Let F be a number field and let $\sigma_1,\dots,\sigma_{r_1}:F\to\mathbb{R}$ be the real embeddings and $\sigma_{1+r_1}=\overline{\sigma_{r_1+1+r_2}},\dots,\sigma_{r_1+r_2}=\overline{\sigma_{r_1+2r_2}}:F\to\mathbb{C}$ be the complex embeddings. For integer $n\ge 2$ let $r(n)=r_2$ if n is odd, $r(n)=r_1+r_2$ if n is even. Then there exists $y_1,\dots,y_{r(n+1)}\in\mathbb{Q}[F\setminus\{0,1\}]$ such that*

$$\zeta_F(n)=\pi^{nr(n)}D_F^{-1/2}\det\Big[\mathcal{L}_n\big(\sigma_i(y_j)\big)\Big]_{1\le i,j\le r(n+1)}.$$

where D_F is the discriminant of F.

1.3 Multiple polylogarithms

Multiple polylogarithms are iterated generalizations of polylogarithms to the multi-variable situation.

Definition 1.3.1. Let d, $n_1, \dots, n_d$ be positive integers, and let $x_1, \dots, x_d$ be complex variables such that $|\prod_{j=1}^{i} x_j| < 1$ for all $i = 1, \dots, d$. The multiple polylogarithm of depth d is defined by

$$Li_{n_1,\dots,n_d}(x_1, \dots, x_d) = \sum_{k_1>k_2>\cdots>k_d\geq 1} \frac{x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}}{k_1^{n_1} k_2^{n_2} \cdots k_d^{n_d}}. \tag{1.7}$$

As in the definition of multiple zeta functions the order of indices in (1.7) is sometimes reversed in research papers, even by the author himself. Note also that occasionally the term "multiple polylogarithm" means the single variable function $Li_{n_1,n_2\dots,n_d}(t, 1, 1, \dots, 1)$ in the literature, which will in fact be needed in this book later (see § 1.11).

Now by setting $a_i = \prod_{j=1}^{i} x_j^{-1}$ it is not hard to see (Exercise 1.3)

$$Li_{n_1,\dots,n_d}(x_1, \dots, x_d) = \int_0^1 \left(\frac{dt}{t}\right)^{n_1-1} \frac{dt}{a_1 - t} \cdots \left(\frac{dt}{t}\right)^{n_d-1} \frac{dt}{a_d - t}. \tag{1.8}$$

It is an iterated path integral in the sense of Chen [73, 74] whose path lies in $\mathbb{C}$. One thus can easily enlarge its domain of definition to some open subset of $\mathbb{C}^n$. However, it is difficult to study the monodromy of the multiple polylogarithms by this integral expression.

1.4 Analytic continuation of multiple logarithms

In this section we describe a method of defining the analytic continuation of multiple logarithms by Chen's iterated integrals. This can be generalized to arbitrary multiple polylogarithms (see [370]) but the notation is too cumbersome so we leave it to the interested reader.

For any positive integer n we set $\mathfrak{L}_n(\mathbf{x}) = Li_{\underbrace{1,\dots,1}_{n\text{ times}}}(\mathbf{x})$, define the index set

$$\mathfrak{S}_n = \{0, 1\}^n = \{\mathbf{i} = (i_1, \dots, i_n) : \ i_t = 0 \text{ or } 1 \ \forall t = 1, \dots, n\}, \tag{1.9}$$

and write $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{1} = (1, \dots, 1)$. The weight function on the indices $|(i_1, \dots, i_n)| = \sum_{t=1}^{n} i_t$ is the number of nonzero components.

Lemma 1.4.1. *Let $\mathbf{x} = (x_1, \dots, x_n)$ be an ordered set of complex variables. Let*

$$\mathbf{x}_j = (x_1, \dots, x_{j-1}, x_j x_{j+1}, x_{j+2}, \dots, x_n)$$

for $j = 1, \dots, n-1$ and $\mathbf{x}_n = (x_1, \dots, x_{n-1})$. Then the total differential

$$d\mathfrak{L}_n(\mathbf{x}) = \sum_{j=1}^{n} d_j \mathfrak{L}_n(\mathbf{x}) = \sum_{j=1}^{n} \Big\{ \mathfrak{L}_{n-1}(\mathbf{x}_j) dx_j/(1 - x_j) + \mathfrak{L}_{n-1}(\mathbf{x}_{j-1}) dx_j/x_j(x_j - 1) \Big\} \tag{1.10}$$

$$= \sum_{j=1}^{n} \mathfrak{L}_{n-1}(\mathbf{x}_j) d \log \left(\frac{1 - x_{j+1}^{-1}}{1 - x_j} \right), \tag{1.11}$$

where $x_{n+1} = \infty$, $\mathfrak{L}_0(\mathbf{x}_1) = 1$ and when $t = 1$ the second term in the sum of (1.10) does not appear.

Proof. Suppose $j > 1$ (the case $j = 1$ is simpler). Then by definition

$$\begin{aligned} d_j\mathfrak{L}_n(\mathbf{x}) =& \sum_{k_1>\cdots>\widehat{k_j}>\cdots>k_{j-1}} \frac{x_1^{k_1}\cdots\widehat{x_j^{k_j}}\cdots x_d^{k_d}}{k_1\cdots\widehat{k_j}\cdots k_d} \sum_{k_j=k_{j+1}+1}^{k_{j-1}-1} x_j^{k_j-1}\,dx_j \\ =& \sum_{k_1>\cdots>\widehat{k_j}>\cdots>k_{j-1}} \frac{x_1^{k_1}\cdots\widehat{x_j^{k_j}}\cdots x_d^{k_d}}{k_1\cdots\widehat{k_j}\cdots k_d}\cdot\frac{x_j^{k_{j-1}-1}-x_j^{k_{j+1}}}{x_j-1}\,dx_j \\ =&\mathfrak{L}_{n-1}(\mathbf{x}_j)dx_j/(1-x_j)+\mathfrak{L}_{n-1}(\mathbf{x}_{j-1})dx_j/x_j(x_j-1), \end{aligned}$$

which proves (1.10). The second equation (1.11) now follows immediately (Exercise 1.4). □

Suppose $\mathbf{i} = (i_1, \dots, i_n)$ has weight k and $i_{\tau_1} = \cdots = i_{\tau_k} = 1$. We define

$$\mathbf{x}(\mathbf{i}) = \mathbf{y} = (y_1, \dots, y_k), \text{ where } y_m = \prod_{\alpha=1+\tau_{m-1}}^{\tau_m} x_\alpha, \quad 1 \le m \le k \tag{1.12}$$

with $\tau_0 = 0$. Set $w_0(\mathbf{x}) = 0$ and

$$w_t(\mathbf{x}) := d\log\Big(\frac{1-x_{t+1}^{-1}}{1-x_t}\Big), \qquad \text{for } \ 1 \le t \le n.$$

We now define a partial order $\prec$ on $\mathfrak{S}_n$ as follows: Given $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{j} = (j_1, \dots, j_n)$, $\mathbf{j} \prec \mathbf{i}$ (or, equivalently, $\mathbf{i} \succ \mathbf{j}$) if $j_t \le i_t$ for every $1 \le t \le n$ and $j_s < i_s$ for some s.

Theorem 1.4.2. *Let $\mathbf{x} = (x_1, \dots, x_n)$ be an ordered set of complex variables. The multiple logarithm $\mathfrak{L}_n(\mathbf{x})$ is a multi-valued holomorphic function on*

$$S_n' = \mathbb{C}^n \setminus \Bigg\{(x_1, \dots, x_n) : \prod_{1\le j\le n}(1-x_j) \prod_{1\le j<k\le n}\big(1-x_j\cdots x_k\big) = 0\Bigg\} \tag{1.13}$$

and can be expressed by

$$\mathfrak{L}_n(\mathbf{x}) = \sum_{\substack{\mathbf{0}\ne\mathbf{j}_1\prec\cdots\prec\mathbf{j}_n,\\ \mathbf{j}_1,\dots,\mathbf{j}_n\in\mathfrak{S}_n}} \int_{\mathbf{0}}^{\mathbf{x}} w_{\mathbf{j}_n-\mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n))\cdots w_{\mathbf{j}_2-\mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2))w_1(\mathbf{x}(\mathbf{j}_1)), \tag{1.14}$$

where the subtraction $- : \mathfrak{S}_n \to \mathbb{Z}$ is a binary operation such that

$$(i_1, \dots, i_n) - (j_1, \dots, j_n) = \begin{cases} s, & \textit{if } i_s = j_s + 1 \textit{ and } i_\ell = j_\ell \quad \forall \ell \ne s; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The proof is straight-forward so we leave it as an exercise. □

It is clear that every nonzero term in the sum of (1.14) is provided by some vector index $(\mathbf{j}_1, \dots, \mathbf{j}_n) \in \mathfrak{S}_n^n$ with $|\mathbf{j}_\ell| = \ell$ for all $\ell = 1, \dots, n$, which we call *admissible* for convenience.

Example 1.4.3. When $n = 1$,

$$Li_1(x) = \int_0^x d\log\Big(\frac{1}{1-x}\Big) = -\log(1-x).$$

When $n = 2$, $\mathfrak{S}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$ and there are two admissible elements in $\mathfrak{S}_2^2$: $((1,0),(1,1))$ and $((0,1),(1,1))$. Let $\mathbf{x} = (x, y)$ then

$$\mathbf{x}(1,0) = x, \quad \mathbf{x}(0,1) = xy, \quad \mathbf{x}(1,1) = (x, y).$$

Thus

$$\begin{aligned} Li_{1,1}(x,y) &= \int_{\mathbf{0}}^{\mathbf{x}} w_2(\mathbf{x})w_1(\mathbf{x}(1,0)) + w_1(\mathbf{x})w_1(\mathbf{x}(0,1)) \\ &= \int_{(0,0)}^{(x,y)} \frac{dy}{1-y}\frac{dx}{1-x} + \left(\frac{dx}{1-x} + \frac{dy}{y(y-1)}\right)\frac{d(xy)}{1-xy}. \end{aligned}$$

The iterated path integrals are understood in the sense of Chen. For example, if $\alpha : [0,1] \to \mathbb{C}^2$ is the path from $(0,0)$ to (x_0,y_0) then

$$\int_{(0,0)}^{(x_0,y_0)} \frac{dy}{1-y}\frac{dx}{1-x} = \int_0^1 \left(\int_0^t \frac{d\,x\circ\alpha(s)}{1-x\circ\alpha(s)}\right)\frac{d\,y\circ\alpha(t)}{1-y\circ\alpha(t)},$$

where $x\circ\alpha(s)$ (resp. $y\circ\alpha(s)$) is the x- (resp. y-) coordinate of $\alpha(s)$. When $n=3$ let $\mathbf{x}=(x,y,z)$. Then

$$\begin{aligned} \mathbf{x}(1,0,0) =& x, \quad \mathbf{x}(0,1,0) = xy, \quad \mathbf{x}(0,0,1) = xyz, \quad \mathbf{x}(1,1,0) = (x,y), \\ \mathbf{x}(1,0,1) =& (x,yz), \quad \mathbf{x}(0,1,1) = (xy,z), \quad \mathbf{x}(1,1,1) = (x,y,z). \end{aligned}$$

Thus

$$\begin{aligned} Li_{1,1,1}(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} & \frac{dz}{1-z}\frac{dy}{1-y}\frac{dx}{1-x} + \frac{dz}{1-z}\left(\frac{dx}{1-x}+\frac{dy}{y(y-1)}\right)\frac{d(xy)}{1-xy} \\ &+ \left(\frac{dx}{1-x}+\frac{dy}{y(y-1)}\right)\frac{dz}{1-z}\frac{d(xy)}{1-xy} + \left(\frac{dy}{1-y}+\frac{dz}{z(z-1)}\right)\frac{d(yz)}{1-yz}\frac{dx}{1-x} \\ &+ \left(\frac{dy}{1-y}+\frac{dz}{z(z-1)}\right)\left(\frac{dx}{1-x}+\frac{d(yz)}{yz(yz-1)}\right)\frac{d(xyz)}{1-xyz} \\ &+ \left(\frac{dx}{1-x}+\frac{dy}{y(y-1)}\right)\left(\frac{d(xy)}{1-xy}+\frac{dz}{z(z-1)}\right)\frac{d(xyz)}{1-xyz}. \end{aligned}$$

1.5 Monodromy of multiple logarithms

The main goal of this section is to define the variation matrix associated with the multiple logarithms by utilizing the vector index set $\mathfrak{S}$ as defined in (1.9), together with the partial order $\prec$.

Write $\mathbf{x}=(x_1,\dots,x_n)$ as before. We know the n-tuple logarithm $\mathfrak{L}_n(\mathbf{x})$ is related to multiple logarithms of lower weights. This can be seen easily, for instance, when we take the derivatives as given by Lemma 1.4.1.

Definition 1.5.1. Suppose $\mathbf{i}\in\mathfrak{S}_n$ has weight k. We define the $\mathbf{i}$-th *retraction* map $\rho_{\mathbf{i}} : \mathfrak{S}_n \to \mathfrak{S}_k$ as follows.

(1) If $\mathbf{i} \not\succ \mathbf{j}$ then $\rho_{\mathbf{i}}(\mathbf{j}) = (0,\dots,0)$

(2) Suppose $\mathbf{i}\succ\mathbf{j}$ and $\mathbf{j}$ has weight l. Suppose further that the 1's occur at the positions $\tau_1,\dots,\tau_k$ in $\mathbf{i}$ and $t_1,\dots,t_l$ in $\mathbf{j}$, respectively. Then we can put $t_r=\tau_{\alpha_r}$ for $1\le r\le l$. Set the entry of $\rho_{\mathbf{i}}(\mathbf{j})$ to be 1 if it is at the α_r-th ($1\le r\le l$) component and 0 otherwise.

For instance $\rho_{(1101)}\big((1100)\big) = (110) \in \mathfrak{S}_3$ and $\rho_{(1101)}\big((0100)\big) = (010)$. In general, $\rho_{\mathbf{i}}(\mathbf{i}) = \mathbf{1}_k$ for all $\mathbf{i}$ with $|\mathbf{i}|=k$.

We are now ready to define the entries of the $2^n\times 2^n$ variation matrix $\mathcal{M}_{[n]}(\mathbf{x}) := \mathcal{M}_{\underbrace{1,\dots,1}_{n\text{ times}}}(\mathbf{x})$ associated with the n-tuple logarithm $\mathfrak{L}_n(\mathbf{x})$.

Definition 1.5.2. Fix a point $\mathbf{x} \in S_n$ where

$$S_n = \mathbb{C}^n \setminus \left\{ (x_1, \dots, x_n) : \prod_{1\le j\le n} x_j(1-x_j) \prod_{1\le j<k\le n} \bigl(1 - x_j \cdots x_k\bigr) = 0 \right\} \tag{1.15}$$

For $1 \le s \le n$ write $a_s = a_s(\mathbf{x}) =: (x_1 \cdots x_s)^{-1}$ and

$$\theta_s = \theta_s(\mathbf{x}) = \frac{dt}{t - a_s} = \frac{dt}{t - a_s(\mathbf{x})}.$$

(1) If $\mathbf{j} \not\prec \mathbf{i}$ then we define the $(\mathbf{i}, \mathbf{j})$-th entry of $\mathcal{M}_{[n]}(\mathbf{x})$ to be 0.

(2) Suppose $\mathbf{i} \succ \mathbf{j}$ as given in Definition 1.5.1(2). Recall from equation (1.12) we have

$$\mathbf{x}(\mathbf{i}) = \mathbf{y} = (y_1, \dots, y_k), \qquad y_m = \prod_{\alpha=1+\tau_{m-1}}^{\tau_m} x_\alpha = \frac{a_{\tau_{m-1}}(\mathbf{x})}{a_{\tau_m}(\mathbf{x})}, \quad 1 \le m \le k.$$

with $\tau_0 = 0$, $a_0(\mathbf{x}) = 1$. Set $t_0 = \alpha_0 = 0$, $t_{l+1} = n+1$, $\alpha_{l+1} = k+1$, $a_{n+1}(\mathbf{x}) = a_{k+1}(\mathbf{y}) = 0$. Define the $(\mathbf{i}, \mathbf{j})$-th entry of $\mathcal{M}_{[n]}(\mathbf{x})$ as $(2\pi i)^l E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ where

$$\begin{aligned} E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{y}) :=& (-1)^{k-l} \prod_{r=0}^{l} \int_{a_{\alpha_{r+1}}(\mathbf{y})}^{a_{\alpha_r}(\mathbf{y})} \theta_{\alpha_r+1}(\mathbf{y}) \cdots \theta_{\alpha_{r+1}-1}(\mathbf{y}) \\ =& (-1)^{k-l} \prod_{r=0}^{l} \int_{p_r} \theta_{\tau_{\alpha_r+1}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) \end{aligned} \tag{1.16}$$

Here the $l+1$ paths $p_0, \dots, p_l$ for the $l+1$ integrals are *independent of* $\mathbf{i}$ where p_r is any fixed contractible path from $a_{t_{r+1}}(\mathbf{x})$ to $a_{t_r}(\mathbf{x})$ in the punctured complex plane $\mathbb{C} \setminus \bigcup_{t_r<s<t_{r+1}} \{a_s(\mathbf{x})\}$, and the integral $\int_{p_r} = 1$ if $\alpha_r + 1 = \alpha_{r+1}$. We get the second equality by observing that

$$a_m(\mathbf{y}) = (y_1 \cdots y_m)^{-1} = a_{\tau_m}(\mathbf{x}) \Longrightarrow a_{\alpha_r}(\mathbf{y}) = a_{\tau_{\alpha_r}}(\mathbf{x}) = a_{t_r}(\mathbf{x}).$$

Proposition 1.5.3. *Suppose* $\mathbf{i}$ *and* $\mathbf{j}$ *are given as in Definition* 1.5.2(2)*. As multi-valued functions*

$$E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \prod_{r=0}^{l} \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1} \left(\frac{a_{t_r}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}, \dots, \frac{a_{\tau_{\alpha_{r+1}-2}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})} \right) \tag{1.17}$$

$$\begin{aligned} =& \mathfrak{L}_{k-\alpha_l}\bigl(x_{1+t_l} \cdots x_{\tau_{\alpha_l+1}}, \dots, x_{1+\tau_{k-1}} \cdots x_{\tau_k}\bigr) \cdot \\ & \cdot \prod_{r=0}^{l-1} \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1} \left(\frac{1 - x_{1+t_r} \cdots x_{t_{r+1}}}{1 - x_{1+\tau_{\alpha_r+1}} \cdots x_{t_{r+1}}}, \dots, \frac{1 - x_{1+\tau_{\alpha_{r+1}-2}} \cdots x_{t_{r+1}}}{1 - x_{1+\tau_{\alpha_{r+1}-1}} \cdots x_{t_{r+1}}} \right). \end{aligned} \tag{1.18}$$

Here $\mathfrak{L}_0 = 1$ *and* $a_0 = 0$.

Proof. By direct and simple calculation using substitution we get

$$\begin{aligned} (-1)^{\alpha_{r+1}-\alpha_r-1} & \int_{p_r} \theta_{\tau_{\alpha_r+1}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) \\ &= \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1} \left(\frac{a_{t_r}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}, \dots, \frac{a_{\tau_{\alpha_{r+1}-2}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})} \right). \end{aligned}$$

The proposition follows immediately. $\square$

Example 1.5.4. On the last row of $\mathcal{M}_{[n]}(\mathbf{x})$ one has

$$\begin{aligned} E_{\mathbf{1},\mathbf{j}}(\mathbf{x}) =&\gamma^n_{\mathbf{j}}(\mathbf{x}) = \prod_{r=0}^{l} \mathfrak{L}_{t_{r+1}-t_r-1}\Big(\frac{a_{t_r}-a_{t_{r+1}}}{a_{t_r+1}-a_{t_{r+1}}},\cdots,\frac{a_{t_{r+1}-2}-a_{t_{r+1}}}{a_{t_{r+1}-1}-a_{t_{r+1}}}\Big) \\ &=\prod_{r=0}^{l} \mathfrak{L}_{t_{r+1}-t_r-1}\Big(\frac{1-x_{1+t_r}\cdots x_{t_{r+1}}}{1-x_{2+t_r}\cdots x_{t_{r+1}}},\cdots,\frac{1-x_{t_{r+1}-1}x_{t_{r+1}}}{1-x_{t_{r+1}}}\Big) \end{aligned} \tag{1.19}$$

where $\mathfrak{L}_0=1$ and $x_{n+1}=\infty$. In particular, $E_{\mathbf{1},\mathbf{0}}=\gamma^n_{\mathbf{0}}(\mathbf{x})=\mathfrak{L}_n(\mathbf{x})$ and $E_{\mathbf{1},\mathbf{1}}=\gamma^n_{\mathbf{1}}(\mathbf{x})=1$.

For any $n\ge 2$ let $L_n=L_n(\mathbf{x})=[C_{\mathbf{0}}\ \dots\ C_{\mathbf{1}}]$ be the matrix with 2^n columns $C_{\mathbf{j}}$ ($\mathbf{j}\in\mathcal{S}_n$) where

$$C_{\mathbf{j}}=\sum_{\mathbf{i}\succ\mathbf{j}}\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e_{\mathbf{i}}$$

where $e_{\mathbf{i}}$ is the $\mathbf{i}$-th standard unit column vector. In listing the columns we have used a complete order "$<$" on $\mathfrak{S}_n$: If $|\mathbf{i}|<|\mathbf{j}|$ then $\mathbf{i}<\mathbf{j}$ (or, equivalently, $\mathbf{j}>\mathbf{i}$). If $|\mathbf{i}|=|\mathbf{j}|$ then the usual lexicographic order from left to right is in force with $1<0$ (not $0<1$!).

We now fix a standard basis $\{e_{\mathbf{i}}:\mathbf{i}\in\mathfrak{S}_n\}$ of $\mathbb{C}^{2^n}$ consisting of column vectors. Suppose $|\mathbf{i}|=k$. It follows from definition that the $\mathbf{i}$-th row is

$$R_{\mathbf{i}}:=\sum_{\mathbf{j}\prec\mathbf{i}}(2\pi i)^{|\mathbf{j}|}\gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e^T_{\mathbf{j}}=(2\pi i)^k e^T_{\mathbf{i}}+\sum_{\mathbf{j}\precneqq\mathbf{i}}(2\pi i)^{|\mathbf{j}|}\gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e^T_{\mathbf{j}} \tag{1.20}$$

where $e^T_{\mathbf{j}}$ are now row vectors. Note that $\gamma^k_{\rho_{\mathbf{i}}(\mathbf{i})}=\gamma^k_{\mathbf{1}_k}=1$ by definition. It is clear that the first entry (i.e. $\mathbf{j}=\mathbf{0}$) of this row is $\mathfrak{L}_k\big(\mathbf{x}(\mathbf{i})\big)$.

Example 1.5.5. Using the above definition we can easily get the variation matrix associated with the double logarithm:

$$\mathcal{M}_{1,1}(x,y)=\begin{bmatrix} 1 & & & \\ Li_1(x) & 2\pi i & & \\ Li_1(xy) & & 2\pi i & \\ Li_{1,1}(x,y) & 2\pi i Li_1(y) & 2\pi i(Li_1(x)-Li_1(y)-\log y) & (2\pi i)^2 \end{bmatrix}.$$

For any $k=0,\dots,n$ let us call the minor of $\mathcal{M}_{[n]}(\mathbf{x})$ consisting of rows beginning with k-tuple logarithms (corresponding to $\mathbf{i}=(\{1\}^k,0,\dots,0)$) and ending just before the $(k+1)$-tuple logarithms *the k-th block*. It has $\binom{n}{k}$ rows with row indices $|\mathbf{i}|=k$.

Lemma 1.5.6. *The matrix $\mathcal{M}_{[n]}(\mathbf{x})$ is a lower triangular matrix. Moreover, the columns with $|\mathbf{j}|=k$ of the k-th block of $\mathcal{M}_{[n]}(\mathbf{x})$ is $(2\pi i)^k$ times the identity matrix of rank $\binom{n}{k}$.*

Proof. The lemma follows directly from equation (1.20) because if $\mathbf{j}\precneqq\mathbf{i}$ then $\mathbf{j}<\mathbf{i}$. □

Lemma 1.5.7. *The $\mathbf{j}$-th column of $\mathcal{M}_{[n]}(\mathbf{x})$ is*

$$(2\pi i)^{|\mathbf{j}|}C_{\mathbf{j}}=(2\pi i)^{|\mathbf{j}|}\sum_{\mathbf{i}\succ\mathbf{j}}\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e_{\mathbf{i}}$$

where $\mathbf{x}(\mathbf{i})$ is defined by equation (1.12) *depending on $\mathbf{i}$.*

Proof. Use equation (1.20). □

Proposition 1.5.8. *The columns of* $\mathcal{M}_{[n]}(\mathbf{x})$ *form the set of the fundamental solutions of the following system of differential equations*

$$\begin{cases} d\,X_{\mathbf{0}} = 0, \\ d\,X_{\mathbf{i}} = \displaystyle\sum_{|\mathbf{k}|=|\mathbf{i}|-1,\,\mathbf{k}\prec\mathbf{i}} X_{\mathbf{k}}\, d\,\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{k})}\big(\mathbf{x}(\mathbf{i})\big) \quad \textit{for all } 1 \le |\mathbf{i}| \le n \end{cases} \tag{1.21}$$

where $\mathbf{x}(\mathbf{i})$ *is determined as in equation* (1.12).

Proof. We prove the proposition by induction on n. It is easy to see the proposition is valid for $n = 1$ and $n = 2$. We assume that $n \ge 3$ and the proposition is true for $\le n-1$. Let us now consider the $\mathbf{j}$-th column as expressed in Lemma 1.5.7. The cases $|\mathbf{i}| = 1$ or $\mathbf{j} > \mathbf{i}$ are obvious. Suppose

(1) $1 < |\mathbf{i}| < n$ and $\mathbf{j} \le \mathbf{i}$. There are two cases. (i) $\mathbf{j} \not\prec \mathbf{i}$. This is trivial because each term of both sides is zero. (ii) $\mathbf{j} \prec \mathbf{i}$. Then there is a t such that $i_t = j_t = 0$. We denote $\mathbf{i}' \in \mathfrak{S}_{n-1}$ the corresponding index after deleting the i_t-th component. By induction

$$\sum_{|\mathbf{k}'|=|\mathbf{i}'|-1,\,\mathbf{j}'\prec\mathbf{k}'\prec\mathbf{i}'} \gamma^{|\mathbf{k}'|}_{\rho_{\mathbf{k}'}(\mathbf{j}')}\big(\mathbf{x}'(\mathbf{k}')\big)\, d\,\gamma^{|\mathbf{i}'|}_{\rho_{\mathbf{i}'}(\mathbf{k}')}\big(\mathbf{x}'(\mathbf{i}')\big) = d\,\gamma^{|\mathbf{i}'|}_{\rho_{\mathbf{i}'}(\mathbf{j}')}\big(\mathbf{x}'(\mathbf{i}')\big)$$

where we set $\mathbf{x}' = (x_1, \dots, x_{i_t-1}, x_{i_t}x_{i_t+1}, x_{i_t+2}, \dots, x_n)$. Since $|\mathbf{i}'| = |\mathbf{i}|$ and $|\mathbf{k}'| = |\mathbf{k}|$ we can get the desired equation by inserting 0 before the i_t-th components of $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$.

(2) $\mathbf{i} = \mathbf{1}$ and $|\mathbf{j}| = l$. We need to show

$$d\gamma^n_{\mathbf{j}}(\mathbf{x}) = \sum_{|\mathbf{k}|=n-1,\,\mathbf{j}\prec\mathbf{k}} \gamma^{n-1}_{\rho_{\mathbf{k}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{k})\big)\, d\gamma^n_{\mathbf{k}}(\mathbf{x}). \tag{1.22}$$

This is trivial when $l = n$. The case $l = 0$ follows from (1.11) of Lemma 1.4.1:

$$d\mathfrak{L}_n(\mathbf{x}) = \sum_{t=1}^{n} \mathfrak{L}_{n-1}(x_1, \dots, x_{t-1}, x_t x_{t+1}, x_{t+2}, \dots, x_n)\, d\log\frac{1 - x_{t+1}^{-1}}{1 - x_t}.$$

So we may assume $0 < l < n$, $j_{t_1} = \cdots = j_{t_l} = 1$ and $j_t = 0$ for all other indices t. By definition (1.19) we have

$$d\gamma^n_{\mathbf{j}}(\mathbf{x}) = \sum_{r=0}^{l} \sum_{t_r<s<t_{r+1}} \gamma^{n-1}_{\rho_{\mathbf{v}_s}(\mathbf{j})}\big(\mathbf{x}(\mathbf{v}_s)\big)\, d\gamma^n_{\mathbf{v}_s}(\mathbf{x})$$

where $t_0 = 0$, $t_{l+1} = n+1$ and

$$\mathbf{v}_s = (1, \dots, \underset{\substack{\uparrow \\ t_r\text{-th place}}}{1}, \dots\dots\dots, 1, \underset{\substack{\uparrow \\ s\text{-th place}}}{0}, 1, \dots\dots\dots, \underset{\substack{\uparrow \\ t_{r+1}\text{-th place}}}{1}, \dots, 1).$$

Under the retraction map $\rho_{\mathbf{v}_s}$ the numbering of the indices changes as follows: $t \rightsquigarrow t$ if $t < s$ and $t \rightsquigarrow t-1$ if $t > s$. We also have

$$a_t\big(\mathbf{x}(\mathbf{v}_s)\big) = \begin{cases} a_t(\mathbf{x}) & \text{if } t < s, \\ a_{t+1}(\mathbf{x}) & \text{if } t > s. \end{cases}$$

Hence for each s such that $t_r < s < t_{r+1}$ the integral expression of $\gamma^{n-1}_{\rho_{\mathbf{k}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{k})\big)$ is unchanged under $\rho_{\mathbf{k}}$ ($\mathbf{j} \prec \mathbf{k}$) except the $\mathbf{v}_s$-term. Equation (1.22) now follows immediately from Leibniz rule and so the proposition is proved. ☐

Theorem 1.5.9. *Let* $\mathcal{M}_{[n]}(\mathbf{x}) = \big[E_{\mathbf{i},\mathbf{j}}(\mathbf{x})\big]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}$ *where* $E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ *are defined by* (1.16). *Let* $1 \le j \le k \le n$ *and* $q_{jk} \in \pi_1(S_n, \mathbf{x})$ *(resp.* $1 \le j < n$ *and* q_{j0}*) enclose* $\mathcal{D}_{jk} = \{x_j \dots x_k = 1\}$*, (resp.* $\mathcal{D}_{j0} = \{x_j = 0\}$*) only once but no other irreducible component of* D_n *such that* $\int_{q_{jk}} d\log(1 - x_j \dots x_k) = 2\pi i$ *(resp.* $\int_{q_{j0}} d\log x_j = 2\pi i$*). Then*

$$M(q_{j0}) = I + \big[n_{\mathbf{i},\mathbf{j}}\big]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}, \quad M(q_{jk}) = I + \big[m_{\mathbf{i},\mathbf{j}}\big]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}$$

where I *is the identity matrix of rank* 2^n*,*

$$n_{\mathbf{i},\mathbf{j}} = \begin{cases} -1 & \text{if } t_r \le n - j \le t_{r+1} - 2,\ r \ge 1,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_{s+1} \text{ and } n - j \le s \le t_{r+1} - 2 \\ 0 & \text{otherwise,} \end{cases} \tag{1.23}$$

and

$$m_{\mathbf{i},\mathbf{j}} = \begin{cases} 1 & \text{if } t_r = j \le n - k \le t_{r+1} - 2,\ r \ge 1,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_{n-k+1} \\ -1 & \text{if } t_r + 1 \le i \le n - k = t_{r+1} - 1,\ r \ge 0,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_i \\ 0 & \text{otherwise.} \end{cases} \tag{1.24}$$

Here $\mathbf{i}$ *and* $\mathbf{j}$ *in the case of* $m_{\mathbf{i},\mathbf{j}} = \pm 1$ *and* $n_{\mathbf{i},\mathbf{j}} = -1$ *satisfy the condition in Definition* 1.5.2(2).

Proof. . If we analytically continue every integral entry of $\mathcal{M}_{[n]}(\mathbf{x})$ along a same loop $q \in \pi_1(S_n, \mathbf{x})$ we will denote this action of q by $\Theta(q)$. By definition it is clear that if $\mathbf{i} \not\succ \mathbf{j}$ then $\Theta(q)E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ which is either 0 or 1. Thus we are only concerned with $E_{\mathbf{i},\mathbf{j}}$ with $\mathbf{i} \succ \mathbf{j}$.

We now fix some $\mathbf{j}$. If $|\mathbf{j}| = n$ then clearly $(\Theta(q) - I)C_{\mathbf{1}} = [0, \dots, 0]^T$ for any loop q. This proves the proposition for $|\mathbf{j}| = n$. We now assume $|\mathbf{j}| < n$. Let $\mathbf{i}$ and $\mathbf{j}$ be given as in Definition 1.5.2(2). Then the theorem follows from the explicit expression (1.18) using the monodromy computations contained in Theorem 4.3, Propositions 5.4 and 5.5 of [370]. □

Corollary 1.5.10. *The monodromy representation of* $\mathcal{M}_{[n]}(\mathbf{x})$

$$\rho_{\mathbf{x}} : \pi_1(S_n, \mathbf{x}) \longrightarrow \mathrm{GL}_{2^n}(\mathbb{Z})$$

is unipotent.

1.6 Varition matrix for multiple polylogarithms

In this section we provide the general rule to construct the variation matrix $\mathcal{M}_{\mathbf{n}}(\mathbf{x})$ associated to general multiple polylogarithm $Li_{\mathbf{n}}(\mathbf{x})$.

Let $\mathbf{n} = (n_1, \dots, n_d)$ and $n = |\mathbf{n}|$. Let $\mathbf{x} = (x_1, \dots, x_d)$ and $a_j = \prod_{k=1}^{j} x_k$ for all $j = 1, \dots, d$. We now give the construction of the entries in the last row of $\mathcal{M}_{\mathbf{n}}(\mathbf{x})$ (say, with column number c) together with the first column entry with row number also equal to c. The other rows can follow the same pattern since every row starts with some multiple polylogarithm by merging some the variables in $\mathbf{x}$ (merging means multiplying some consecutive variables).

Set a_0 and rename the ordered tuple $(1, \{0\}^{n_1-1}, a_1, \dots, \{0\}^{n_d-1}, a_d, 0)$ as $(b_0, b_1, \dots, b_n, b_{n+1})$. Now we choose $i_0 < i_1 < \dots < i_\ell < i_{\ell+1}$ subject to the following conditions:

(i) $i_0 = 0$, b_0 is the leading 1 and, $i_{\ell+1} = n + 1$ and b_{n+1} is the last 0.

(ii) For all $1 \le j \le l$, either $i_j + 1 = i_{j+1}$ or one of b_{i_j} and $b_{i_{j+1}}$ is nonzero.

(iii) For fixed $1 \le j \le d$, when choosing the 0's between a_{j-1} and a_j (if there are any) always start from the first 0 (denoted by b_{i_s}) immediately after a_{j-1} and choose consecutively until, say, b_{i_t}.

Set $b'_{i_s} = b_{i_s}$ except in case (iii) if b_{i_t} is not the last 0 immediately before a_j then we define $b'_{i_s} = a_{j-1}$ (one may think of a collapse of these consecutive 0's onto a_{j-1} if not all of the 0's between a_{j-1} and a_j are chosen). For each such choice of $1 \le i_1 < \cdots < i_\ell \le n$ satisfying (i)-(iii) above we set the corresponding column entry of $\mathcal{M}_{\mathbf{n}}(\mathbf{x})$ as

$$(-1)^{\#\{j : b_{i_j} \neq 0\}} \int_0^1 \frac{dt}{t - b_{i_1}} \cdots \frac{dt}{t - b_{i_\ell}} \tag{1.25}$$

and the corresponding row entry as

$$I(b_0; b_1, \dots, b_{i_1-1}; b_{i_1}) I(b'_{i_1}; b_{i_1+1}, \dots, b_{i_2-1}; b_{i_2}) \cdots I(b'_{i_\ell}; b_{i_\ell+1}, \dots, b_n; b_{n+1}), \tag{1.26}$$

where

$$I(b'_k; b_{k+1}, \dots, b_{m-1}; b_m) = \begin{cases} 1, & \text{if } m = k+1; \\ (-1)^{\#\{j : b_j \neq 0, k<j<m\}} \int_{b_m}^{b'_k} \frac{dt}{t-b_{k+1}} \cdots \frac{dt}{t-b_{m-1}}, & \text{if } m > k+1. \end{cases}$$

As a check, the value in (1.26) should always be nonzero. Notice that in the above iterated integral $I(b'_k; \cdots; b_m)$ we have the two boundary values reversed. Sometimes different choices of subsets of b_j's may give rise to the same column entry, then we need to combine all the corresponding row entries into one entry by adding them up (for example, try Exercise 1.10 for $\mathcal{M}_{2,2}(x, y)$). A general choice looks like the following:

$$(1, \{0\}^{m_1-1}, a_{i_1}, \dots, \{0\}^{m_j-1}, a_{i_j}, \dots, \{0\}^{m_\ell-1}, a_{i_\ell}, 0)$$

where $1 \le i_1 < \cdots < i_\ell \le d$ and the block of zeros $\{0\}^{m_j-1}$ (for $m_j > 1$) must be chosen from a block of consecutive 0's between a_r and a_{r+1} (starting from the 0 immediately after a_r) for some r satisfying $0 \le i_{j-1} \le r < i_j \le d$. The correspond column entry is then given by $Li_{m_1,\dots,m_\ell}(1/a_{i_1}, a_{i_1}/a_{i_2}, \dots, a_{i_{\ell-1}}/a_{i_\ell})$. We order these entries by the following complete ordering: (i) $\mathbf{m}_1 < \mathbf{m}_2$ if $|\mathbf{m}_1| < |\mathbf{m}_2|$, and (ii) if $|\mathbf{m}_1| = |\mathbf{m}_2|$ then we order by the indices $\mathbf{i}$ using the lexicographic order with $1 < 2 < \cdots$. For example, $Li_2(x)$ (with $(i_1) = (1)$) comes before $Li_{1,1}(x, y)$ (with $(i_1, i_2) = (1, 1)$) which in turn comes before $Li_2(xy)$ (with $(i_1) = (2)$).

Usually it is tedious to write down explicitly the variation matrix associated with any given multiple polylogarithm $Li_{m_1,\dots,m_n}(\mathbf{x})$. However, the following general result has been proved by Deligne and Goncharov [94] (see next section for relevant definitions):

The multiple polylogarithm $Li_{n_1,\dots,n_d}(\mathbf{x})$ underlies a good unipotent graded-polarizable variation of mixed Hodge-Tate structures $(V_{n_1,\dots,n_d}, W_\bullet, \mathcal{F}^\bullet)$ over

$$S_d = \mathbb{C}^d \setminus \Big\{ \prod_{i=1}^{d} x_i(1 - x_i) \prod_{1 \le i < j \le d} (1 - x_i \dots x_j) = 0 \Big\}$$

with the weight-graded quotients gr^W_{-2k} being given by c_k copies of the Tate structure $\mathbb{Z}(k)$ which are nonzero only for $0 \le k \le n := n_1 + \cdots + n_d$.

Here c_k is the number of different ways to pick ordered $(k+2)$-tuples $(b_{i_0}, \dots, b_{i_{k+1}})$ from the ordered numbers $(b_0, \dots, b_{n+1})$ in the following tableau

$$\Big| \, b_0 \Big| \cdots \Big| \, b_{n+1} \Big| = \Big| \, a_0 \Big| \underbrace{0 \Big| \cdots \Big| 0}_{n_1 - 1 \text{ times}} \Big| \, a_1 \Big| \cdots\cdots \Big| \, a_{d-1} \Big| \underbrace{0 \Big| \cdots \Big| 0}_{n_d - 1 \text{ times}} \Big| \, a_d \Big| \, 0 \Big| \tag{1.27}$$

such that all three conditions (i)-(iii) above are satisfied. It is apparent that

$$c_k \geq d_k(n_1, \dots, n_d) := \sum_{\substack{k_1+\cdots+k_d=k \\ 0\leq k_j<n_j \ \forall j}} 1.$$

Each term in the sum corresponds to the following choice: for every $j = 1, \dots, n$, choose k_j 0's immediately after a_{j-1}.

Example 1.6.1. By the definition, we always have $c_0 = c_n = 1$. When $n_1 = \cdots = n_d = 1$ tableau (1.27) becomes

$$\left| \, b_0 \right| \cdots \left| \, b_{n+1} \, \right| = \left| \, \underbrace{1 \left| \, \cdots \, \right| 1}_{n+1 \text{ times}} \, \right| \, 0 \, \right|$$

Hence c_k is the number of ways to choose k elements from the set $\{b_1, \dots, b_n\}$, i.e., $c_k = \binom{n}{k}$.

For ease of statement let us put a box $\boxed{}$ on a number whenever we choose it.

Example 1.6.2. This example will help you to do Exercise 1.8 and 1.9. For let's consider $Li_{2,1}$. There are six nontrivial ways to put boxes on $\left| \; 1 \; \right| \; a_1 \; \left| \; a_2 \; \right| \; 0 \; \left| \; 0 \; \right|$:

$$(1) \ \left|\boxed{1}\right| 0 \left|\boxed{a_1}\right| a_2 \left|\boxed{0}\right| \qquad (2) \ \left|\boxed{1}\right| 0 \left| a_1 \right|\boxed{a_2}\left|\boxed{0}\right| \qquad (3) \ \left|\boxed{1}\right|\boxed{0}\left|\boxed{a_1}\right| a_2 \left|\boxed{0}\right|$$

$$(4) \ \left|\boxed{1}\right| 0 \left|\boxed{a_1}\right|\boxed{a_2}\left|\boxed{0}\right| \qquad (5) \ \left|\boxed{1}\right|\boxed{0}\left| a_1 \right|\boxed{a_2}\left|\boxed{0}\right| \qquad (6) \ \left|\boxed{1}\right|\boxed{0}\left|\boxed{a_1}\right|\boxed{a_2}\left|\boxed{0}\right|$$

Together with the trivial choice $\left|\boxed{1}\right| 0 \left| a_1 \right| a_2 \left|\boxed{0}\right|$ we get $c_0 = c_3 = 1$, $c_1 = 2$ and $c_2 = 3$.

However, for $Li_{1,2}$ we have altogether only six ways to do this:

$$(1) \ \left|\boxed{1}\right| a_1 \left| 0 \right| a_2 \left|\boxed{0}\right| \qquad (2) \ \left|\boxed{1}\right|\boxed{a_1}\left| 0 \right| a_2 \left|\boxed{0}\right| \qquad (3) \ \left|\boxed{1}\right| a_1 \left| 0 \right|\boxed{a_2}\left|\boxed{0}\right|$$

$$(4) \ \left|\boxed{1}\right|\boxed{a_1}\left| 0 \right|\boxed{a_2}\left|\boxed{0}\right| \qquad (5) \ \left|\boxed{1}\right| a_1 \left|\boxed{0}\right|\boxed{a_2}\left|\boxed{0}\right| \qquad (6) \ \left|\boxed{1}\right|\boxed{a_1}\left|\boxed{0}\right|\boxed{a_2}\left|\boxed{0}\right|$$

Thus $c_0 = c_3 = 1$, $c_1 = c_2 = 2$.

1.7 Variations of mixed Hodge structures of multiple polylogarithms

1.7.1 Definition of variations of MHS: a review

In this section we briefly review the theory of variations of mixed Hodge structures (for which we use the abbreviate MHS only in this chapter).

A pure ($\mathbb{Z}$-)Hodge structure (HS) of weight k consists of a finitely generated abelian group $H(\mathbb{Z})$ and a decreasing Hodge filtration $\mathcal{F}^\bullet$ on $H(\mathbb{C}) := H(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ such that $H(\mathbb{C}) = \mathcal{F}^p \oplus \overline{\mathcal{F}^{k-p+1}}$ for all integers p. Here the "bar" is the complex conjugation on the second factor of the tensor product. A special example is the Tate structure $\mathbb{Z}(-k)$ of weight $2k$ consists of $H(\mathbb{Z}) = \mathbb{Z}$ and the filtration $\mathcal{F}^p = 0$ for $p > k$ and $\mathcal{F}^p = H(\mathbb{C})$ for $p \leq k$. If we replace $\mathbb{Z}$ by $\mathbb{Q}$ in the above then we get a pure ($\mathbb{Q}$-)HS of weight k.

A MHS consists of a finitely generated abelian group $H(\mathbb{Z})$ and two filtrations: an increasing weight filtration $W_\bullet$ on $H(\mathbb{Q}) := H(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and a decreasing filtration $\mathcal{F}^\bullet$ on $H(\mathbb{C})$, which are compatible in the following sense. On each graded piece of the weight filtration $\mathrm{gr}_k^W = W_k/W_{k-1}$ the induced Hodge filtration determined by

$$\mathcal{F}^p(\mathrm{gr}_k^W)(\mathbb{C}) = {(\mathcal{F}^p \cap W_k(\mathbb{C}) + W_{k-1}(\mathbb{C}))}\Big/{W_{k-1}(\mathbb{C})}$$

is a pure Hodge structure of weight k where $W_k(\mathbb{C}) := W_k \otimes_{\mathbb{Z}} \mathbb{C}$. If all the pure Hodge structures induced as above are direct sums of Tate structures then we say the MHS is a Tate structure. For a mixed Hodge-Tate structure we can put a framing as in [22, §1.3.4, §1.4].

Following Steenbrink and Zucker [308, Definitions 3.1, 3.2 and 3.4] we have

Definition 1.7.1. A *variation of HS of weight* k defined over $\mathbb{Q}$ and a complex manifold S is a collection of data $(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^\bullet)$ where

(a) $\mathbb{V}_{\mathbb{Q}}$ is a locally constant sheaf (local system) of $\mathbb{Q}$-vector spaces on S,

(b) $\mathcal{F}^\bullet$ is a decreasing filtration by holomorphic subbundles of the locally free sheaf $\mathcal{V} = \mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{V}_{\mathbb{Q}}$.

(c) At each $s \in S$, $\mathcal{F}^\bullet$ induces the Hodge filtration $\mathcal{F}_s^\bullet$ of a Hodge structure of weight k on the fiber $\mathcal{V}_s$ of $\mathcal{V}$ such that

(i) whenever $p+q=k$ one has $\mathcal{V}_s = \mathcal{F}_s^p \oplus \overline{\mathcal{F}_s^{q+1}}$, where the "bar" denotes the complex conjugation,

(ii) equivalently, one has $\mathcal{V}_s = \bigoplus_{p+q=k} H_s^{p,q}$ where $H_s^{p,q} = \mathcal{F}_s^p \cap \overline{\mathcal{F}_s^q}$.

(d) (Griffiths transversality) Under the connection ∇ in $\mathcal{V}$,

$$\nabla \mathcal{F}^p \subset \Omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{F}^{p+1} \qquad \text{for all } p.$$

Definition 1.7.2. *A polarization over* $\mathbb{Q}$ of a variation of Hodge structure of weight k over $\mathbb{Q}$ is a non-degenerated and flat bilinear pairing:

$$\beta : \mathbb{V}_{\mathbb{Q}} \times \mathbb{V}_{\mathbb{Q}} \longrightarrow \mathbb{Q},$$

such that β is $(-1)^k$-symmetric, and the Hermitian form $\beta_s(C_s v, \bar{w})$ is positive on each fiber. Here C_s denotes the Weil operator with respect to $\mathcal{F}_s$, namely the direct sum of multiplication by i^{p-1} on $H_s^{p,q}$. A variation is called *polarizable* (over $\mathbb{Q}$) if it admits a polarization (over $\mathbb{Q}$).

Definition 1.7.3. A *variation of MHS* defined over $\mathbb{Q}$ and a complex manifold S is a collection of data $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ where

(a) $\mathbb{V}_{\mathbb{Q}}$ is a local system of $\mathbb{Q}$-vector spaces on S,

(b) $W_\bullet$ is an increasing filtration of the $\mathbb{V}_{\mathbb{Q}}$ by local subsystems,

(c) $\mathcal{F}^\bullet$ is a decreasing filtration by holomorphic subbundles of $\mathcal{V} = \mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{V}_{\mathbb{Q}}$.

(d) $\nabla \mathcal{F}^p \subset \Omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{F}^{p+1}$ for all p.

(e) The data

$$\big(\mathrm{gr}_k^W \mathbb{V}_{\mathbb{Q}}, \mathcal{F}^\bullet\big({\mathcal{O}_S \otimes_E W_k}\big/{\mathcal{O}_S \otimes_E W_{k-1}}\big)\big) \tag{1.28}$$

is a variation of HS of weight k defined over $\mathbb{Q}$; or equivalently, on the fiber over $s \in S$, $(V_s, W_s, \mathcal{F}_s)$ is a MHS defined over $\mathbb{Q}$.

(f) If the induced collection of variations of HS (1.28) are all polarizable then the MHS is called graded-polarizable.

Remark 1.7.4. By extension of scalars in $\mathbb{V}_{\mathbb{Q}}$ one can define $\mathbb{V}_{\mathbb{F}}$ for any field $\mathbb{F}$ such that $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{R}$.

Giving a local system $\mathbb{V}_{\mathbb{Q}}$ is equivalent to specifying its monodromy representation

$$\rho_x : \pi_1(S, x) \longrightarrow \operatorname{Aut}_{\mathbb{Q}} \mathcal{V}_x.$$

A variation is called *unipotent* if this representation is unipotent. From Proposition 1.3 of [165] we know that a variation of MHS $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ is unipotent if and only if each of the variations of Hodge structure $\operatorname{gr}^W_k \mathbb{V}_{\mathbb{Q}}$ is constant.

In general, the behavior of a variation of MHS over a non-compact base S at "infinity" is very hard to control. Steenbrink and Zucker [308] consider the case when S is a curve and define the admissibility condition at infinity. For higher dimensional S, Kashiwara, M. Saito, and others define a variation over S to be admissible if its restriction to every curve is admissible in the sense of Steenbrink-Zucker.

However, the behavior of *unipotent* variations of MHS at infinity can be controlled rather easily. We have the classical result of Deligne [87, Proposition 5.2] which defines the *canonical extension* $\tilde{\mathcal{V}}$ of $\mathcal{V}$.

Theorem 1.7.5. (Deligne) *Let $\tilde{S}$ be a normalization of S. Let $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ be a unipotent variation of MHS over S. Then*
(a) There is a unique extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ over $\tilde{S}$ satisfying the following equivalent conditions:

> *(i) Inside every section of $\tilde{\mathcal{V}}$, every flat section of $\mathcal{V}$ increases at most at the rate of $O(\log^k \|x\|)$ (k large enough) on every compact set of $D = \tilde{S} - S$.*
>
> *(ii) Similarly, every flat section of $\mathcal{V}^\vee$ (the dual) increases at most at the rate of $O(\log^k \|x\|)$ (k large enough).*

(b) The combination of the two conditions (i) and (ii) is equivalent to the combination of the following two conditions:

> *(iii) In terms of any local basis of $\tilde{\mathcal{V}}$ the connection matrix $\boldsymbol{\omega}$ of $\mathcal{V}$ has at most logarithmic singularities along D.*
>
> *(iv) The residue of $\boldsymbol{\omega}$ along any irreducible component of D is nilpotent.*

We will verify conditions (iii) and (iv) by Proposition 1.5.8 for the multiple logarithm variations of MHS. They are unipotent variations by Theorem 1.5.9.

Definition 1.7.6. Let $\tilde{S}$ be a compactification of S. Then a unipotent variation of MHS $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ over S is said to be *good* if it satisfies the following conditions at infinity

> (1) the Hodge filtration bundles $\mathcal{F}^\bullet$ extend over $\tilde{S}$ to sub-bundles $\tilde{\mathcal{F}}^\bullet$ of the canonical extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ such that they induce the corresponding thing for each pure subquotient $\operatorname{gr}^W_k \mathbb{V}_{\mathbb{Q}}$,
>
> (2) for the nilpotent logarithm N_j of a local monodromy transformation about a component D_j of D, the weight filtration of N_j relative to $W_\bullet$ exists.

A slightly different definition first appeared in [164, 165] with the extra assumption that $D = \tilde{S} - S$ is a normal crossing divisor. In these papers Hain and Zucker classified good unipotent variations of MHS on algebraic manifolds. With constant pure weight subquotients these variations behave well at infinity.

1.7.2 MHS of multiple logarithms

Fix an embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$. Let $\mathcal{D}_n = D_n \cup (\mathbb{CP}^n \setminus \mathbb{C}^n)$. Let $M_{2^n}(\mathbb{C})$ be the set of $2^n \times 2^n$ matrices over $\mathbb{C}$. Put

$$\boldsymbol{\omega} = \Big(c_{\mathbf{i},\mathbf{j}}\Big)_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n} \in H^0(\mathbb{CP}^n, \Omega^1_{\mathbb{CP}^n}(\log(\mathcal{D}_n))) \otimes M_{2^n}(\mathbb{C}) \tag{1.29}$$

where

$$c_{\mathbf{i},\mathbf{j}} = \begin{cases} d\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big) & \text{if } |\mathbf{j}| = |\mathbf{i}| - 1,\ \mathbf{j} \prec \mathbf{i}, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check that all of the 1-forms in $\boldsymbol{\omega}$ have logarithmic singularity on $\mathcal{D}_n$. Moreover, because $\mathcal{M}_{[n]}(\mathbf{x})$ is invertible and $\boldsymbol{\omega}$ is closed we get

$$d\boldsymbol{\omega} = 0, \quad \boldsymbol{\omega}\wedge\boldsymbol{\omega} = 0. \tag{1.30}$$

This shows that $\boldsymbol{\omega}$ is integrable.

Now we define a meromorphic connection ∇ on the trivial bundle

$$\mathbb{CP}^n \times \mathbb{C}^{2^n} \longrightarrow \mathbb{CP}^n \tag{1.31}$$

by

$$\nabla f = df - \boldsymbol{\omega} f$$

where $f : S_n \to \mathbb{C}^{2^n}$ is a section. This connection has regular singularities along $\mathcal{D}_n$ because $\boldsymbol{\omega}$ is integrable by (1.30) and all the 1-forms in $\boldsymbol{\omega}$ are logarithmic in any compactification of S_n. By the explicit construction of $\boldsymbol{\omega}$ we see immediately that the conditions (iii) and (iv) of Theorem 1.7.5 are satisfied. Proposition 1.5.8 further implies that the columns $(2\pi i)^{|\mathbf{j}|} C_{\mathbf{j}}(\mathbf{x})$ of $\mathcal{M}_{[n]}(\mathbf{x})$ satisfy $\nabla f = 0$ and are therefore flat sections of (1.31). Even though they are multi-valued, their $\mathbb{Z}$-linear span is well defined thanks to Theorem 1.5.9. Hence $V_{[n]}(\mathbf{x})$ forms a local system over S_n.

Definition 1.7.7. The local system $V_{[n]}(\mathbf{x})$ is called the *n-tuple logarithm local system.*

To define the MHS on $V_{[n]}$ we can define the weight filtration by putting $W_{2k+1} = W_{2k}$ and

$$W_{-2k} V_{[n]}(\mathbf{x}) = \langle (2\pi i)^{|\mathbf{i}|} C_{\mathbf{i}} : \ |\mathbf{i}| \geq k \rangle_{\mathbb{Q}}$$

which is the $\mathbb{Q}$ vector space with basis $\{(2\pi i)^{|\mathbf{i}|} C_{\mathbf{i}} : \ |\mathbf{i}| \geq k\}$. In particular, $W_{-2k} V_{[n]}(\mathbf{x}) = 0$ if $k > n$ and $W_{2k} V_{[n]}(\mathbf{x}) = V_{[n]}(\mathbf{x})$ if $k \geq 0$. By regarding $e_{\mathbf{i}}$'s as column vectors one can define the Hodge filtration on $V_{[n]}(\mathbf{x}) \otimes \mathbb{C} = V_{[n],\mathbb{C}}$ as follows:

$$\mathcal{F}^{-k} V_{[n],\mathbb{C}} := \langle e_{\mathbf{i}} : \ |\mathbf{i}| \leq k \rangle_{\mathbb{C}}.$$

So in particular, $\mathcal{F}^{-k} V_{[n],\mathbb{C}} = 0$ for $k < 0$ and $\mathcal{F}^{-k} V_{[n],\mathbb{C}} = V_{[n],\mathbb{C}}$ for $k \geq n$.

By induction on n and using Lemma 1.5.6 it is easy to show that

$$\mathcal{F}^{-p} \cap W_{-2k} V_{[n],\mathbb{C}} = \begin{cases} 0 & \text{if } p \leq k-1 \\ \langle (2\pi i)^{|\mathbf{i}|} e_{\mathbf{i}} : \ k \leq |\mathbf{i}| \leq p \rangle & \text{if } k \leq p \leq n\ . \\ \langle (2\pi i)^{|\mathbf{i}|} e_{\mathbf{i}} : \ k \leq |\mathbf{i}| \leq n \rangle & \text{if } p \geq n \end{cases}$$

This implies that

$$\mathcal{F}^{-p}\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = \begin{cases} 0 & \text{if } p \le k-1 \\ W_{-2k}V_{[n],\mathbb{C}}/W_{-2k-1}V_{[n],\mathbb{C}} & \text{if } p \ge k. \end{cases}$$

In other words, $\mathcal{F}^q \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = 0$ for $q \ge -k+1$ and $\mathcal{F}^q \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}}$ for $q \le -k$. This means that the Hodge filtration induces a pure HS of weight $-2k$ on each weight graded piece. Furthermore, it is not hard to see by checking the powers of $2\pi i$ appearing on the diagonal of $\mathcal{M}_{[n]}(\mathbf{x})$ that this induced structure on $\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}}$ is isomorphic to the direct sum of $\binom{n}{k}$ copies of the Tate structure $\mathbb{Z}(k)$ by Lemma 1.5.6.

1.8 Limit MHS of multiple logarithms

Let the monodromy of $\mathcal{M}_{[n]}(\mathbf{x})$ at any subvariety $\mathcal{D}$ of $\mathbb{CP}^n$ be given by the matrix $T_{\mathcal{D}}$ and the local monodromy logarithm by $N_{\mathcal{D}} = \log T_{\mathcal{D}}/(2\pi i)$. Note that $T_{\mathcal{D}}$ is unipotent so $N_{\mathcal{D}}$ is well-defined.

Now let us recall the construction of the unipotent variations of limit MHS at the “infinity” with *normal crossing*. Let S be a complex manifold of dimension d. Suppose that S is embedded in $\tilde{S}$, via the mapping j, such that $D = \tilde{S} - S$ is a divisor with normal crossings. Let $\mathbb{V}$ be any unipotent local system of complex vector spaces on S, and $\mathcal{V}$ the corresponding vector bundle. According to Theorem 1.7.5 by Deligne there is a canonical extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ over $\tilde{S}$. Moreover, when the local monodromy is nilpotent $\tilde{\mathcal{V}}$ is a subsheaf of $j_*\mathcal{V}$. The local picture of $S \subset \tilde{S}$ is $(\Delta^*)^r \times \Delta^{d-r} \subset \Delta^d$ where Δ is the unit disk and Δ^* is the punctured one. We let $t_1, \dots, t_r$ denote the variables on $(\Delta^*)^r$, and $N_1, \dots, N_r$ the (commuting) local nilpotent logarithms of the associated monodromy transformations of the fibre. For $z_1, \dots, z_r$ in the upper half-plane, the universal covering mapping for $(\Delta^*)^r$ is given by

$$t_j = \exp(2\pi i z_j), \quad j = 1, \cdots, r.$$

Let $v_1, \dots, v_m$ be a basis of the multi-valued sections of $\mathbb{V}$ over $(\Delta^*)^r \times \Delta^{d-r}$, the formula

$$[\tilde{v}_1, \dots, \tilde{v}_m] = [v_1, \dots, v_m] \exp\left(-\sum_{j=1}^r 2\pi i z_j N_j \right) = [v_1, \dots, v_m] \prod_{j=1}^r t_j^{-N_j}$$

determines a basis of the sections of $\mathcal{V}$ over Δ^d and these provide, by definition, the generators of $\tilde{\mathcal{V}}$ over Δ^d.

In our situation, although the divisor D_n is not normal crossing Theorem 1.7.5 is still valid because the image of the global holomorphic logarithmic forms in the complex of smooth forms on S is independent of the normal crossings compactification (see [163, Prop. (3.2)]). In fact, the forms we are considering lie in the subcomplex generated by 1-forms of the type df/f where f is a rational function. Such forms are automatically logarithmic in any compactification and therefore our connection is automatically regular. Hence the admissibility and the existence of the limit MHS is an automatic consequence of the admissibility of our variations restricted to every curve in S_n. Moreover, the pullback to $\tilde{S}_n$ of our trivial bundle (1.31) restricted to S_n is exactly Deligne's canonical extension of (1.31), and the pullbacks of the subbundles $\mathcal{F}^\bullet$ and $W_\bullet$ are the correct extended Hodge and weight subbundles. Therefore we have

Theorem 1.8.1. *The n-tuple logarithm underlies a good unipotent graded-polarizable variation of mixed Hodge-Tate structures $(V_{[n]}, W_\bullet, \mathcal{F}^\bullet)$ over*

$$S_n = \mathbb{C}^n \setminus \Big\{ \prod_{1\le j\le n} x_j(1-x_j) \prod_{1\le i<j\le n} (1-x_i\dots x_j) = 0 \Big\}.$$

with the weight-graded quotients gr^W_{-2k} being given by $\binom{n}{k}$ copies of the Tate structure $\mathbb{Z}(k)$.

Proof. It is clearly that all the odd graded weight quotients are zero so that we can let the polarizations on the weight graded quotients gr^W_{-2k} be the ones that give each vector $2\pi i e_{\mathbf{j}}$ ($|\mathbf{j}| = k$) length 1. Then everything is clear except the Griffiths transversality condition. But this condition is also satisfied because $dC_{\mathbf{j}} = \boldsymbol{\omega} C_{\mathbf{j}}$ for every $\mathbf{j} \in \mathfrak{S}_n$ by Proposition 1.5.8 since all the entries of $\boldsymbol{\omega}$ on and above the main diagonal are zero. $\square$

If we want to determine the limit MHS of multiple logarithms explicitly we can still apply the techniques used in the normal crossing case. We will carry this out only for the depth two and three cases. The general picture is similar but much more complicated.

1.8.1 Limit MHS of double logarithm

First we look at the double logarithm variation of MHS. We have

$$\mathcal{M}_{1,1}(x,y) = \begin{bmatrix} 1 & & & \\ \mathfrak{L}_1(x) & 2\pi i & & \\ \mathfrak{L}_1(xy) & 0 & 2\pi i & \\ \mathfrak{L}_2(x,y) & 2\pi i \mathfrak{L}_1(y) & 2\pi i h(x,y) & (2\pi i)^2 \end{bmatrix}$$

where $h(x,y) = \mathfrak{L}_1(x) - \mathfrak{L}_1(y) - \log y$.
(i) Let us first try to extend the MHS to the divisor $\mathcal{D}_{20} = \{y=0\}$ along the tangent vector $\partial/\partial y$. We have

$$T_{\{y=0\}} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad N_{\{x=0\}} = \frac{\log T_{\{x=0\}}}{2\pi i} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & -\frac{1}{2\pi i} & 0 \end{bmatrix}.$$

Let $\mathcal{M}_{1,1}(x,y) = [C_0(x,y) \ \cdots C_3(x,y)]$. Define

$$[s_0 \ s_1 \ s_2 \ s_3] = \lim_{t\to 0} \mathcal{M}_{1,1}(x,t) \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & \log t/(2\pi i) & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & & & \\ \mathfrak{L}_1(y) & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ 0 & 0 & 2\pi i \mathfrak{L}_1(x) & (2\pi i)^2 \end{bmatrix}.$$

Let $V_{\mathbb{Q},\{y=0\}}$ be the $\mathbb{Q}$-linear span of s_0, s_1, s_2, s_3, and $V_{\mathbb{C},\{y=0\}} = \mathbb{C} \otimes V_{\mathbb{Q},\{y=0\}}$. Let $\{e_j : j = 0, \cdots, 3\}$ be the standard basis of $\mathbb{C}^4$ where the only nonzero entry of e_j is at the $(j+1)$st component. Then the limit MHS on $\{(x,y) : y = 0, x \ne 1\}$ along $\partial/\partial y$ are given by

$$((V_{\mathbb{Q},\{y=0\}}, W_\bullet), (V_{\mathbb{C},\{y=0\}}, F^\bullet))$$

where for $k = 0, \ldots, 3$

$$W_{-2k}V_{\mathbb{Q},\{x=0\}} = \langle s_k, \ldots, s_3\rangle, W_{-2k} = W_{-2k+1} \tag{1.32}$$

and

$$F^{-k}V_{\mathbb{C},\{x=0\}} = \langle e_0, \ldots, e_k\rangle. \tag{1.33}$$

(ii) A similar calculation shows that along the tangent vector $\partial/\partial y$ the limit MHS on the divisor $\mathcal{D}_{22} = \{(x,1) : x \neq 1\}$ is the $\mathbb{Q}$-linear span of $s_0, \ldots, s_3$ where

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & & & \\ \mathfrak{L}_1(x) & 2\pi i & & \\ \mathfrak{L}_1(x) & 0 & 2\pi i & \\ \mathfrak{L}_2(x,1) & 0 & 2\pi i\mathfrak{L}_1(x) & (2\pi i)^2 \end{bmatrix}.$$

It is easy to see by differentiation that $\mathfrak{L}_2(x,1) = \big(\mathfrak{L}_1(x)\big)^2/2$.
(iii) The extension of MHS to $\mathcal{D}_{11} = \{(1,y) : y \neq 0,1\}$ along the tangent vector $\partial/\partial x$ is given by the $\mathbb{Q}$-linear span of $s_0, \ldots, s_3$ where

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & & & \\ 0 & 2\pi i & & \\ -Li_1(\frac{y}{y-1}) & 0 & 2\pi i & \\ Li_2(\frac{y}{y-1}) & 2\pi i Li_1(y) & -2\pi i\log\frac{y}{y-1} & (2\pi i)^2 \end{bmatrix}.$$

(iv) Limit MHS on $\mathcal{D}_{12} = \{(x,1/x) : x \neq 0,1\}$ along the tangent vector $\partial/\partial y$ is given by the $\mathbb{Q}$-linear span of $s_0, \ldots, s_3$ where

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & & & \\ -Li_1(\frac{x}{x-1}) & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ -Li_2(\frac{x}{x-1}) & 2\pi i\log\frac{x}{x-1} & 0 & (2\pi i)^2 \end{bmatrix}.$$

(v) $\mathcal{D}_{20} \cap \mathcal{D}_{11} = (1,0)$. From (i) we see that there are limit MHS on the open set $\mathcal{D}_{20} \setminus \{(1,0)\}$ of $\mathcal{D}_{20}$. We now can easily extend these MHS to $(1,0)$ along the vector $\partial/\partial x$ and find the limit MHS at $(1,0)$ to be the $\mathbb{Q}$-linear span of $s_0, \cdots, s_3$ where

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & & & \\ 0 & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ 0 & 0 & 0 & (2\pi i)^2 \end{bmatrix}.$$

If we start from (iii) and then extend the MHS to $(1,0)$ along tangent vector $\partial/\partial x$ we will get the same limit MHS.
(vi) $\mathcal{D}_{22} \cap \mathcal{D}_{12} = \mathcal{D}_{12} \cap \mathcal{D}_{11} = \mathcal{D}_{11} \cap \mathcal{D}_{22} = (1,1)$. We can start from either case (ii) or (iii) or (iv). Extending the limit MHS of case (ii) we see immediately that the along the tangent vector $\partial/\partial y$ the limit MHS at $(1,1)$ is given by the $\mathbb{Q}$-linear span of

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & & & \\ 0 & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ E_{4,1} & 0 & 0 & (2\pi i)^2 \end{bmatrix}. \tag{1.34}$$

If we extend the limit MHS of case (iii) to $(1,1)$ along tangent vector $\partial/\partial x$ we find that only the lower left corner entry is different from the above. Instead of 0 it is

$$E_{4,1}=\lim_{y\to 1} Li_2(\frac{y}{y-1})+\frac{1}{2}\log^2(1-y)-\log y\log(1-y)=-Li_2(1)=-\frac{\pi^2}{12},$$

since

$$Li_2(1-t)+Li_2(1-1/t)+\frac{1}{2}\log^2 t=0 \quad \forall t\neq 0. \tag{1.35}$$

But if we take $s_0'=s_0-s_3/48$ we get the same basis as in (1.34). The same phenomenon occurs if we start from case (iv) and then use tangent vector $\partial/\partial y$.

If we extend the limit MHS of (iv) to the point $(1,1)$ along the tangent vector $\partial/\partial x$ then we find that

$$E_{4,1}=\lim_{x\to 1} -Li_2(\frac{x}{x-1})-\frac{1}{2}\log^2(1-x)=Li_2(1)=\frac{\pi^2}{12}$$

by taking $t=1-x$ in (1.35). Now if we let $s_0'=s_0+\frac{1}{48}s_3$ then we get the same basis as in (1.34). This phenomenon happens in higher logarithm cases too.

In Exercise 1.11 you will see another example of limit MHS. From all these examples we put forward the following

Conjecture 1.8.2. *The variations of mixed Hodge-Tate structures associated to every multiple polylogarithm can be produced as the variations of some limit mixed Hodge-Tate structures related to some suitable choice of multiple logarithm.*

1.9 Single-valued multiple polylogarithms

Using the monodromy of multiple polylogarithms obtained from section 1.5 we can construct single-valued multiple polylogarithms. Although this procedure can be carried out in general the computation in each particular case will involve very complicated notations. So we will just treat several lower weight cases in detail to illustrate the main idea.

1.9.1 General procedure of construction

In this subsection we first describe the general procedure to define the single-valued real analytic version of $Li_{m_1,\dots,m_n}(\mathbf{s})$, denoted by $\mathcal{L}_{m_1,\dots,m_n}(\mathbf{x})$, provided the variation matrix $\mathcal{M}_{m_1,\dots,m_n}(\mathbf{x})$ is already found. We will work with the multiple logarithms below, but the idea is exactly the same for the general case.

Let $\mathcal{M}_{[n]}=\mathcal{M}_{[n]}(\mathbf{x})=L_n(\mathbf{x})\tau_n(2\pi i)$ where

$$\tau_n(\lambda)=\mathrm{diag}\big[\lambda^{|\mathbf{j}|}\big]_{\mathbf{j}\in\mathcal{S}_n}.$$

Define the matrix

$$B_{[n]}=\tau_n(i)\mathcal{M}_{[n]}\overline{\mathcal{M}}_n^{-1}\tau_n(i)$$

where $\overline{\mathcal{M}}_n$ is the complex conjugation of $\mathcal{M}_{[n]}$. By the monodromy computation we see that $B_{[n]}$ is a single-valued matrix function defined over S_n because all the monodromy automorphisms are given by real matrices. Now both $\mathcal{M}_{[n]}$ and $\overline{\mathcal{M}}_{[n]}^{-1}$ are lower triangular matrices, so is $B_{[n]}$. Moreover its diagonal is given by $\tau_n(i)\tau_n(2\pi i)\overline{\tau_n(2\pi i)}^{-1}\tau_n(i)=I$ where I is the identity matrix. Hence we may assume $B_{[n]}=I+N$ where N a nilpotent matrix. We see that $\log B$ is well defined and since $\overline{B}_n=B_{[n]}^{-1}$ it satisfies

$$\overline{\log B_{[n]}}=-\log B_{[n]},$$

namely, $\log B_{[n]}$ is a pure imaginary matrix. Then we define $i^{2[n/2]-1}/2$ times the lower left corner entry of $\log B_{[n]}$ to be $\mathfrak{L}_n(\mathbf{x})$ which is a single-valued real analytic version of the multiple logarithm $\mathfrak{L}_n(\mathbf{x})$.

Remark 1.9.1. Our method is slightly different from that in [21]. In fact in the polylogarithm case the matrix $B_{[n]}$ constructed as above is the conjugate of the one in [21] by $\tau(i)$.

1.9.2 Single-valued double logarithms

We have seen from Example 1.5.5 that $\mathcal{M}_{1,1}(x,y) = L_{1,1}(x,y)\tau_{1,1}(2\pi i)$ where

$$L_{1,1}(x,y) = \begin{bmatrix} 1 & & & \\ Li_1(x) & 1 & & \\ Li_1(xy) & & 1 & \\ Li_{1,1}(x,y) & Li_1(y) & \log\frac{y-1}{y(1-x)} & 1 \end{bmatrix} \text{ and } \tau_{1,1}(\lambda) = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda^2 \end{bmatrix}.$$

Let $B_{1,1}(x,y) = \tau_{1,1}(i)L_{1,1}(x,y)\tau_{1,1}(-1)\overline{L_{1,1}(x,y)}^{-1}\tau_{1,1}(i)$. Then $B_{1,1}(x,y)$ is unipotent and single-valued. An easy calculation shows

$$\log B_{1,1}(x,y) = \begin{bmatrix} 0 & & & \\ -2i\log|1-x| & 0 & & \\ -2i\log|1-xy| & & 0 & \\ -2i\mathcal{L}_{1,1}(x,y) & -2i\log|1-y| & 2i\log\left|\frac{y-1}{y(1-x)}\right| & 0 \end{bmatrix}$$

where

$$\mathcal{L}_{1,1}(x,y) = \mathrm{Im}\,\big(Li_{1,1}(x,y)\big) - \arg(1-x)\log|1-y| + \arg(1-xy)\log\left|\frac{y-1}{y(1-x)}\right| \tag{1.36}$$

is the single-valued real analytic version of $Li_{1,1}(x,y)$.

By differentiation it is easy to check that

$$Li_{1,1}(x,y) = Li_2\Big(\frac{xy-x}{1-x}\Big) - Li_2\Big(\frac{x}{x-1}\Big) - Li_2(xy). \tag{1.37}$$

So by using the single-valued dilogarithm function $\mathcal{L}_2(z) = \mathrm{Im}\,\big(Li_2(z)\big) + \arg(1-z)\log|z|$ we can also recover (1.36) as

$$\mathcal{L}_{1,1}(x,y) = \mathcal{L}_2\Big(\frac{xy-x}{1-x}\Big) - \mathcal{L}_2\Big(\frac{x}{x-1}\Big) - \mathcal{L}_2(xy). \tag{1.38}$$

Further, using the five term relation we find

$$\mathcal{L}_{1,1}(x,y) + \mathcal{L}_{1,1}(y,x) + \mathcal{L}_2(xy) = 0.$$

One can check quickly $\mathcal{L}_{1,1}$ satisfies the functional equation

$$\mathcal{L}_{1,1}(x,y) = -\mathcal{L}_{1,1}\Big(\frac{x}{x-1}, 1-y\Big)$$

by using the functional equation $\mathcal{L}_2(x) = -\mathcal{L}_2(1-x) = -\mathcal{L}_2(1/x)$.

1.9.3 Single-valued double polylogarithms $\mathcal{L}_{1,2}$ and $\mathcal{L}_{2,1}$

By [352] a single-valued version of $Li_3(x)$ can be defined as

$$\mathcal{L}_3(z) = \mathrm{Re}\,\big(Li_3(z)\big) - \log|z|\,\mathrm{Re}\,\big(Li_2(z)\big) - \frac{1}{3}(\log|z|)^2\log|1-z|. \tag{1.39}$$

We now consider $Li_{2,1}(x,y)$ and $Li_{1,2}(x,y)$. By the procedure outlined in section 1.9.1 we find that the single-valued version of $Li_{2,1}(x,y)$ is (see Exercise 1.8):

$$\begin{aligned}\mathcal{L}_{2,1}(x,y) &= \operatorname{Re} Li_{2,1}(x,y) - \arg(1-xy)\big[\mathcal{L}_2(x)+\mathcal{L}_2(y)\big] + \log|1-y| \operatorname{Re} Li_2(x) \\ &- \log|x| \operatorname{Re} Li_{1,1}(x,y) - \log|1-y^{-1}| \operatorname{Re} Li_2(xy) - \frac{1}{3}\log|x^2y| \log|1-xy| \log\Big|1-y^{-1}\Big| \\ &+ \frac{1}{3}\log|x|\big(2\log|1-x|\log|1-y| + \log|1-xy|\log|y(1-x)|\big).\end{aligned}$$

The single-valued version of $Li_{1,2}(x,y)$ is (see Exercise 1.9):

$$\begin{aligned}\mathcal{L}_{1,2}(x,y) &= \operatorname{Re} Li_{1,2}(x,y) + \arg(1-xy)\big[\mathcal{L}_2(x)+\mathcal{L}_2(y)\big] - \arg(1-x)\mathcal{L}_2(y) \\ &+ \log|1-x| \operatorname{Re} Li_2(xy) - \log|y| \operatorname{Re} Li_{1,1}(x,y) + \frac{1}{3}\log|1-x|\log|xy|\log|1-xy| \\ &+ \frac{1}{3}\log|y|\Big[\log|1-x|\log|1-y| + \log|1-xy|\log\Big|\frac{y(1-x)}{1-y}\Big|\Big].\end{aligned}$$

Using the single-valued versions of dilogarithm $\mathcal{L}_2(z)$ and trilogarithm $\mathcal{L}_3(z)$ we can express $\mathcal{L}_{1,2}(y,x)$ by the trilogarithms

$$\mathcal{L}_{1,2}(y,x) \;=\; \mathcal{L}_3(1-xy) + \mathcal{L}_3(1-y) - \mathcal{L}_3\Big(\frac{1-y}{1-xy}\Big) - \mathcal{L}_3(x) + \mathcal{L}_3\Big(\frac{x-xy}{1-xy}\Big) - \mathcal{L}_3(1),$$

where $\mathcal{L}_3$ is the single-valued trilogarithm given by (1.39). This follows from the relation first discovered by Zagier after Goncharov's conviction that such identity should exist:

$$\begin{aligned}Li_{1,2}(x,y) &= Li_3(1-xy) + Li_3(1-x) - Li_3\Big(\frac{1-x}{1-xy}\Big) - Li_3(y) + Li_3\Big(\frac{y-xy}{1-xy}\Big) - Li_3(1) \\ &- \log(1-xy)\big(Li_2(1)+Li_2(1-x)\big) - \log\Big(\frac{1-x}{1-xy}\Big)Li_2(y) + \frac{1}{2}\log(y)\log^2(1-xy). \qquad (1.40)\end{aligned}$$

The geometric origin of this identity can be found in [362, Remark 3.18]. By straightforward computation one can find the following interesting formula:

$$\mathcal{L}_{1,2}(x,y) + \mathcal{L}_{2,1}(y,x) + \mathcal{L}_3(xy) = 2\log|y| \operatorname{Re}(Li_1(y)Li_1(x)).$$

One should compare this with

$$Li_{1,2}(x,y) + Li_{2,1}(y,x) + Li_3(xy) = Li_2(y)Li_1(x).$$

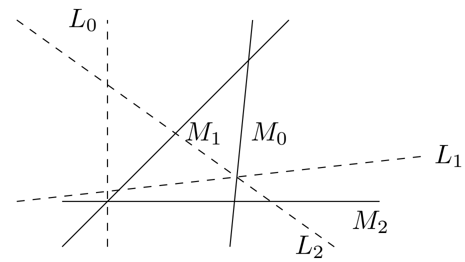
Finally, one can expressed the triple logarithm $\mathcal{L}_{1,1,1}(x,y,z)$ using the trilogarithm (see Exercise 1.13). Thus in weight three we essentially only need trilogarithm. But it may be a little surprising that similar identities in higher weight cases do not exist in general from geometric considerations (see [362, p. 146]). For example, $\mathcal{L}_{2,2}(x,y)$ cannot be expressed by only tetralogarithms $\mathcal{L}_4$.

1.10 Aomoto polylogarithms

For many people it is much easier to grasp a concept when geometric figures are attached to it. This is certainly true when we study multiple polylogarithms. Many deep results have been obtained for them up to weight three especially because in these lower weights we can draw pictures and visualize/discover various relations between them. Moving up to higher dimensions it is natural to consider the so-called Aomoto polylogarithms.

Definition 1.10.1. Let F be a fixed field of at least two elements. Fix a positive integer n. By a *simplex* in the projective spaces $\mathbb{P}^n_F$ we mean an ordered set of hyperplanes $L = (L_0, \dots, L_n)$. It is *nondegenerate* if the intersection of all the hyperplanes L_i is empty. A *face* of L is any nonempty intersection of the hyperplanes. A pair of simplices is *admissible* if they do not have common faces of the same dimension. It is a *generic pair* if all the faces of the two simplices are in general position.

Table 1.1: An admissible non-generic pair of triangles.



By projective invariance to be defined below, given a nondegenerate simplex L we may choose the coordinate system $[t_0, \dots, t_n]$ in $\mathbb{P}^n_F$ such that $L_i = \{t_i = 0\}$ for $0 \le i \le n$. If $F = \mathbb{C}$ then we define the canonical differential form associated to L as

$$\omega_L = \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_n}{t_n}.$$

The Aomoto n-logarithm is a multi-valued function on configurations of nondegenerate admissible pairs of simplices $(L;\, M)$ in $\mathbb{CP}^n$ defined as follows:

$$\Lambda_n(L;\, M) = \int_{\Delta_M} \omega_L,$$

where Δ_M is the n-cycle representing a generator of the relative homology $H_n(\mathbb{CP}^n, \cup_{j=0}^n M_j; \mathbb{Z})$.

We now define the double scissors congruence groups $A_n(F)$ whose defining relations should reflect (conjecturally all of) the functional equations of Aomoto polylogarithms. Furthermore, the associated graded object should form a Hopf algebra which provides a bridge to the study of motivic polylogarithmic complexes.

Definition 1.10.2. Define $A_0(F) = \mathbb{Z}$. If $n > 0$ then $A_n(F)$ is the abelian group generated by admissible pairs of n-simplices $(L;\, M)$ subject to the following relations:

(R1) *Nondegeneracy.* $(L;\, M) = 0$ if and only if L or M is degenerate.

(R2) *Trivial intersection.* Suppose $L_0, \dots, L_n$ and $M_0, \dots, M_n$ are hyperplanes in $\mathbb{P}^{n+1}_F$. If $N = L_i$ or $N = M_i$ for some i then

$$(N|L;\, M) := \big((L_0 \cap N, \dots, L_n \cap N);\, (M_0 \cap N \dots, M_n \cap N)\big) = 0.$$

(R3) *Skew symmetry.* For every permutation σ of $\{0, \dots, n\}$

$$(\sigma L;\, M) = (L; \sigma M) = \mathrm{sgn}(\sigma)(L;\, M),$$

where $\sigma(L_0, \dots, L_n) = (L_{\sigma(0)}, \dots, L_{\sigma(n)})$.

(R4) *Additivity in L and M.* For any hyperplanes $L_0, \dots, L_{n+1}$ and n-simplex M in $\mathbb{P}^n_F$

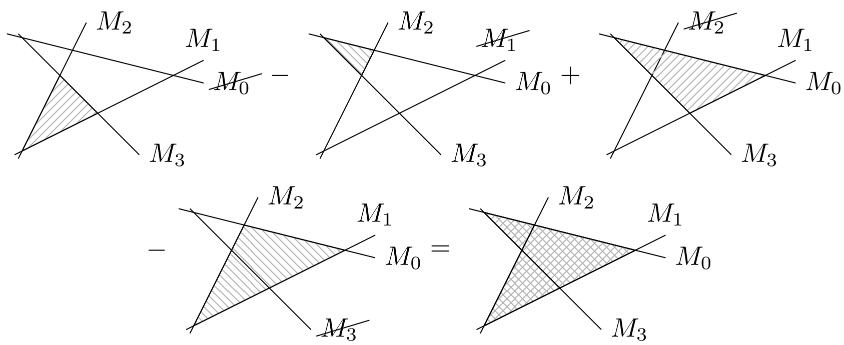
$$\sum_{j=0}^{n+1}(-1)^j((L_0, \dots, \widehat{L_j}, \dots, L_{n+1});\, M) = 0$$

if every pair $((L_0, \dots, \widehat{L_j}, \dots, L_{n+1});\, M)$ is admissible. A similar relation is satisfied by any n-simplex L and hyperplanes $M_0, \dots, M_{n+1}$.

(R5) *Projective invariance.* For every $g \in \mathrm{PGL}_{n+1}(F)$

$$(gL;\, gM) = (L;\, M).$$

Remark 1.10.3. The additivity relation (R4) can be visualized in dimension two by the following picture:

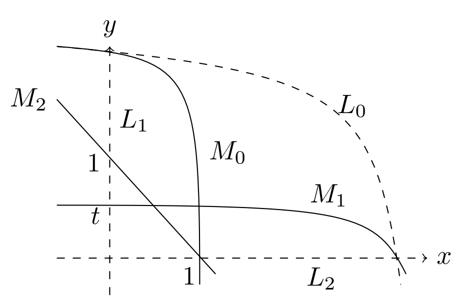


Example 1.10.4. We now consider the simplest case: $n = 1$. By projective invariance (R5) we may assume without loss of generality that $L = (\infty, 0)$ and $M = (1, a)$ for some $a \neq 0, 1, \infty$. Then de Rham cohomology $H^1_{\mathrm{dR}}(\mathbb{CP}^1 \setminus L) = H^1(\mathbb{C}^*)$ is generated by $[dz/z]$ where the relative singular homology $H_1(\mathbb{CP}^1, M)$ is generated by $[\Delta_M]$ where Δ_M is the line segment from 1 to a. So we get the period

$$\int_1^a \frac{dz}{z} = \log a.$$

Example 1.10.5. Let $n = 2$. Take $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$ and $L_0 = \{z = 0\}$. Let $M_0 = \{x = z\}$, $M_1 = \{y = tz\}$, and $M_2 = \{x + y = z\}$. We have the following picture:

Table 1.2: A real picture for dilogarithm.



Hence the period is given by

$$\int_{0<1-x<y<t} \frac{dx}{x}\frac{dy}{y} = -\int_{0<x<y<t} \frac{dy}{y}\frac{dx}{1-x} = -Li_2(t).$$

Definition 1.10.6. The n-simplex L whose faces are $L_i = \{t_i = 0\}$ $(0 \le i \le n)$ is called the *standard (coordinate) simplex*. For an arbitrary field F we denote by $\Lambda_{n_1,\dots,n_d}$ the subgroup of A_n $(n = n_1 + \cdots + n_d)$ generated by all *multiple polylogarithmic pairs* $\Lambda_{n_1,\dots,n_d}(x_1,\dots,x_d)$ corresponding to the multiple polylogarithms (all possible $x_1,\dots,x_d$). Specifically, we can represent $\Lambda_{n_1,\dots,n_d}(x_1,\dots,x_d)$ by $(L;\, M)$ where L is the standard simplex in $\mathbb{P}^n_F$ and M is determined as follows: Take $a_j = 1/(x_1 \dots x_j)$ and the vertex facing M_0 to be

$$m_0 = [z_0,\dots,z_n] = [\,\underbrace{0,\dots,0}_{n_1-1 \text{ times}}, -a_1,\, \underbrace{0,\dots,0}_{n_2-1 \text{ times}}, -a_2, \dots,\, \underbrace{0,\dots,0}_{n_d-1 \text{ times}}, -a_d, 1].$$

The vertex m_j $(1 \le j \le n)$ facing M_j is given by

$$m_i = [z_1+1,\dots,z_j+1, z_{j+1},\dots,z_n, 1].$$

If $F = \mathbb{C}$ and $M = \Lambda_{n_1,\dots,n_d}(x_1,\dots,x_d)$ a real simplex then the integral

$$\int_{\Delta_M} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} = (-1)^d Li_{n_1,\dots,n_d}(x_1,\dots,x_d), \tag{1.41}$$

where Δ_M is the convex real span by the faces of M. This is the origin of the name *multiple polylogarithmic pair.*

The key to the study of the double scissors congruence group A_n is to equip it with a good Hopf algebra structure. In lower weight cases (≤ 4) this can be done explicitly using some geometric combinatorial argument (see [362]) but in general this is still unclear. The product μ is easy to define but the coproduct is highly nontrivial. If this can be done for arbitrary n then we have the following surprising conjecture.

Conjecture 1.10.7. *The restricted coproduct provides a complex*

$$A_{>0} \longrightarrow A_{>0} \otimes A_{>0} \longrightarrow A_{>0} \otimes A_{>0} \otimes A_{>0} \longrightarrow \cdots$$

whose graded n-piece

$$A_n \longrightarrow \bigoplus_{k=1}^{n-1} A_k \otimes A_{n-k} \longrightarrow \cdots$$

gives

$$H^i_{(n)}(A_\bullet, \mathbb{Q}) \cong gr^\gamma_n K_{2n-i}(F)_\mathbb{Q}$$

where $gr^\gamma_n K_{2n-i}(F)_\mathbb{Q}$ is the γ-filtration of K-groups.

In fact, according to the Tannakian formalism the category $\mathsf{MTM}(F)$ of mixed Tate motives over a field F is supposed to be equivalent to the category of graded modules over a certain graded commutative Hopf algebra $\mathcal{A}_\bullet$ (see [24] and [145, Ch. 3]). Therefore the Ext groups in the category $\mathsf{MTM}(F)$ of mixed Tate motives over $\mathrm{Spec}(F)$ are isomorphic to the cohomology of the Hopf algebra $\mathcal{A}_\bullet$. Beilinson et al. conjecture that $A_\bullet$ is isomorphic to $\mathcal{A}_\bullet$ and therefore the groups $A_n \otimes \mathbb{Q}$ should have a Hopf algebra structure over $\mathbb{Q}$. This is the primary motivation to study the groups A_n. Further, by Beilinson's conjecture, there is a negatively graded Lie algebra $L_\bullet(F)$ over $\mathbb{Q}$ such that

$$\mathrm{Ext}^i_{\mathsf{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong H^i_{(n)}(L_\bullet(F)) \cong \mathrm{gr}^n_\gamma\, K_{2n-i}(F) \otimes \mathbb{Q}.$$

This is why we have Conjecture 1.10.7.

One hopes to construct the motivic Lie algebra $L_\bullet(F)$ explicitly, at least when the weight is not too large. To this end, let $\mathcal{B}_n(F) = \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$ where the subgroup $\mathcal{R}_n(F)$ reflects (conjecturally all of) the functional equations of the single-valued polylogarithm $\mathcal{L}_n(z)$ defined by (1.3). A lot of evidence [141] shows that $\mathcal{B}_n(F)\otimes\mathbb{Q}$ (for $n=1,2,3$) is dual to the motivic Lie algebra $L_{-n}(F)$ (the dual is between the ind and pro $\mathbb{Q}$-vector spaces). It follows from this line of thought [142] that the following conjecture should be true.

Conjecture 1.10.8. *Let* $\Pi_n = \bigoplus_{j=1}^{n-1}\mu(A_j\otimes A_{n-j})$ *be the subgroup of prisms of* A_n *then*

$$(A_n/\Pi_n)\otimes\mathbb{Q}\cong L_{-n}(F)^\vee$$

and for $n=1,2,3$

$$(A_n/\Pi_n)\otimes\mathbb{Q}\cong L_{-n}(F)^\vee\cong\mathcal{B}_n(F)\otimes\mathbb{Q}.$$

The first isomorphism means that the dual to the Hopf algebra $A_\bullet\otimes\mathbb{Q}$ is isomorphic to the universal enveloping algebra of the Lie algebra $L_\bullet(F)$. This conjecture is trivial when $n=1$ since all are isomorphic to $F^\times\otimes\mathbb{Q}$ by the cross ratio (see Exercise 1.2). For $n=2$ the first isomorphism is proved in [23] by taking $L_{-2}(F)^\vee$ to be the Bloch group $B_2(F)$ which is the quotient group of $\mathbb{Z}[\mathbb{P}^1_F]$ modulo the five-term relations.[2] It is known that for number fields $\mathcal{B}_2(F)\cong B_2(F)$.

Similar to $B_2(F)$ the group $B_3(F)$ is defined by Goncharov [142] as the quotient of $\mathbb{Z}[\mathbb{P}^1_F]$ by the subgroup generated by the generic seven-term relations of some generalized cross ratio r_3 and the Kummer-Spence relations (see [362, section 2.4] for details). The main result of [362] is that modulo torsions one has

$$A_3(F)/\Pi_3(F)\cong B_3(F).$$

1.11 Other multiple polylogarithms

In the study of multiple zeta values one often needs to consider the one variable multiple polylogarithms. Let $\mathbf{s}=(s_1,\dots,s_d)\in\mathbb{N}^d$ and define

$$\widetilde{\mathrm{Li}}_{\mathbf{s}}(z)=\int_0^z\left(\frac{dt}{t}\right)^{s_1-1}\frac{dt}{1-t}\cdots\left(\frac{dt}{t}\right)^{s_d-1}\frac{dt}{1-t}=\sum_{k_1>\dots>k_d\ge1}\frac{z^{k_1}}{k_1^{s_1}\cdots k_d^{s_d}},\quad |z|<1. \tag{1.42}$$

It is easy to see that $\widetilde{\mathrm{Li}}_{\mathbf{s}}(z)=Li_{\mathbf{s}}(z,1,\dots,1))$.

Proposition 1.11.1. *We have*

$$\frac{d}{dz}\widetilde{\mathrm{Li}}_{\mathbf{s}}(z)=\begin{cases}\widetilde{\mathrm{Li}}_{s_1-1,s_2,\dots,s_d}(z)/z, & \text{if } s_1>1;\\ \widetilde{\mathrm{Li}}_{s_2,s_3,\dots,s_d}(z)/(1-z), & \text{if } s_1=1.\end{cases}$$

Proof. The case $s_1>1$ is obvious. An easy application of geometric series summation formula proves the case $s_1=1$ immediately. □

To study the so called multiple zeta star values (see Chapter **??**) it is useful to consider the following variation of one variable multiple polylogarithms. $\mathbf{s}=(s_1,\dots,s_d)\in\mathbb{N}^d$ and define

$$Le_{\mathbf{s}}(z)=\sum_{k_1\ge\dots\ge k_d\ge1}\frac{z^{k_1}}{k_1^{s_1}\cdots k_d^{s_d}},\quad |z|<1. \tag{1.43}$$

[2]In the literature the Bloch group sometimes refers to the subgroup of $B_2(F)$ which is isomorphic to the indecomposable part of $K_3(F)$ modulo torsions.

1.12 Historical notes

It is well-known that logarithm was first invented by John Napier in 1614. Although the original intention was to simplify the multiplication and the division to the level of addition and subtraction logarithms have found many other applications in modern sciences.

Polylogarithms found their applications in modern physics as early as in the 1950s although they sometimes appeared in disguised forms (see, for example, [200, Appendix] and [18, p. 916]). It was discovered later that they should play an important role in the computation of Feynman integrals in perturbative quantum field theory [54]. In recent years, multiple polylogarithms are related to the study of conformal field theory [119, 211], amplitudes [56, 151] and Feynman graphs [39, 65, 249, 299] in physics as well as Hopf and Lie algebras, combinatorics (double shuffle relations) [264, 265], algebraic geometry [133, 132, 141, 142, 362, 363, 367], modular forms [134], and cluster algebras [118, 140] in mathematics.

The first serious study of classical polylogarithms seemed to be carried out by Jonquière [197]. Starting from 1950s Leonard Lewin, an engineer, systematically studied polylogarithms by generalizing many results for dilogarithm [233, 234]. Apparently he chose the notation Li because of "poly**l**ogarithm **i**ntegral". Later on with his collaborators he found many very interesting ladder relations such as the following one of order 12 [2]: let $\rho = (\sqrt{5}-1)/2$ then

$$\begin{aligned}\frac{Li_n(\rho^{12})}{12^{n-1}} - \frac{3}{2}\frac{Li_n(\rho^6)}{6^{n-1}} - \frac{Li_n(\rho^4)}{4^{n-1}} + \frac{11}{48}\frac{Li_n(\rho^2)}{2^{n-1}}\\ = \frac{13}{48}\frac{\log^n(\rho)}{n!} - \frac{\zeta(2)}{48}\frac{\log^{n-2}(\rho)}{(n-2)!} + \frac{19\zeta(4)}{1728}\frac{\log^{n-4}(\rho)}{(n-4)!} + \frac{67\zeta(5)}{6912}\frac{\log^{n-5}(\rho)}{(n-5)!}, \quad 1 \le n \le 5.\end{aligned}$$

A few higher order relations have been discovered since, some of which are only numerically (see, for example, [16, 81]).

In a series of papers spanning over twenty years K.-T. Chen developed his far-reaching theory of iterated integrals [72, 73, 74] which was unfortunately overlooked for several decades. Since 1990s, especially because of its applications in multiple zeta and multiple polylogarithms, its importance has been gradually recognized by more and more mathematicians. Richard Hain, one of Chen's Ph.D. students, has played a pivotal role in this process.

During his study of monodromy properties of $Li_n(x)$ Ramakrishnan [287] implicitly came upon the functions defined in (1.2) (written donw by Zagier explicitly in [351] and by Wojtkowiak in a modified form in [341]). When $n = 2$ this coincides with the Bloch-Wigner dilogarithm which was brought to prominence in Bloch's famous Irvine's lectures [32]. Notice that the function (1.3) also generalizes the Bloch-Wigner function.

Around 2000 Herbert Gangl investigated various functional equations of polylogarithms $Li_n(x)$ for $n \le 7$ (see [129, 130]). Although there are a lot of information available for these functional equations when n is small the most general functional equation of $Li_n(x)$ is still a mystery whose complete solution will provide key information in the construction of some motivic complexes [141, 142].

In [133] Gangl and Müller-Stach discovered a cycle-version of the five-term relation (1.5) satisfied by the dilogarithm in some higher Chow group. They also proved a certain Kummer relation in the cycle-version holds for the trilogarithm. This has been generalized to trilogarithm's 22-term relation (1.6) in [367].

Turning to the multiple polylogarithms, E. Kummer, H. Poincaré, I. Lappo-Danilevsky and K.-T. Chen all studied them in various disguised forms from different points of view ranging from differential equations to iterated integrals. In the mid 1990s, Goncharov utilized these functions to solve some difficult problems in number theory and algebraic geometry. Sometimes, the term "hyperlogarithm" (see, for e.g. [210]) is used instead of multiple polylogarithm, but this is not standard since for other people (see, for example [285]) hyperlogarithm means different functions.

Back in early 1980s Deligne [88] noticed that the dilogarithm gives rise to a good unipotent variation of mixed Hodge-Tate structures. In his survey paper [162] Richard Hain gave a detailed account of the polylogarithm local systems as variations of mixed Hodge structure building on Ramakrishna's explicit computation of the monodromy of polylogarithms [287]. The monodromy computation also yields the single-valued variant $\mathcal{L}_n(z)$ of the polylogarithms [21, 31]. These functions in turn have significant applications in arithmetic algebraic geometry such as Zagier's Conjecture 1.2.1. On the other hand, as pointed out by Goncharov in [143], "higher cyclotomy theory" should study the multiple polylogarithm motives at roots of unity, not only those of the polylogarithms. This is the reason why it is important to consider the variations of mixed Hodge structures associated with the multiple polylogarithms.

The idea in section 1.9 follows closely that of Beilinson and Deligne [21] as given in [31]. By this approach one can easily construct the single-valued version of any multiple polylogarithm as long as one can determine its associated variation matrix explicitly.

In order to understand the motivic origin of multiple polylogarithms it is beneficial to work with Aomoto polylogarithms [9] which are geometrically more intuitive. Our treatment here is quite superficial and avoids a lot of technical details but hopeful still gives the reader a glance of what can possibly be done with this powerful concept. In particular, the double scissors congruence groups are first introduced in [24] and then studied by Beilinson et al. in two companion papers [22, 23]. They were able to construct the weight two motivic complex modulo torsions using the geometric combinatorial properties satisfied by the double scissors congruence group of pairs of triangles on the projective plane. This is generalized to dimension three in [362]. Notice that in their original definition, Beilinson et al. imposed another type of relations called *Intersection Additivity.* However, in [361] we showed that these relations follow from the others.

Exercises

Exercise 1.1. Verify (1.4) by differentiation .

Exercise 1.2. Define the cross-ratio

$$r(a,b,c,d)=\frac{(a-c)(b-d)}{(a-d)(b-c)}$$

for all distinct $a,b,c,d\in\mathbb{CP}^1$. Show that for any five distinct points $x_0,\dots,x_4\in\mathbb{CP}^1$ we have the five term relation of dilogarithm

$$\sum_{i=0}^{4}(-1)^i\mathcal{L}_2(r(x_0,\dots,\widehat{x_i},\dots,x_4))=0,$$

where $\widehat{x_i}$ means x_i is removed.

Exercise 1.3. Prove (1.8) by geometric series expansion.

Exercise 1.4. Derive (1.11) from (1.10) by breaking it into two sums and making an index substitution.

Exercise 1.5. Prove Theorem 1.4.2 by induction on n using Lemma 1.4.1.

Exercise 1.6. Derive (1.18) from (1.17) in Proposition 1.5.3, using substitutions and the integral expression of multiple polylogarithms given by (1.8).

Exercise 1.7. Verify the double logarithm variation matrix in Example 1.5.5 by Definition 1.5.2.

Exercise 1.8. Check that the variation matrix $\mathcal{M}_{2,1}(x,y)$ associated with $Li_{2,1}(x,y)$ over S_2 is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ Li_1(x) & 1 & 0 & 0 & 0 & 0 & 0 \\ Li_1(xy) & 0 & 1 & 0 & 0 & 0 & 0 \\ Li_2(x) & \log(x) & 0 & 1 & 0 & 0 & 0 \\ Li_{1,1}(x,y) & Li_1(y) & Li_1\left(\frac{1-xy}{1-y}\right) & 0 & 1 & 0 & 0 \\ Li_2(xy) & 0 & \log(xy) & 0 & 0 & 1 & 0 \\ Li_{2,1}(x,y) & Li_1(y)\log(x) & g(x,y) & Li_1(y) & \log x & -Li_1(y^{-1}) & 1 \end{bmatrix} \tau_{2,1}(2\pi i)$$

where

$$g(x,y) = -\int_{a_2}^{1} \frac{dt}{t}\frac{dt}{t-a_1} = Li_2(x) - Li_2(y^{-1}) - \log(xy)Li_1(y^{-1}).$$

The columns of $\mathcal{M}_{2,1}(x,y)$ form the fundamental solutions of the differential equation over S_2

$$d\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \\ dLi_1(x) & 0 & 0 & 0 & 0 & 0 & \\ dLi_1(xy) & 0 & 0 & 0 & 0 & 0 & \\ 0 & dLi_1(y) & dLi_1\left(\frac{1-xy}{1-y}\right) & 0 & 0 & 0 & \\ 0 & d\log(x) & 0 & 0 & 0 & 0 & \\ 0 & 0 & d\log(xy) & 0 & 0 & 0 & \\ 0 & 0 & 0 & d\log(x) & dLi_1(y) & -dLi_1(y^{-1}) & 0 \end{bmatrix} \lambda$$

Exercise 1.9. Check the variation matrix $\mathcal{M}_{1,2}(x,y)$ associated with $Li_{1,2}(x,y)$ over S_2 is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ Li_1(x) & 1 & 0 & 0 & 0 & 0 \\ Li_1(xy) & 0 & 1 & 0 & 0 & 0 \\ Li_{1,1}(x,y) & Li_1(y) & Li_1\left(\frac{1-xy}{1-y}\right) & 1 & 0 & 0 \\ Li_2(xy) & 0 & \log(xy) & 0 & 1 & 0 \\ Li_{1,2}(x,y) & Li_2(y) & f(x,y) & \log y & Li_1(x) & 1 \end{bmatrix} \tau_{1,2}(2\pi i)$$

where

$$f(x,y) = -\int_{a_2}^{1} \frac{dt}{t-a_1}\frac{dt}{t} = Li_2(y^{-1}) - Li_2(x) + \log(xy)Li_1(x).$$

The columns of $\mathcal{M}_{1,2}(x,y)$ form the fundamental solutions of the differential equation over S_2

$$d\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ dLi_1(x) & 0 & 0 & 0 & 0 & 0 \\ dLi_1(xy) & 0 & 0 & 0 & 0 & 0 \\ 0 & dLi_1(y) & dLi_1\left(\frac{1-xy}{1-y}\right) & 0 & 0 & 0 \\ 0 & 0 & d\log(xy) & 0 & 0 & 0 \\ 0 & 0 & 0 & d\log y & dLi_1(x) & 0 \end{bmatrix} \lambda$$

Exercise 1.10. Write down the variation matrices $\mathcal{M}_{2,2}(x,y)$ and $\mathcal{M}_{3,1}(x,y)$ associated with $Li_{2,2}(x,y)$ and $Li_{3,1}(x,y)$, respectively, over S_2 using the rule given in section 1.6.

Exercise 1.11. Compute the limit MHS of the triple logarithm $\mathfrak{L}_3(x,y,z)$ on $\mathcal{D}_{11} = \{x=1\}$ over $\{(1,y,z) : yz(1-y)(1-z)(1-yz) \neq 0\}$ along the vector $\partial/\partial x$. Show that the variation matrix is essentially $\mathcal{M}_{2,1}\left(\frac{y}{y-1}, \frac{z(y-1)}{yz-1}\right)$. This gives some evidence for Conjecture 1.8.2.

Exercise 1.12. Prove the functional equation (1.35) of dilogarithm by differentiation.

Exercise 1.13. Prove the identity

$$\begin{aligned} Li_{1,1,1}(x,y,z) = Li_{1,2}\Big(\frac{1}{1-y}, \frac{z(1-y)}{z-1}\Big) - Li_{1,2}\Big(\frac{1-yx}{1-y}, \frac{z(1-y)}{z-1}\Big) \\ - \log(1-x)Li_2\Big(\frac{z}{z-1}\Big) - \log(1-z)Li_{1,1}(x,y) \end{aligned}$$

by differentiation. Then use (1.40) to show that

$$\begin{aligned} \mathcal{L}_{1,1,1}(x,y,z) =& \mathcal{L}_3\Big(\frac{(y-1)(1-xyz)}{y(1-x)(1-z)}\Big) + \mathcal{L}_3\Big(\frac{y}{y-1}\Big) + \mathcal{L}_3(yz) - \mathcal{L}_3\Big(\frac{1-xyz}{1-z}\Big) \\ & - \mathcal{L}_3\Big(\frac{1-xyz}{yz(1-x)}\Big) - \mathcal{L}_3\Big(\frac{y-xy}{y-1}\Big) - \mathcal{L}_3\Big(\frac{y-yz}{y-1}\Big) + \mathcal{L}_3(1-z) \end{aligned}$$

modulo products of logs and dilogs.

Exercise 1.14. Prove (1.41) for $d=2$ and $d=3$.

Appendix A

Answers to Some Exercise Problems

1.10. In this solution we use the notation $\ell_n(x) = \log^n(x)/n!$. For $Li_{2,2}(x,y)$ set $a_0 = 1$, $a_1 = 1/x$ and $a_2 = 1/(xy)$. Then we have

$$Li_{2,2}(x,y) = I(1;0,a_1,0,a_2;0) = I(1;0_1,a_1,0_2,a_2;0).$$

Here, in order to distinguish the two interior 0's we have used the set $\{0_1, a_1, 0_2, a_2\}$. For each choice of its subset we write $\alpha \otimes \beta$ to mean the corresponding column entry α and row entry β. Here are the possible choices:

1. $\emptyset$: $1 \otimes Li_{2,2}(x,y)$
2. $\{a_1\}$: $-I(1;a_1;0) \otimes -I(1;0_1;a_1)I(a_1;0_2,a_2;0) = Li_1(x) \otimes \ell_1(x)Li_2(y)$
3. $\{a_2\}$: $-I(1;a_2;0) \otimes -I(1;0_1,a_1,0_2;a_2) = Li_1(xy) \otimes k(x,y)$
4. $\{0_1,a_1\}$: $-I(1;0_1,a_1;0) \otimes -I(a_1;0_2,a_2;0) = Li_2(x) \otimes Li_2(y)$
5. $\{0_1,a_2\}$: $-I(1;0_1,a_2;0) \otimes -I(0_1;a_1,0_2;a_2) = Li_2(xy) \otimes Li_2(y^{-1})$ (Note there is no collapse of 0's! Row entry $Li_2(y^{-1})$ to be combined with $Li_2(x)$ from the next case.)
6. $\{0_2,a_2\}$: $-I(1;0_2,a_2;0) \otimes -I(1;0_1,a_1;0_2) = Li_2(xy) \otimes Li_2(x)$
7. $\{a_1,a_2\}$: $I(1;a_1,a_2;0) \otimes I(1;0_1;a_1)I(a_1;0_2;a_2) = Li_{1,1}(x,y) \otimes \ell_1(x)\ell_1(y)$
8. $\{a_1,0_2,a_2\}$: $I(1;a_1,0_2,a_2;0) \otimes I(1;0_1;a_1) = Li_{1,2}(x,y) \otimes \ell_1(x)$
9. $\{0_1,a_1,a_2\}$: $I(1;0_1,a_1,a_2;0) \otimes I(a_1;0_2;a_2) = Li_{2,1}(x,y) \otimes \ell_1(y)$
10. $\{0,a_1,0,a_2\}$: $Li_{2,2}(x,y) \otimes 1$

Thus the variation matrix $\mathcal{M}_{3,1}(x,y)$ associated with $Li_{3,1}(x,y)$ over S_2 is

$$\begin{bmatrix}
1 \\
Li_1(x) & 1 \\
Li_1(xy) & 0 & 1 \\
Li_2(x) & \ell_1(x) & 0 & 1 \\
Li_{1,1}(x,y) & Li_1(y) & h(x,y) & 0 & 1 \\
Li_2(xy) & 0 & \ell_1(xy) & 0 & 0 & 1 \\
Li_{2,1}(x,y) & \ell_1(x)Li_1(y) & g(x,y) & Li_1(y) & \ell_1(x) & \alpha(y) & 1 \\
Li_{1,2}(x,y) & Li_2(y) & f(x,y) & 0 & \ell_1(y) & Li_1(x) & 0 & 1 \\
Li_{2,2}(x,y) & \ell_1(x)Li_2(y) & k(x,y) & Li_2(y) & \ell_1(x)\ell_1(y) & \begin{matrix} Li_2(y^{-1}) \\ +Li_2(x) \end{matrix} & \ell_1(y) & \ell_1(x) & 1
\end{bmatrix} \tau_{2,2}(2\pi i)$$

where

$$\begin{aligned}
\alpha(y) =& -Li_1(y^{-1}) = -Li_1(y) - \ell_1(y), \\
h(x,y) =& -I(1;a_1;a_2) = -I\Big(1;\frac{a_1-a_2}{1-a_2};0\Big) = Li_1\Big(\frac{1-xy}{1-y}\Big) = Li_1(x) - Li_1(y) - \ell_1(y), \\
f(x,y) =& -I(1;a_1,0_2;a_2) = -I\Big(1;\frac{a_1-a_2}{1-a_2},\frac{-a_2}{1-a_2};0\Big) = Li_{1,1}\Big(\frac{1-xy}{1-y},1-y\Big) \\
=& Li_2(y^{-1}) - Li_2(x) + \ell_1(xy)Li_1(x), \\
g(x,y) =& -I(1;0_1,a_1;a_2) = -I\Big(1;\frac{-a_2}{1-a_2},\frac{a_1-a_2}{1-a_2};0\Big) = Li_{1,1}\Big(1-xy,\frac{1}{1-y}\Big) \\
=& Li_2(x) - Li_2(y^{-1}) - \ell_1(xy)Li_1(y^{-1})
\end{aligned}$$

by using (1.37), and

$$\begin{aligned}
k(x,y) =& -I(1;0_1,a_1,0_2;a_2) = -I\Big(1;\frac{-a_2}{1-a_2},\frac{a_1-a_2}{1-a_2},\frac{-a_2}{1-a_2};0\Big) \\
=& -Li_{1,1,1}\Big(1-xy,\frac{1}{1-y},1-y\Big) \\
=& 2Li_3(x) - \ell_1(x)Li_2(x) - 2Li_3(y^{-1}) + \ell_1(y)Li_2\Big(\frac{1-xy}{x(1-y)}\Big) - \ell_1(y)Li_2(y^{-1}) \\
& -\ell_1(xy)\Big(Li_2(y^{-1}) - \ell_1(y)Li_1(y^{-1})\Big) - \ell_1(y)Li_2\Big(1-\frac{1}{xy}\Big) - \ell_1(y)Li_2\Big(\frac{1-xy}{1-y}\Big)
\end{aligned}$$

by using Exercise 1.13, equation (1.40) and the following identities:

$$Li_3(y^{-1}) - Li_3(y) = \ell_3(y), \quad Li_2(y^{-1}) + Li_2(y) = 2Li_2(1) - \ell_2(y). \tag{A.1}$$

For $Li_{3,1}(x,y)$ set $a_0 = 1$, $a_1 = 1/x$ and $a_2 = 1/(xy)$. Then we have

$$Li_{3,1}(x,y) = I(a_0;0,0,a_1,a_2;0) = I(1;0,0,a_1,a_2;0).$$

We have the following possible choices of the subsets of $\{0,0,a_1,a_2\} = \{0_1,0_2,a_1,a_2\}$:

1. $\emptyset$: $1 \otimes Li_{3,1}(x,y)$
2. $\{a_1\}$: $-I(1;a_1;0) \otimes -I(1;0,0;a_1)I(a_1;a_2;0) = Li_1(x) \otimes \ell_2(x)Li_1(y)$
3. $\{a_2\}$: $-I(1;a_2;0) \otimes -I(1;0,0,a_1;a_2) = Li_1(xy) \otimes l(x,y)$
4. $\{0_1,a_1\}$: $-I(1;0_1,a_1;0) \otimes -I(1;0_2;a_1)I(a_1;a_2;0) = Li_2(x) \otimes \ell_1(x)Li_1(y)$ (collapse of 0_1 onto $a_0 = 1$)

5. $\{0_1, a_2\}$: $-I(1; 0_1, a_2; 0) \otimes -I(1; 0_2, a_1; a_2) = Li_2(xy) \otimes g(x, y)$ (collapse of 0_1)

6. $\{a_1, a_2\}$: $I(1; a_1, a_2; 0) \otimes I(1; 0, 0; a_1) = Li_{1,1}(x, y) \otimes \ell_2(x)$

7. $\{0_1, a_1, a_2\}$: $I(1; 0_1, a_1, a_2; 0) \otimes I(1; 0_2; a_1) = Li_{2,1}(x, y) \otimes \ell_1(x)$ (collapse of 0_1)

8. $\{0, 0, a_1\}$: $-I(1; 0, 0, a_1; 0) \otimes -I(a_1; a_2; 0) = Li_3(xy) \otimes Li_1(y)$

9. $\{0, 0, a_2\}$: $-I(1; 0, 0, a_2; 0) \otimes -I(0; a_1; a_2)$

10. $\{0, 0, a_1, a_2\}$: $Li_{3,1}(x, y) \otimes 1$

Thus the variation matrix $\mathcal{M}_{3,1}(x, y)$ associated with $Li_{3,1}(x, y)$ over S_2 is

$$\begin{bmatrix}
1 \\
Li_1(x) & 1 \\
Li_1(xy) & 0 & 1 \\
Li_2(x) & \ell_1(x) & 0 & 1 \\
Li_{1,1}(x,y) & Li_1(y) & h(x,y) & 0 & 1 \\
Li_2(xy) & 0 & \ell_1(xy) & 0 & 0 & 1 \\
Li_3(x) & \ell_2(x) & 0 & \ell_1(x) & 0 & 0 & 1 \\
Li_{2,1}(x,y) & \ell_1(x)Li_1(y) & g(x,y) & Li_1(y) & \ell_1(x) & \alpha(y) & 0 & 1 \\
Li_3(xy) & 0 & \ell_2(xy) & 0 & 0 & \ell_1(xy) & 0 & 0 & 1 \\
Li_{3,1}(x,y) & \ell_2(x)Li_1(y) & l(x,y) & \ell_1(x)Li_1(y) & \ell_2(x) & \tilde{g}(x,y) & Li_1(y) & \ell_1(x) & \alpha(y) & 1
\end{bmatrix} \tau_{3,1}(2\pi i)$$

where $\tilde{g}(x, y) = g(x, y) - Li_2(x)$ and

$$\begin{aligned}
l(x, y) =& -I(1; 0, 0, a_1; a_2) = -I\Big(1; \frac{-a_2}{1-a_2}, \frac{-a_2}{1-a_2}, \frac{a_1 - a_2}{1-a_2}; 0\Big) \\
=& -Li_{1,1,1}\Big(1 - xy, 1, \frac{1}{1-y}\Big) \\
=& Li_3(x) - Li_3(y^{-1}) - \ell_1(xy)Li_2(y^{-1}) - \ell_2(xy)Li_1(y^{-1}).
\end{aligned}$$

by using Exercise 1.13, (1.40), (A.1) and the following identities:

$$\begin{aligned}
Li_3(1 - y^{-1}) + Li_3(1 - y) + Li_3(y) =& Li_3(1) + \ell_2(y)Li_1(y) + \ell_3(y), \\
Li_2(1 - y^{-1}) + Li_2(1 - y) =& -\ell_2(y).
\end{aligned}$$

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