Expander graphs from groups of Kac-Moody type and generalizations

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Durham 2013

1

[1] joint work with R Blok and A Vdovina

Finite simple groups as expanders

Theorem (Kassabov-Lubotzky-Nikolov)

There exist k and $\epsilon > 0$ so that any finite simple group G (which is not a Suzuki group) admits a generating set S of size k so that the Cayley graph with respect to this is an ϵ-expander.

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Presentations of simple groups

Theorem (Guralnick-Kantor-Kassabov-Lubotzky)

All finite (quasi)simple groups of Lie type, with the possible exception of the Ree groups ${}^2G_2(3^{2e+1})$ *have presentations with 2 generators and 51 relations. All symmetric and alternating groups have presentations with 2 generators and 8 relations.*

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The approach is quite similar to the above mentioned result of Kassabov, Nikolov and Lubotzky, one generates mall rank subgroups and then the Weyl groups. The presentation is not very uniform even for a given series of groups.

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Surely this is more than enough (although it is not an absolutely trivial question to decide if two groups are isomorphic based on the Cayley table).

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*	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
b	b	a	d	c	f	e	h	g	j	i	l	k	n	m	p	o	r	q	t	s	v	u	x	w
c	c	e	a	f	b	d	i	k	g	l	h	j	o	q	m	r	n	p	u	w	s	x	t	v
d	d	f	b	e	a	c	j	l	h	k	g	i	p	r	n	q	m	o	v	x	t	w	s	u
e	e	c	f	a	d	b	k	i	l	g	j	h	q	o	r	m	p	n	w	u	x	s	v	t
f	f	d	e	b	c	a	l	j	k	h	i	g	r	p	q	n	o	m	x	v	w	t	u	s
g	g	h	m	n	s	t	a	b	o	p	u	v	c	d	i	j	w	x	e	f	k	l	q	r
h	h	g	n	m	t	s	b	a	p	o	v	u	d	c	j	i	x	w	f	e	l	k	r	q
i	i	k	o	q	u	w	c	e	m	r	s	x	a	f	g	l	t	v	b	d	h	j	n	p
j	j	l	p	r	v	x	d	f	n	q	t	w	b	e	h	k	s	u	a	c	g	i	m	o
k	k	i	q	o	w	u	e	c	r	m	x	s	f	a	l	g	v	t	d	b	j	h	p	n
l	l	j	r	p	x	v	f	d	q	n	w	t	e	b	k	h	u	s	c	a	i	g	o	m
m	m	s	g	t	h	n	o	u	a	v	b	p	i	w	c	x	d	j	k	q	e	r	f	l
n	n	t	h	s	g	m	p	v	b	u	a	o	j	x	d	w	c	i	l	r	f	q	e	k
o	o	u	i	w	k	q	m	s	c	x	e	r	g	t	a	v	f	l	h	n	b	p	d	j
p	p	v	j	x	l	r	n	t	d	w	f	q	h	s	b	u	e	k	g	m	a	o	c	i
q	q	w	k	u	i	o	r	x	e	s	c	m	l	v	f	t	a	g	j	p	d	n	b	h
r	r	x	l	v	j	p	q	w	f	t	d	n	k	u	e	s	b	h	i	o	c	m	a	g
s	s	m	t	g	n	h	u	o	v	a	p	b	w	i	x	c	j	d	q	k	r	e	l	f
t	t	n	s	h	m	g	v	p	u	b	o	a	x	j	w	d	i	c	r	l	q	f	k	e
u	u	o	w	i	q	k	s	m	x	c	r	e	t	g	v	a	l	f	n	h	p	b	j	d
v	v	p	x	j	r	l	t	n	w	d	q	f	s	h	u	b	k	e	m	g	o	a	i	c
w	w	q	u	k	o	i	x	r	s	e	m	c	v	l	t	f	g	a	p	j	n	d	h	b
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b	b	a	d	c	f	e	h	g	j	i	l	k	n	m	p	o	r	q	t	s	v	u	x	w
c	c	e	a	f	b	d	i	k	g	l	h	j	o	q	m	r	n	p	u	w	s	x	t	v
d	d	f	b	e	a	c	j	l	h	k	g	i	p	r	n	q	m	o	v	x	t	w	s	u
e	e	c	f	a	d	b	k	i	l	g	j	h	q	o	r	m	p	n	w	u	x	s	v	t
f	f	d	e	b	c	a	l	j	k	h	i	g	r	p	q	n	o	m	x	v	w	t	u	s
g	g	h	m	n	s	t	a	b	o	p	u	v	c	d	i	j	w	x	e	f	k	l	q	r
h	h	g	n	m	t	s	b	a	p	o	v	u	d	c	j	i	x	w	f	e	l	k	r	q
i	i	k	o	q	u	w	c	e	m	r	s	x	a	f	g	l	t	v	b	d	h	j	n	p
j	j	l	p	r	v	x	d	f	n	q	t	w	b	e	h	k	s	u	a	c	g	i	m	o
k	k	i	q	o	w	u	e	c	r	m	x	s	f	a	l	g	v	t	d	b	j	h	p	n
l	l	j	r	p	x	v	f	d	q	n	w	t	e	b	k	h	u	s	c	a	i	g	o	m
m	m	s	g	t	h	n	o	u	a	v	b	p	i	w	c	x	d	j	k	q	e	r	f	l
n	n	t	h	s	g	m	p	v	b	u	a	o	j	x	d	w	c	i	l	r	f	q	e	k
o	o	u	i	w	k	q	m	s	c	x	e	r	g	t	a	v	f	l	h	n	b	p	d	j
p	p	v	j	x	l	r	n	t	d	w	f	q	h	s	b	u	e	k	g	m	a	o	c	i
q	q	w	k	u	i	o	r	x	e	s	c	m	l	v	f	t	a	g	j	p	d	n	b	h
r	r	x	l	v	j	p	q	w	f	t	d	n	k	u	e	s	b	h	i	o	c	m	a	g
s	s	m	t	g	n	h	u	o	v	a	p	b	w	i	x	c	j	d	q	k	r	e	l	f
t	t	n	s	h	m	g	v	p	u	b	o	a	x	j	w	d	i	c	r	l	q	f	k	e
u	u	o	w	i	q	k	s	m	x	c	r	e	t	g	v	a	l	f	n	h	p	b	j	d
v	v	p	x	j	r	l	t	n	w	d	q	f	s	h	u	b	k	e	m	g	o	a	i	c
w	w	q	u	k	o	i	x	r	s	e	m	c	v	l	t	f	g	a	p	j	n	d	h	b
x	x	r	v	l	p	j	w	q	t	f	n	d	u	k	s	e	h	b	o	i	m	c	g	a

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b	b	a	d	c	f	e	h	g	j	i	l	k	n	m	p	o	r	q	t	s	v	u	x	w
c	c	e	a	f	b	d	i	k	g	l	h	j	o	q	m	r	n	p	u	w	s	x	t	v
d	d	f	b	e	a	c	j	l	h	k	g	i	p	r	n	q	m	o	v	x	t	w	s	u
e	e	c	f	a	d	b	k	i	l	g	j	h	q	o	r	m	p	n	w	u	x	s	v	t
f	f	d	e	b	c	a	l	j	k	h	i	g	r	p	q	n	o	m	x	v	w	t	u	s
g	g	h	m	n	s	t	a	b	o	p	u	v	c	d	i	j	w	x	e	f	k	l	q	r
h	h	g	n	m	t	s	b	a	p	o	v	u	d	c	j	i	x	w	f	e	l	k	r	q
i	i	k	o	q	u	w	c	e	m	r	s	x	a	f	g	l	t	v	b	d	h	j	n	p
j	j	l	p	r	v	x	d	f	n	q	t	w	b	e	h	k	s	u	a	c	g	i	m	o
k	k	i	q	o	w	u	e	c	r	m	x	s	f	a	l	g	v	t	d	b	j	h	p	n
l	l	j	r	p	x	v	f	d	q	n	w	t	e	b	k	h	u	s	c	a	i	g	o	m
m	m	s	g	t	h	n	o	u	a	v	b	p	i	w	c	x	d	j	k	q	e	r	f	l
n	n	t	h	s	g	m	p	v	b	u	a	o	j	x	d	w	c	i	l	r	f	q	e	k
o	o	u	i	w	k	q	m	s	c	x	e	r	g	t	a	v	f	l	h	n	b	p	d	j
p	p	v	j	x	l	r	n	t	d	w	f	q	h	s	b	u	e	k	g	m	a	o	c	i
q	q	w	k	u	i	o	r	x	e	s	c	m	l	v	f	t	a	g	j	p	d	n	b	h
r	r	x	l	v	j	p	q	w	f	t	d	n	k	u	e	s	b	h	i	o	c	m	a	g
s	s	m	t	g	n	h	u	o	v	a	p	b	w	i	x	c	j	d	q	k	r	e	l	f
t	t	n	s	h	m	g	v	p	u	b	o	a	x	j	w	d	i	c	r	l	q	f	k	e
u	u	o	w	i	q	k	s	m	x	c	r	e	t	g	v	a	l	f	n	h	p	b	j	d
v	v	p	x	j	r	l	t	n	w	d	q	f	s	h	u	b	k	e	m	g	o	a	i	c
w	w	q	u	k	o	i	x	r	s	e	m	c	v	l	t	f	g	a	p	j	n	d	h	b
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b	b	a	d	c	f	e	h	g	j	i	l	k	n	m	p	o	r	q	t	s	v	u	x	w
c	c	e	a	f	b	d	i	k	g	l	h	j	o	q	m	r	n	p	u	w	s	x	t	v
d	d	f	b	e	a	c	j	l	h	k	g	i	p	r	n	q	m	o	v	x	t	w	s	u
e	e	c	f	a	d	b	k	i	l	g	j	h	q	o	r	m	p	n	w	u	x	s	v	t
f	f	d	e	b	c	a	l	j	k	h	i	g	r	p	q	n	o	m	x	v	w	t	u	s
g	g	h	m	n	s	t	a	b	o	p	u	v	c	d	i	j	w	x	e	f	k	l	q	r
h	h	g	n	m	t	s	b	a	p	o	v	u	d	c	j	i	x	w	f	e	l	k	r	q
i	i	k	o	q	u	w	c	e	m	r	s	x	a	f	g	l	t	v	b	d	h	j	n	p
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m	m	s	g	t	h	n	o	u	a	v	b	p	i	w	c	x	d	j	k	q	e	r	f	l
n	n	t	h	s	g	m	p	v	b	u	a	o	j	x	d	w	c	i	l	r	f	q	e	k
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p	p	v	j	x	l	r	n	t	d	w	f	q	h	s	b	u	e	k	g	m	a	o	c	i
q	q	w	k	u	i	o	r	x	e	s	c	m	l	v	f	t	a	g	j	p	d	n	b	h
r	r	x	l	v	j	p	q	w	f	t	d	n	k	u	e	s	b	h	i	o	c	m	a	g
s	s	m	t	g	n	h	u	o	v	a	p	b	w	i	x	c	j	d	q	k	r	e	l	f
t	t	n	s	h	m	g	v	p	u	b	o	a	x	j	w	d	i	c	r	l	q	f	k	e
u	u	o	w	i	q	k	s	m	x	c	r	e	t	g	v	a	l	f	n	h	p	b	j	d
v	v	p	x	j	r	l	t	n	w	d	q	f	s	h	u	b	k	e	m	g	o	a	i	c
w	w	q	u	k	o	i	x	r	s	e	m	c	v	l	t	f	g	a	p	j	n	d	h	b
x	x	r	v	l	p	j	w	q	t	f	n	d	u	k	s	e	h	b	o	i	m	c	g	a

Not so pleasant ... you need 576 entries.If we do know this is a group, can we cut this down?

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b	b	a	d	c	f	e	h	g							
c	c	e	a	f	b	d	i		g				o		m
d	d	f	b	e	a	c									
e	e	c	f	a	d	b									
f	f	d	e	b	c	a									
g	g	h	m				a	b	o				c		i
h	h	g					b	a							
i	i		o				c		m				a		g
m	m		g				o		a				i		c
o	o		i				m		c				g		a

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b	b	a	d	c	f	e	h	g							
c	c	e	a	f	b	d	i		g				o		m
d	d	f	b	e	a	c									
e	e	c	f	a	d	b									
f	f	d	e	b	c	a									
g	g	h	m				a	b	o				c		i
h	h	g					b	a							
i	i		o				c		m				a		g
m	m		g				o		a				i		c
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Although we only give about 13% of the entries, the only group containing such a structure inside its Cayley table is S_4.

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c	c	e	a	f	b	d	i		g				o		m
d	d	f	b	e	a	c									
e	e	c	f	a	d	b									
f	f	d	e	b	c	a									
g	g	h	m				a	b	o				c		i
h	h	g					b	a							
i	i		o				c		m				a		g
m	m		g				o		a				i		c
o	o		i				m		c				g		a

Although we only give about 13% of the entries, the only group containing such a structure inside its Cayley table is S_4. Moreover the solution does not even require the condition that the "sudoku table" is 24×24.

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- for any $x, y \in \mathcal{A}$, $x * y$ is defined if and only if there exists $i \in I$ so that $x, y \in G_i$

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It is not very hard to show that any amalgam admits a universal completion (direct limit), that is, a completion $\hat{G}$ so that any other completion G is a canonical quotient of $\hat{G}$.

Recall the example above

*	a	b	c	d	e	f	g	h	i				m		o
a	a	b	c	d	e	f	g	h	i				m		o
b	b	a	d	c	f	e	h	g							
c	c	e	a	f	b	d	i		g				o		m
d	d	f	b	e	a	c									
e	e	c	f	a	d	b									
f	f	d	e	b	c	a									
g	g	h	m				a	b	o				c		i
h	h	g					b	a							
i	i		o				c		m				a		g
m	m		g				o		a				i		c
o	o		i				m		c				g		a

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b	b	a	d	c	f	e	h	g			
c	c	e	a	f	b	d	i		g	o	m
d	d	f	b	e	a	c					
e	e	c	f	a	d	b					
f	f	d	e	b	c	a					
g	g	h	m				a	b	o	c	i
h	h	g					b	a			
i	i		o				c		m	a	g
m	m		g				o		a	i	c
o	o		i				m		c	g	a

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I will discuss problems 1 and 3. Problem 2 is very hard and we know very few things in that direction. In some sense is related to the "word problem" in finitely presented groups.

Theorem

(Curtis-Tits) Let G be the universal version of a finite Chevalley group of twisted rank at least 3 with root system Σ, fundamental system Π and root groups X_α. For any set $J \subseteq \Pi$ take $G_J = < X_\alpha | \pm \alpha \in J >$. Then G is the universal completion of the amalgam

$$\mathcal{A} = \bigcup_{|J| \leq 2} G_J$$

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In each of these cases, the groups G_i and $G_{i,j}$ determine the amalgam uniquely. This will not be the case in general.

These theorems have generalisations (due to Abramenko Caprace and Mühlherr for the linear groups and to Bennet, Devillers, H, Köhl Mühlherr and Shpectorov for the unitary case) to a much more general class of groups, the so called Kac-Moody groups.

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Here the embeddings $\phi_{i,\{i,j\}} : G_i \to G_{i,j}$ are the natural ones. It follows that the universal completion of this amalgam is a central extension of the universal (split) Kac-Moody group with diagram Γ.

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Are all such amalgams isomorphic?

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Are all such amalgams isomorphic?

What can we say about the universal completions?

Given a CT structure, the automorphism groups of the groups in the amalgam provide a graph of groups in the sense of Bass-Serre.

Given a CT structure, the automorphism groups of the groups in the amalgam provide a graph of groups in the sense of Bass-Serre. In fact there is also a "pointing" of the graph of groups and one can prove that isomorphism classes of amalgams are in bijection to a certain section of the fundamental group of this graph of groups.

Theorem (Blok-H.)

Let Γ be a simply laced Dynkin diagram with no triangles and k a field with at least 4 elements. There is a natural bijection between isomorphism classes of CT-structures over the field k on a graph Γ and elements of the set

$\{\Phi\colon \pi(\Gamma, i_0) \to \mathbb{Z}_2 \times Aut(k) |\ \Phi \text{ is a group homomorphism}\}$

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In particular the Kac-Moody groups are those coming from a homomorphism into $Aut(k)$.

Moreover all such amalgams are non-collapsing and their universal completions are certain "unitary" subgroups of Kac-Moody groups.

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Theorem (Blok-H)

In the case of an $\tilde{A}_n$ geometry, the groups are in bijection with the group $\mathbb{Z}_2 \times Aut(k)$. The resulting groups are infinite.

The groups in the theorem above are matrix groups over the field $k\{t, t^{-1}\}_\sigma$, the (skew) Laurent polynomial ring over the field k.

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The corresponding twin building can be described in terms of vector bundles over the non commutative projective line. As such they seem to be related to number theoretic notions such as Drinfel'd modules.

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Theorem (Blok-H.-Vdovina)

G^τ has Kazdan property (T) and admits a series of quotients isomorphic to $SU_{2n}(q^s)$ with fixed n and variable s. Once we pick S, a generating set for G^τ, the resulting finite Cayley graphs form a series of expander graphs. The "new girth" of this series is not bounded.

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- Since these presentation form expanders they conform to the diameter requirements in the third motivating problem. A more detailed investigation might lead to an answer to the conjecture.
- Also some of the computational methods from Guralnick, Kantor, Kassabov and Lubotzky can be translated here and we hope they will give efficient uniform presentations with a small number of relations.