Spectral properties of Platonic graphs and a new family of trivalent expanders

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Overview

Geometric group theory preliminaries

Properties of Platonic graphs

A new family of trivalent expanders

The modular group Γ

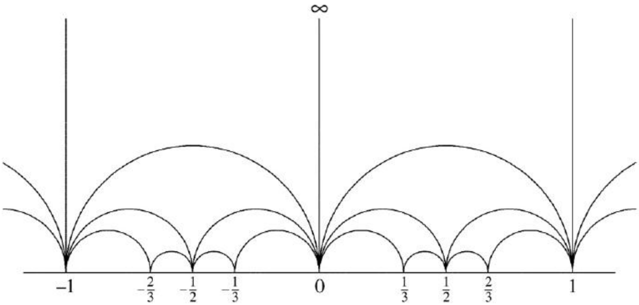
$$SL(2,\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,b,c,d \in \mathbf{Z},\ ad-bc=1 \right\}$$

$$\Gamma = PSL(2,\mathbf{Z}) = SL(2,\mathbf{Z})/\{\pm I\}$$

The elements of Γ can be seen as Möbius tansformations acting on the hyperbolic upper half plane $\mathcal{U}$.

$$z \to \frac{az+b}{cz+d}$$

The Farey tessellation



Set of vertices the extended rationals $\mathbf{Q} \cup \{\infty\}$.

Two vertices a/c and b/d are joined by an edge, a geodesic of $\mathcal{U}$, if and only if $ad - bc = \pm 1$.

The Farey tessellation

Γ is the group of Möbius tansformations leaving the Farey tessellation $\mathcal{F}$ invariant.

Every regular triangular map can be obtained as the quotient of $\mathcal{F}$ by a normal subgroup of Γ [Singerman 1988].

Principal congruence subgroups of Γ

The principal congruence subgroups are the normal subgroups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | a \equiv d \equiv \pm 1 \bmod N,\ b \equiv c \equiv 0 \bmod N \right\}$$

The special congruence subgroups are:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | c \equiv 0 \bmod N \right\}$$

Platonic graphs

We want to study the triangular tessellation $\mathcal{F}/\Gamma(N)$ on the surface $S_N = \mathcal{U}/\Gamma(N)$.

The underlying graphs $\mathcal{G}_N$ of these tessellations are often called Platonic graphs.

Arithmetic structure on Platonic graphs

[I., Singerman 2005] The vertices of $\mathcal{G}_N$ correspond to pairs $(a, b)^T \in \mathbf{Z}_N \times \mathbf{Z}_N$, with $(a, b) = 1$, meaning that (a, b) is a unitary pair of the ring $\mathbf{Z}_N$, after the identification $(a, b)^T = (-a, -b)^T$.

There is a 1-1 correspondence between the vertices of $\mathcal{G}_N$ and the cosets of $\Gamma_1(N)$ in Γ.

The number of vertices of $\mathcal{G}_N$ is:

$$|\Gamma : \Gamma_1(N)| = \frac{N^2}{2} \prod_{p|N} (1 - \frac{1}{p^2})$$

Arithmetic structure on Platonic graphs

Two vertices $(a, b)^T$ and $(c, d)^T$ are connected with an edge if and only if $ad - bc = \pm 1$.

There is a 1-1 correspondence between the set of directed edges of $\mathcal{G}_N$, and $\Gamma(N)/\Gamma \simeq PSL(2, \mathbf{Z}_N)$.

The number of directed edges of $\mathcal{G}_N$ is:

$$|\Gamma : \Gamma(N)| = \frac{N^3}{2} \prod_{p|N} (1 - \frac{1}{p^2})$$

Arithmetic structure on Platonic graphs

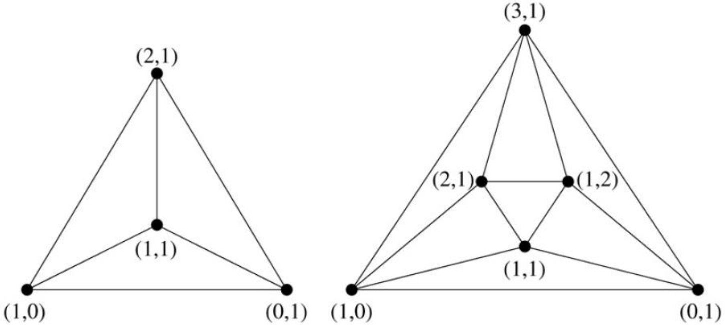
If an automorphism of $\mathcal{G}_N$ leaves a vertex (a, b) invariant, then any vertex (c, d) with $ad - cb = 0$ is also invariant under the same element.

The set of all vertices (c, d) with that property, which will be called an *axis* of $\mathcal{G}_N$, corresponds to a coset of $\Gamma_0(N)$ in Γ.

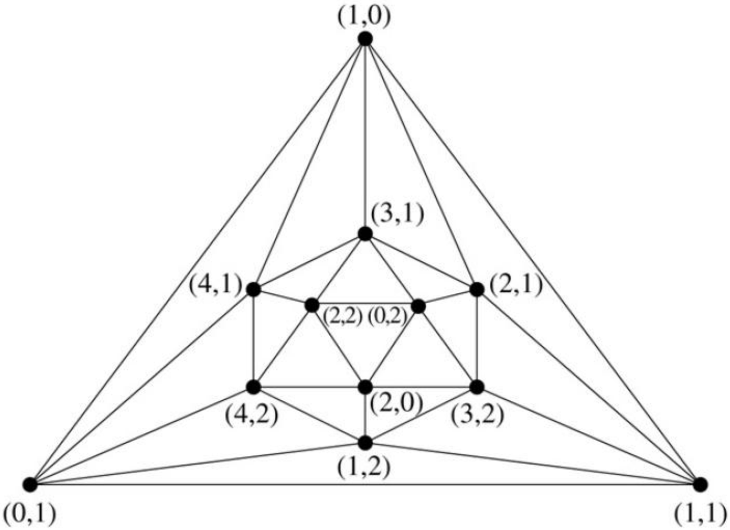
The number of axes of $\mathcal{G}_N$ is:

$$|\Gamma : \Gamma_0(N)| = N \prod_{p|N} (1 + \frac{1}{p})$$

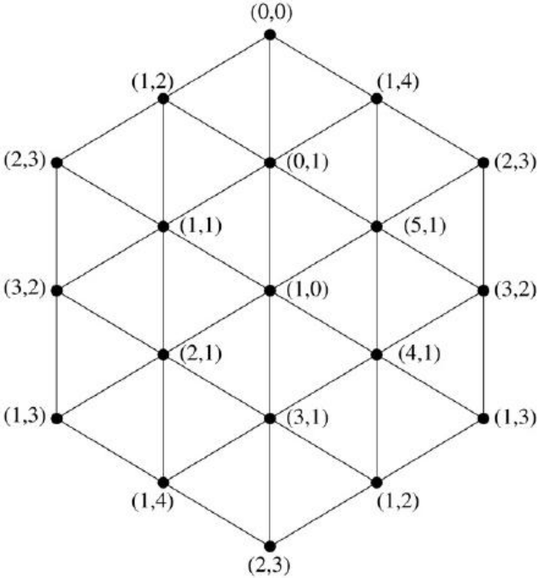
Example N=3,4



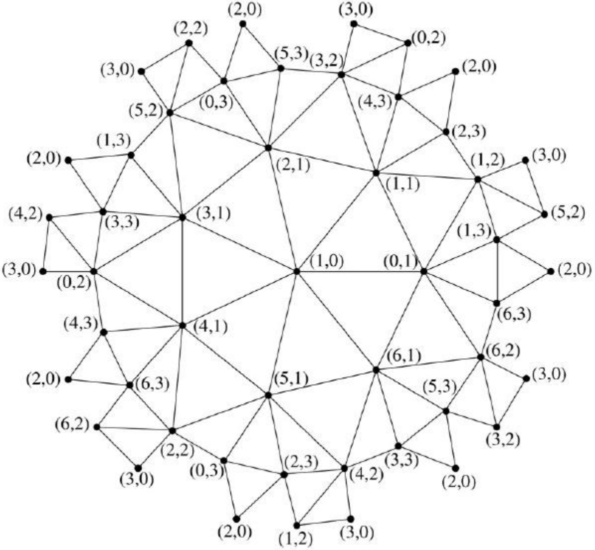
Example N=5



Example N=6



Example N=7



(2,0)
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(1,2)
(3,0)

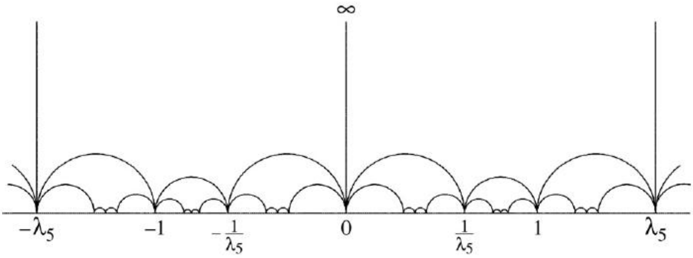
Hecke groups

The Hecke group H^q, $q = 3, 4, 5, \ldots$ is generated by the two Möbius transformations

$$H^q = \left\{ \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \lambda_q = 2\cos\frac{\pi}{q}$$

For $q = 3$ we get the modular group Γ.

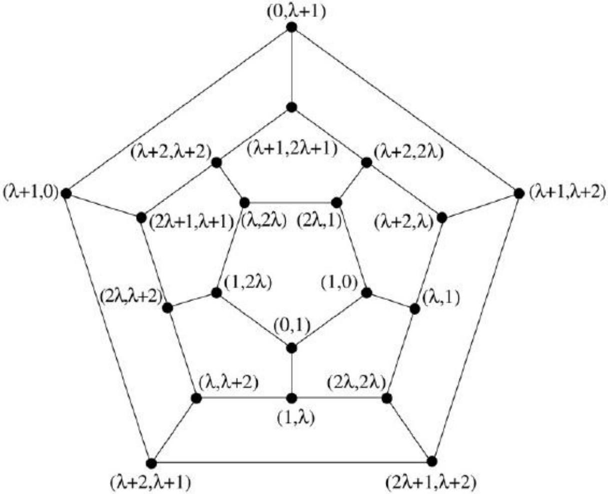
Pentagonal Hecke-Farey tessellation



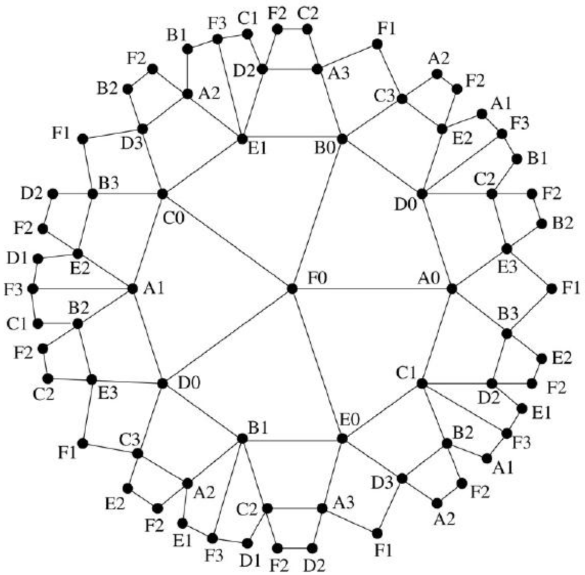
Set of vertices the set $\mathbf{Q}[\lambda_q] \cup \{\infty\}$.

Two vertices a/c and b/d are joined by an edge, a geodesic of $\mathcal{U}$, if and only if $ad - bc = \pm 1$.

Example $H^5(3)$


(0,λ+1)
(λ+2,λ+2)
(λ+1,2λ+1)
(λ+2,2λ)
(λ+1,0)
(λ+1,λ+2)
(2λ+1,λ+1)
(λ,2λ)
(2λ,1)
(λ+2,λ)
(1,2λ)
(1,0)
(2λ,λ+2)
(λ,1)
(0,1)
(λ,λ+2)
(2λ,2λ)
(1,λ)
(λ+2,λ+1)
(2λ+1,λ+2)

Example $H^4(5)$


F2
C2
C1
F3
B1
F1
F2
D2
A3
A2
B2
A2
C3
F2
A1
F3
F1
D3
E1
B0
E2
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B3
C2
D2
F2
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A3
F2
E2
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F2
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F3
D1
F2
D2

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Properties of Platonic graphs for p prime

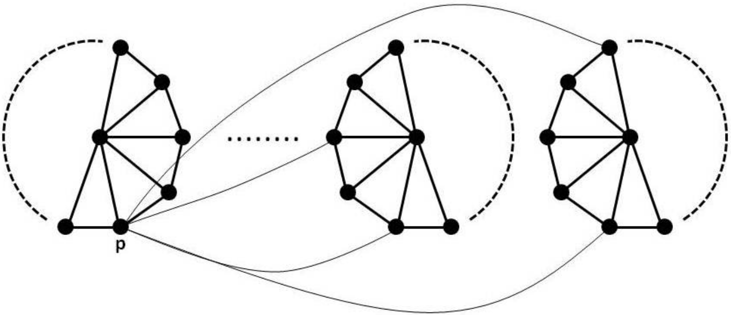
[I., Peyerimhoff, Vdovina, 2011] $\mathcal{G}_p$ is p-vertex connected

Proof of the spectral theorem in [Lanphier & Rosenhouse, 2004] without number theory.

Computation of the spectrum of a family of related graphs which are also Ramanujan.

The wheel structure

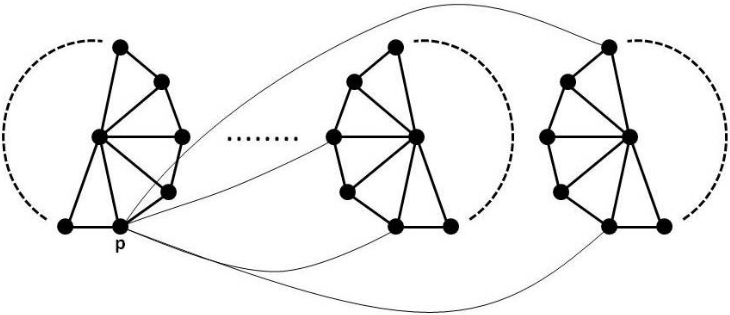
[Lanphier & Rosenhouse, 2004]



$\mathcal{G}_p$ has $p+1$ axes, $(p+1)(p-1)/2$ vertices and $p(p+1)(p-1)/2$ directed edges.

The union of the centers of the wheels form an axis.

Lemma



Every $x \in \partial W_i$ has presicely two neighbours in ∂W_j

There is a bijective map from ∂W_i to ∂W_j.

Proof

By Menger's theorem it suffices to find p vertex disjoint paths between any two vertices.

Separating three cases we find p vertex disjoint paths from [1,0] to:

- the vertices in ∂W_1
- the vertices in any ∂W_j, $j \neq 1$
- the other vertices in the axis of [1,0]

Properties of Platonic graphs for p prime

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Spectra of Platonic graphs

[Lanphier & Rosenhouse, 2004] The eigenvalues of Δ on $\mathcal{G}_p$ are:

(i) p with multiplicity 1

(ii) -1 with multiplicity p

(iii) $\sqrt{p}$ and $-\sqrt{p}$ with multiplicity $(p^2 - 2p - 3)/2$ in total

Proof

The projection

$$\pi([\lambda, \mu]) = \lambda\mu^{-1}$$

maps $\mathcal{G}_p$ onto K_p giving the eigenvalues in cases (i) and (ii).

The eigenvalues of Δ^2 are the solutions of the system

$$\Delta^2 f(v) = pf(v)$$

for all vertices of $\mathcal{G}_p$. But, all vertices corresponding to the same axis give linearly dependent equations, giving $(p+1)$ linearly indpendent equations and a

$$\frac{p^2-1}{2} - (p+1) = \frac{(p+1)(p-3)}{2}$$

dimensional solution.

Proof

Finally, prove the equality in the dimension of the eigenspaces $\mathcal{E}(\Delta, \sqrt(p)$ and $\mathcal{E}(\Delta, -\sqrt(p)$

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The modified Platonic graph $\mathcal{G}'_p$

Let $\mathcal{G}'_p$ be the Platonic graph obtained from $\mathcal{G}_p$ after the removal of an axis.

$\mathcal{G}'_p$ is a Cayley graph over $\Gamma_0(p)/\Gamma(p)$.

The modified Platonic graph $\mathcal{G}'_p$

$\mathcal{G}'_p$ is $(p-1)$-vertex connected.

The eigenvalues of Δ on $\mathcal{G}'_p$ are:

(i) $p-1$ with multiplicity 1
(ii) -1 with multiplicity p
(iii) 0 with multiplicity $(p-3)/2$
(iv) $\sqrt{p}$ and $-\sqrt{p}$ with multiplicity $(p-1)(p-3)/4$, each.

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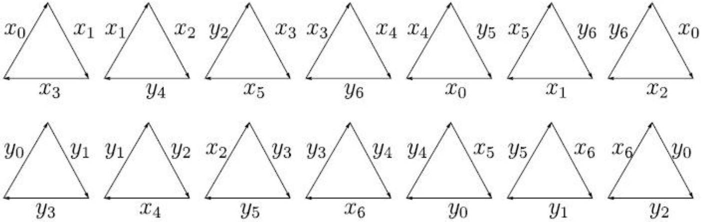
The construction of the trivalent expander

[Peyerimhoff & Vdovina, 2011]



Explicit construction of a simplicial complex of 14 triangles.

The fundamental group

[Cartwright et. al, 1993]

$$G =< x_0, x_1 | r_1, r_2, r_3 >$$

$$r_1 = x_1 x_0 x_1 x_0 x_1 x_0 x_1^{-3} x_0^{-3}$$

$$r_2 = x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_1^2 x_0^{-1} x_1 x_0 x_1$$

$$r_3 = x_1^3 x_0^{-1} x_1 x_0 x_1 x_0^2 x_1^2 x_0 x_1 x_0$$

From the structure of the simplicial complex we infer that the fundamental group G has the Kazhdan T property [Bekka, de la Harpe & Valette, 2008].

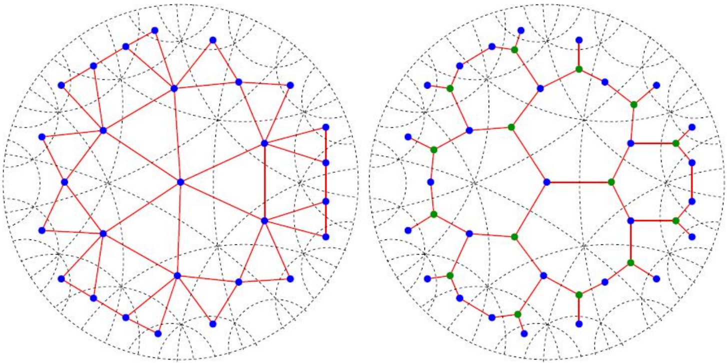
A six-valent expander

A six-valent expander is obtained as a sequence of Cayley graphs of finite index normal subgroups of G, with generators

$$S = \{x_0^{\pm 1}, x_1^{\pm 1}, (x_1^{-1} x_0^{-1})^{\pm 1}\}$$

[Ballmann & Swiatkowski, 1997] and [Bekka, de la Harpe & Valette, 2008].

The Y-*Delta* transformation



A trivalent expander

[I., Peyerimhoff, Vdovina, 2011]

(i) The graphs X_k are six-valent expanders with spectrum in $[-3, C] \cup \{6\}$ with $C < 6$.

(ii) The graphs T_k are trivalent expanders with spectrum in $[-\sqrt{C+3}, \sqrt{C+3}] \cup \{\pm 3\}$.

The intersection with Platonic graphs

[I., Peyerimhoff, Vdovina, 2011]

The graph T_2 is the dual of $\mathcal{G}_8$ in the unique surface of genus 5 with maximal automorphism group of order 192. There is no other isomorphism between T_k and a dual Platonic graph.

The $\mathcal{G}_8$