DISCONTINUOUS PETROV-GALERKIN (DPG) METHOD WITH OPTIMAL TEST FUNCTIONS
Fundamentals

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Drama in Four Acts

Act 1: The Big (Functional Analysis) Picture

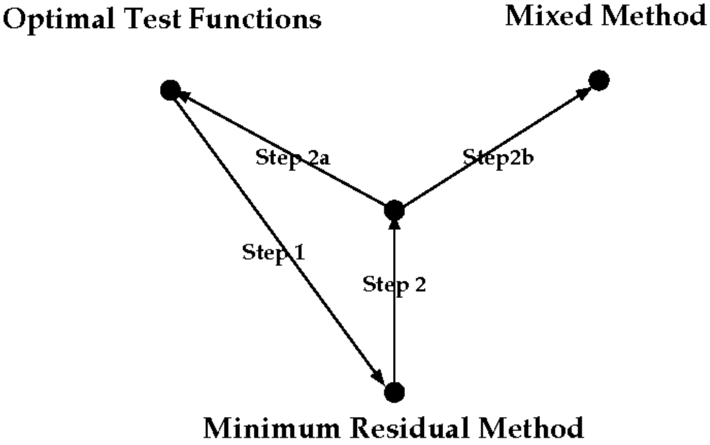
Act 2: Broken Test Spaces and Primal DPG Method

Act 3: Robust Primal DPG Method: Controlling the Convergence (Trial) Norm (new!)

Act 4: Ultraweak Variational Formulation

The Big (Functional Analysis) Picture

Three Interpretations of DPG



Abstract variational problem

U, V - Hilbert spaces,
$b(u,v)$ - bilinear (sesquilinear) continuous form on $U \times V$,

$$|b(u,v)| \le \underbrace{\|b\|}_{=:M} \|u\|_U \, \|v\|_V ,$$

$l(v)$ - linear (antilinear) continuous functional on V,

$$|l(v)\| \le \|l\|_{V'} \, \|v\|$$

The abstract variational problem:

$$\left\{ \begin{array}{l} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \end{array} \right. \quad \Leftrightarrow \quad \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \quad v \in V \end{array}$$

Banach - Babuška - Nečas Theorem

If b satisfies the inf-sup condition ($\Leftrightarrow B$ is bounded below),

$$\inf_{\|u\|_U=1} \sup_{\|v\|_V=1} |b(u,v)| =: \gamma > 0 \quad \Leftrightarrow \quad \sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

and l satisfies the compatibility condition:

$$l(v) = 0 \quad \forall v \in V_0$$

where

$$V_0 := \mathcal{N}(B') = \{v \in V \ : \ b(u,v) = 0 \quad \forall u \in U\}$$

then the variational problem has a unique solution u that satisfies the stability estimate:

$$\|u\| \leq \frac{1}{\gamma} \|l\|_{V'} \,.$$

Proof: Direct interpretation of Banach Closed Range Theorem*.

*see e.g. Oden, D, *Functional Analysis*, Chapman & Hall, 2nd ed., 2010, p.518

Petrov-Galerkin Method and Babuška Theorem

$U_h \subset U, V_h \subset V$, $\dim U_h = \dim V_h$ - finite-dimensional trial and test (sub)spaces

$$\begin{cases} u_h \in U_h \\ b(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h \end{cases}$$

Theorem (Babuška[†]).
The *discrete inf-sup condition*

$$\sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U$$

implies existence, uniqueness and discrete stability

$$\|u_h\|_U \leq \gamma_h^{-1} \|l\|_{V_h'}$$

[†] I. Babuska, "Error-bounds for Finite Element Method.", *Numer. Math*, **16**, 1970/1971.

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$$\|u_h\|_U \leq \gamma_h^{-1} \|l\|_{V_h'}$$

and convergence

$$\|u - u_h\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|u - w_h\|_U$$

(Uniform) discrete stability and approximability imply convergence.

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Optimal test functions

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unless [‡] we employ special test functions that *realize* the supremum in the inf-sup condition:

$$v_h = \text{arg max}_{v \in V} \frac{|b(u_h, v)|}{\|v\|}$$

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Recall that the Riesz operator $R_V : V \to V'$ is an isometry. Then:

$$\begin{aligned} \sup_v \frac{|b(u_h,v)|}{\|v_h\|} &= \|Bu_h\|_{V'} = \| \underbrace{R_V^{-1} B u_h}_{=v_h} \|_V = \frac{(R_V^{-1} B u_h, v_h)_V}{\|v\|_V} \\ &= \frac{\langle B u_h, v_h \rangle}{\|v\|_V} = \frac{b(u_h, v_h)}{\|v\|_V} \end{aligned}$$

Variational definition of v_h: $\quad \begin{cases} v_h \in V \\ (v, \delta v)_V = b(u_h, \delta v) \quad \forall \delta v \in V \,. \end{cases}$

The operator $T := R_V^{-1} B \; : \; U_h \to V$ will be called the *trial to test operator*.

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DPG is a Minimum Residual Method

With the optimal test functions in place, $\gamma_h \geq \gamma$, and the Galerkin method is automatically stable. Trade now the original norm in U for an *energy norm*§:

$$\|u\|_E := \|R_V^{-1} Bu\|_V = \|Bu\|_{V'} = \sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V}$$

Two points:

- With respect to the new, energy norm, *both* continuity constant M and inf-sup constant γ are unity.
- The use of optimal test functions (their construction is independent of the choice of trial norm) implies that $\gamma_h \geq \gamma = 1$.

Thus, by the Babuška Theorem,

$$\|u - u_h\|_E \leq \underbrace{\frac{M}{\gamma_h}}_{=1} \inf_{w_h \in U_h} \|u - w_h\|_E \,.$$

In other words, FE solution u_h *is the best approximation* of the exact solution u in the energy norm. We have arrived through a back door at a *Minimum Residual Method*.

§Residual norm really...

Moral of the story

The minimum residual method,
with the residual measured in the dual test norm,
is *the most stable* Petrov-Galerkin method
you can have.

DPG is a minimum residual method ¶

$$\left\{\begin{array}{l} u \in U \\ b(u,v) = l(v) \quad v \in V \end{array}\right. \quad \Leftrightarrow \quad \begin{array}{l} Bu = l \quad B : U \to V' \\ \langle Bu, v\rangle = b(u,v) \end{array}$$

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$$\frac{1}{2}\|Bu_h - l\|^2_{V'} \to \min_{u_h \in U_h}$$

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- **Minimum residual method reformulated:**

$$\frac{1}{2} \| Bu_h - l \|_{V'}^2 = \frac{1}{2} \| R_V^{-1} (Bu_h - l) \|_V^2 \to \min_{u_h \in U_h}$$

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where

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DPG is a mixed method

An alternate route [‖],

$$\Big(\underbrace{R_V^{-1}(Bu_h - l)}_{=:\psi(\text{error representation function})}, R_V^{-1} B\delta u_h \Big)_V = 0 \quad \delta u_h \in U_h$$

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$$\begin{cases} \psi = R_V^{-1}(Bu_h - l) \\ (\psi, R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h \end{cases}$$

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or

$$\begin{cases} (\psi, \delta v)_V - b(u_h, \delta v) & = -l(\delta v) \quad \forall \delta v \in V \\ b(\delta u_h, \psi) & = 0 \qquad\qquad \forall \delta u_h \in U_h \end{cases}$$

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DPG method, a summary so far

- Stiffness matrix is always hermitian and positive-definite (it is a generalization of the least squares method...).

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- The energy norm of the FE error $u - u_h$ equals the residual and can be computed,

$$\|u - u_h\|_E = \|Bu - Bu_h\|_{V'} = \|l - Bu_h\|_{V'} = \|R_V^{-1}(l - Bu_h)\|_V = \|\psi\|_V$$

  where the *error representation function* ψ comes from

$$\left\{ \begin{array}{l} \psi \in V \\ (\psi, \delta v)_V = \langle l - Bu_h, \delta v \rangle = l(\delta v) - b(u_h, \delta v), \quad \delta v \in V \end{array} \right.$$

  No need for a-posteriori error estimation, note the connection with implicit a-posteriori error estimation techniques **

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- Is the inversion of Riesz operator (computation of the optimal test functions, energy error) feasible ?

DPG method, a summary

- A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get get different methods.
- How to choose the test norm in a systematic way ?
- Is the inversion of Riesz operator (computation of the optimal test functions, energy error) feasible ?
- Being a Ritz method, DPG does not experience any preasymptotic limitations.

Wait a minute!

You cannot compute the optimal test functions!

Approximate optimal test functions

Take a finite-dimensional *enriched* test space: $\tilde{V} \subset V$, dim $\tilde{V} >>$ dim U_h, and invert the Riesz operator approximately,

$$\left\{ \begin{array}{l} \tilde{v}_h \in \tilde{V} \\ (\tilde{v}_h, \widetilde{\delta v})_V = b(u_h, \widetilde{\delta v}) \quad \forall \widetilde{\delta v} \in \tilde{V} \, . \end{array} \right.$$

This leads to an *approximate trial to test operator:*

$$\tilde{T} \, : \, U_h \to \tilde{V} \quad \tilde{T} u_h := \tilde{v}_h$$

and *approximate optimal test space*:

$$\tilde{V}_h := \tilde{T} U_h \, .$$

Some stability must be lost. How much ?

Approximate mixed problem

$$\left\{\begin{array}{lll} \tilde{\psi} \in \tilde{V},\ \tilde{u}_h \in U_h & & \\ (\tilde{\psi}, \widetilde{\delta\psi})_V - b(\tilde{u}_h, \widetilde{\delta\psi}) & = -l(\widetilde{\delta\psi}) & \widetilde{\delta\psi} \in \tilde{V} \\ b(\delta u_h, \tilde{\psi}) & = 0 & \delta u_h \in U_h \end{array}\right.$$

The (discrete) inf sup condition must be satisfied:

$$\sup_{\tilde{\delta\psi} \in \tilde{V}} \frac{|b(u_h, \tilde{\delta\psi})|}{\|\tilde{\delta\psi}\|} \geq \gamma_h \|u_h\|$$

Back to square one ??

Fortin operator

Coming up with a Fortin operator

$$\tilde{\Pi} : V \to \tilde{V}$$

such that

$$\|\tilde{\Pi}v\|_V \leq C\|v\|_V$$

and

$$b(u_h, \tilde{\Pi}v - v) = 0 \quad \forall u_h \in U_h$$

solves the problem [††]

[††] J. Gopalakrishnan and W. Qiu. "An analysis of the practical DPG method.", *Math. Comp.*, 2013 (posted May 31, 2013).

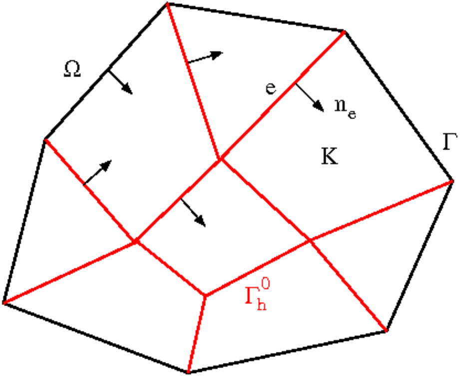
Act Two

Broken Test Spaces and

Primal DPG Method

Primal DPG method

Standard assumptions: $\Omega \subset \mathbb{R}^N$ Lipschitz domain,



Elements:K
Edges:e
Skeleton:$\Gamma_h = \bigcup_K \partial K$
Internal skeleton:$\Gamma_h^0 = \Gamma_h - \partial\Omega$

Primal DPG method

Given $f \in L^2(\Omega)$, consider the model problem,

$$\left\{ \begin{array}{rll} u & = 0 & \text{on } \Gamma := \partial\Omega \\ -\Delta u & = f & \text{in } \Omega \end{array} \right.$$

Multiply the PDE with a test function v, integrate over each element K, integrate by parts and sum up over all elements,

$$\sum_K \int_K \nabla u \cdot \nabla v + \sum_K \int_{\partial K} \frac{\partial u}{\partial n} v = \sum_K \int f\, v$$

The boundary term represents jumps,

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v = \sum_e \int_e \frac{\partial u}{\partial n_e} [v]_e$$

where

$$[v]_e = \left\{ \begin{array}{ll} v_+ - v_- & e \subset \Omega \\ v & e \subset \Gamma \end{array} \right.$$

Primal DPG method

This leads to the variational problem:

$$\begin{cases} u \in H^1(\Omega),\, \hat{t} \in H^{-1/2}(\Gamma_h) \\ (\boldsymbol{\nabla} u, \boldsymbol{\nabla}_h v) - \langle \hat{t}, v \rangle_{\Gamma_h} = (f, v) \quad v \in H^1(\Omega_h) \end{cases}$$

where

$$H^{-1/2}(\Gamma_h) = \text{trace of } H(\mathrm{div}, \Omega) \text{ on } \Gamma_h$$

equipped with the quotient norm.

Theorem ‡‡

The variational problem above is well posed with a mesh independent inf-sup constant γ.

‡‡ L. Demkowicz and J. Gopalakrishnan. "A primal DPG method without a first order reformulation", *Comput. Math. Appl.*, 66(6):1058–1064, 2013

The main point

The test norm is *localizable*, i.e.

$$\underbrace{\|v\|^2_{H^1(\Omega_h)}}_{\text{global norm}} = \sum_K \underbrace{\|v|_K\|^2_{H^1(K)}}_{\text{local norm}} .$$

The (approximate) inversion of the Riesz operator is done locally (elementwise)

DPG element stiffness matrix and load vector

$$u_h = \sum_{i=1}^{N} u_i e_i, \quad v_h \approx \sum_{j=1}^{M} v_j g_j, \quad M >> N$$

Computation of (approximate) optimal test function $v = Te_i$,

$$\sum_j \underbrace{(g_j, g_l)}_{\text{Gram matrix } \boldsymbol{G}} \; v_j^i = \underbrace{b(e_i, g_l)}_{\text{expanded stiffness matrix } \boldsymbol{B}}, \quad l = 1, \ldots, M$$

or

$$\boldsymbol{v} = \boldsymbol{G}^{-1}\boldsymbol{B}\delta\boldsymbol{u}$$

The DPG stiffness matrix and load vector:

$$\boldsymbol{v}^T\boldsymbol{B}\boldsymbol{u} = (\boldsymbol{G}^{-1}\boldsymbol{B}\delta\boldsymbol{u})^T\boldsymbol{B}\boldsymbol{u} = (\delta\boldsymbol{u})^T\boldsymbol{B}^T\boldsymbol{G}^{-1}\boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{v}^T\boldsymbol{b} = (\boldsymbol{G}^{-1}\boldsymbol{B}\delta\boldsymbol{u})^T\boldsymbol{b} = (\delta\boldsymbol{u})^T\boldsymbol{B}^T\boldsymbol{G}^{-1}\boldsymbol{b}$$

Same result with the mixed method interpretation

$$\begin{pmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ \boldsymbol{B}^T & \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$$

Condensing out error indication function $\boldsymbol{\psi}$,

$$\boldsymbol{\psi} = \boldsymbol{G}^{-1}(\boldsymbol{B}\boldsymbol{u} - \boldsymbol{b})$$

we get again,

$$\boldsymbol{B}^T\boldsymbol{G}^{-1}\boldsymbol{B}\,\boldsymbol{u} = \boldsymbol{G}^{-1}\boldsymbol{B}\boldsymbol{b}$$

Primal DPG Formulation for the Poisson problem

Group unknown (watch for the overloaded symbol):

$$u_h := (\underbrace{u_h}_{\text{field}}, \underbrace{\hat{t}_h}_{\text{flux}})$$

Mixed system:

$$\begin{pmatrix} \boldsymbol{G} & -\boldsymbol{B}_1 & -\boldsymbol{B}_2 \\ \boldsymbol{B}_1^T & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{B}_2^T & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{u} \\ \hat{\boldsymbol{t}} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{b} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$$

where $\boldsymbol{B}_1, \boldsymbol{B}_2$ correspond to $(\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}_h \tilde{v})$ and $-\langle \hat{t}_h, \tilde{v} \rangle$, resp.
Eliminate $\boldsymbol{\psi}$ to get the DPG system:

$$\begin{pmatrix} \boldsymbol{B}_1^T \boldsymbol{G}^{-1} \boldsymbol{B}_1 & \boldsymbol{B}_1^T \boldsymbol{G}^{-1} \boldsymbol{B}_2 \\ \boldsymbol{B}_2^T \boldsymbol{G}^{-1} \boldsymbol{B}_1 & \boldsymbol{B}_2^T \boldsymbol{G}^{-1} \boldsymbol{B}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \hat{\boldsymbol{t}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_1^T \boldsymbol{G}^{-1} \boldsymbol{b} \\ \boldsymbol{B}_2^T \boldsymbol{G}^{-1} \boldsymbol{b} \end{pmatrix}$$

Primal DPG method

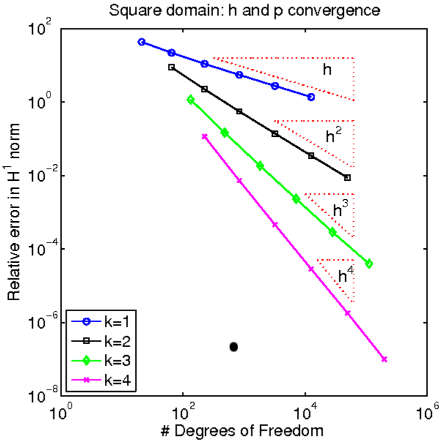
Neglecting the error steming from the approximation of optimal test function (computation of residual), we have,

$$\left(\|u-u_h\|^2_{H^1(\Omega)} \quad +\|\hat{t}-\hat{t}_h\|^2_{H^{-1/2}(\Gamma_h)}\right)^{1/2}$$
$$\leq \frac{1}{\gamma} \underbrace{\inf_{w_h, r_h} \left(\|u-w_h\|^2_{H^1(\Omega)} + \|\hat{t}-r_h\|^2_{H^{-1/2}(\Gamma_h)}\right)^{1/2}}_{\text{best approximation error}}$$

Additionally,

$$\left(\|u-u_h\|^2_{H^1(\Omega)} \quad +\|\hat{t}-\hat{t}_h\|^2_{H^{-1/2}(\Gamma_h)}\right)^{1/2}$$
$$\leq \frac{1}{\gamma} \underbrace{\sup_{v\in H^1(\Omega_h)} \frac{|(\boldsymbol{\nabla} u_h, \boldsymbol{\nabla}_h v) - \langle \hat{t}_h, v\rangle_{\Gamma_h}|}{\|v\|_{H^1(\Omega_h)}}}_{\text{computable residual}}$$
$$= \frac{1}{\gamma}\left(\sum_K \|\psi_K\|^2_{H^1(K)}\right)^{1/2}$$

2D convergence rates



Primal DPG method, 3D examples

- $$\left.\begin{array}{l}\text{Poisson problem}\\ \text{Reaction-dominated diffusion}\\ \text{Convection-dominated diffusion}\end{array}\right\}\quad \text{div-grad problems}$$
- Maxwell equations - curl-curl problem

All examples have been implemented within *hp3d*, a general 3D FE code supporting:

- coupled problems involving $H^1, H(\text{curl})$ and $H(\text{div})$-conforming elements.
- hybrid meshes consisting of hexas, tets, prisms and pyramids,
- *anisotropic* hp-refinements.

Ask me about the code...

FE discretization for div-grad problems

Hexahedral meshes
H^1 element for field u_h:

$$\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^p,$$

Trace of $H(\text{div})$ element:

$$(\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^{p-1}) \times (\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1}) \times (\mathcal{P}^{p-1} \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p)$$

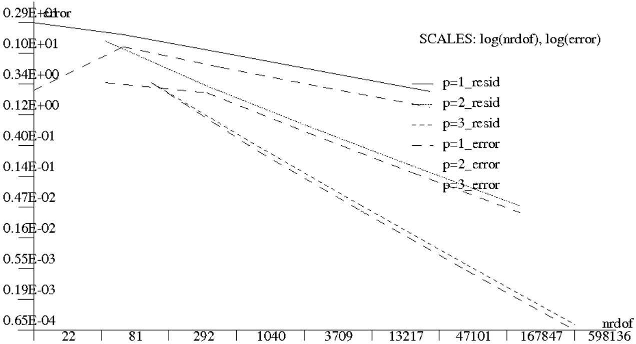
for flux $\hat{t}_h$, and the enriched element:

$$\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p},$$

for test function v_h.
In reported experiments: $p = 1, 2, 3$, $\Delta p = 2$.

Poisson problem, smooth solution, uniform refinements

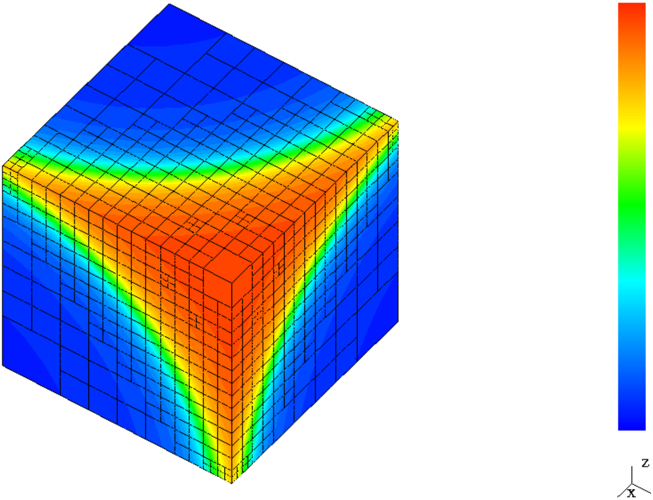
Rectangular domain $\Omega = (0,1) \times (0,2) \times (0,1)$,
Smooth solution: $u = \sin \pi x \sin \pi y \sin \pi z$
Boundary condition: $u = 0$.

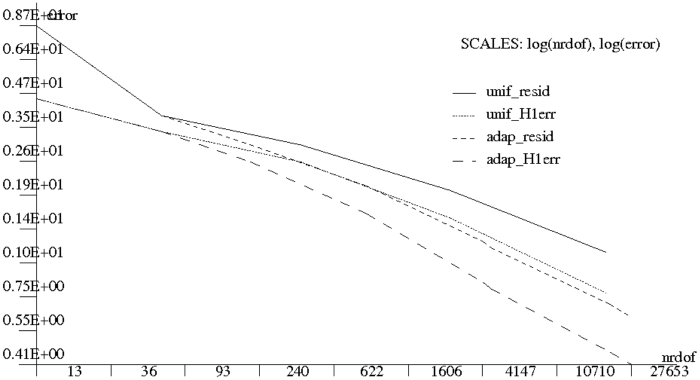


Residual versus H^1 error.

Poisson problem, manufactured shock solution

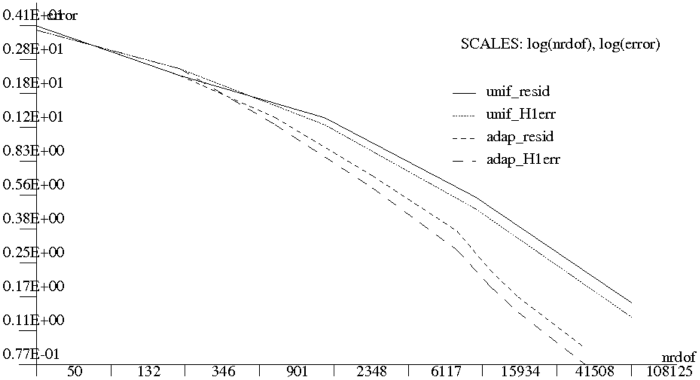
BC: $u = u_0$.





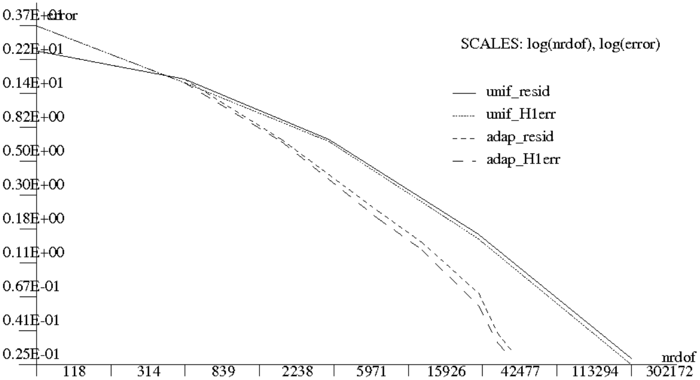
Convergence history for the residual and H^1 error

Shock solution, uniform and h-adaptive refinements, $p = 2$



Convergence history for the residual and H^1 error

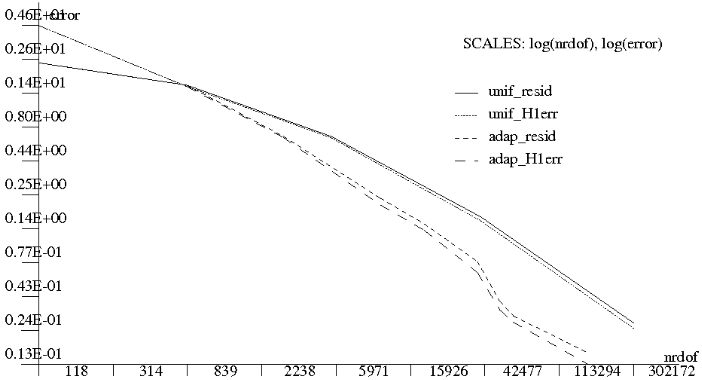
Shock solution, uniform and h-adaptive refinements, $p = 3$



Convergence history for the residual and H^1 error

Shock solution, $p = 3$, Mixed BC

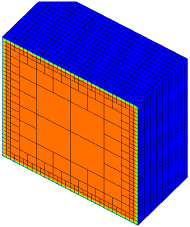
Mixed BC: trace: bottom, top, flux: sides.

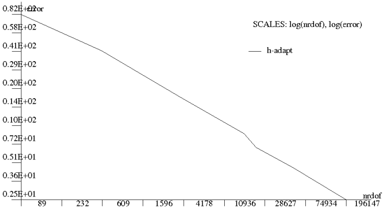


Convergence history for the residual and H^1 error

Reaction-dominated diffusion, $p = 2$.

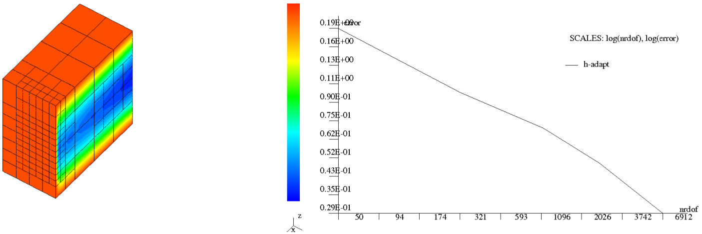
$$\begin{cases} u &= 0 \quad \text{on } \Gamma \\ -\epsilon^2 \Delta u + u &= 1 \quad \text{in } \Omega \end{cases}$$





$\epsilon = 0.01$, left: solution after 7 iterations, right: convergence history

$$
\begin{cases}
-\epsilon^2 \Delta u - u & = \sin \pi y \, \sin \pi z \qquad \text{at } x = 0 \\
u & = 0 \qquad\qquad\qquad\quad \text{on the rest of } \Gamma \\
-\epsilon^2 \Delta u + \frac{\partial u}{\partial x} & = 0 \qquad\qquad\qquad\quad \text{in } \Omega
\end{cases}
$$



$\epsilon = 0.01$, left: solution after 5 iterations, right: convergence history

Maxwell equations.

Assume

$$J_S^{\text{imp}} = n \times H^{imp}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\begin{cases} E \in H(\text{curl}, \Omega),\ n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega),\ n \times \hat{h} = n \times (-i\omega H^{imp}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \nabla \times E, \nabla_h \times F) + ((-\omega^2 \epsilon + i\omega\sigma)E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega(J^{\text{imp}}, F) \\ \hfill \forall F \in H(\text{curl}, \Omega_h) \,. \end{cases}$$

FE discretization for curl-curl problem

Hexahedral meshes
$H(\text{curl})$ element for electric field E:

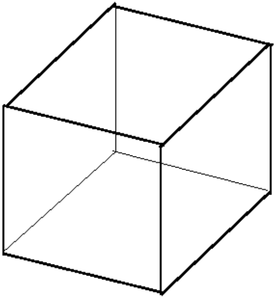
$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1})$$

and trace of the same element for flux (surface current) $\hat{h}$.
Same element for the enriched space but with order $p + \Delta p$.
In reported experiments: $p = 2$, $\Delta p = 2$.

DPG Supports Adaptivity with No Preasymptotic Behavior

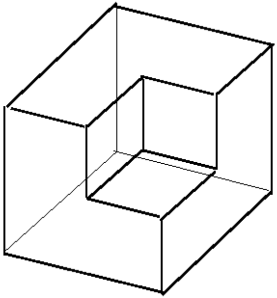
A 3D Maxwell example

Take a cube $(0, 2)^3$



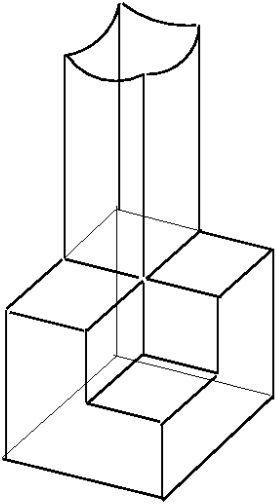
Fichera corner

Divide it into eight smaller cubes and remove one:



Fichera corner microwave

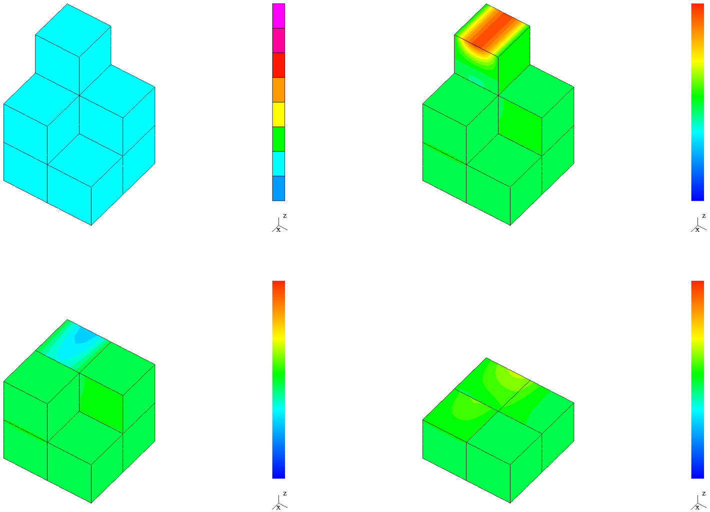
Attach a waveguide:



$\epsilon = \mu = 1, \sigma = 0$
$\omega = 5$(1.6 wavelengths in the cube)

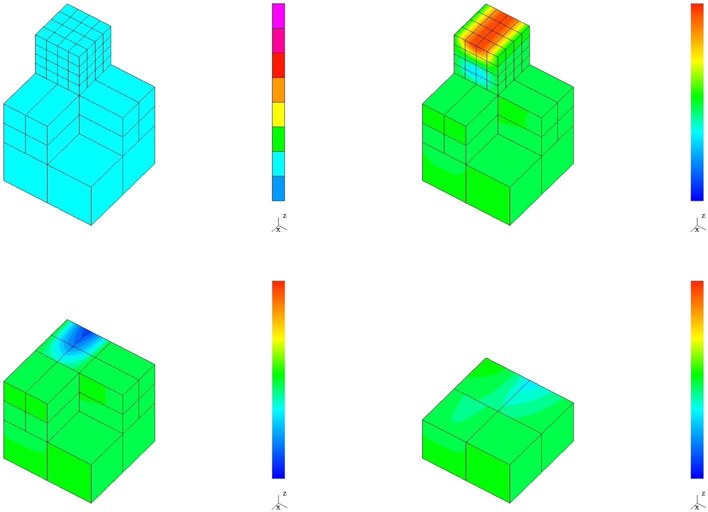
Cut the waveguide and use the lowest propagating mode for BC along the cut.

Fichera corner microwave, $p = 2$.



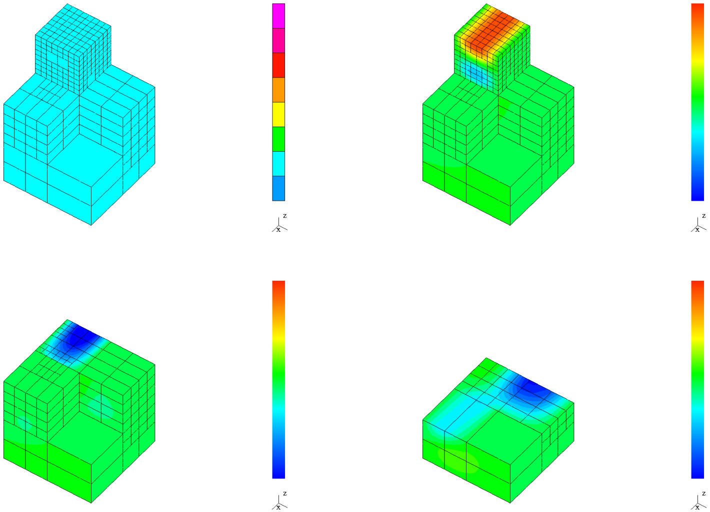
Initial mesh and real part of E_1

Fichera corner microwave, $p = 2$.



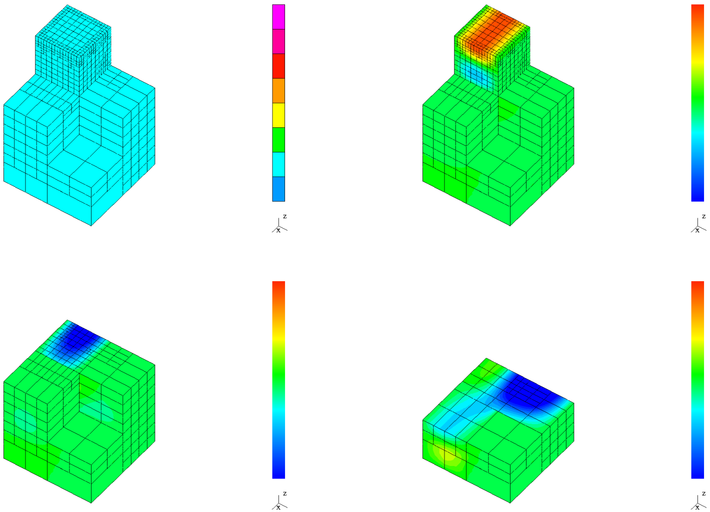
Mesh and real part of E_1 after two refinements

Fichera corner microwave, $p = 2$.



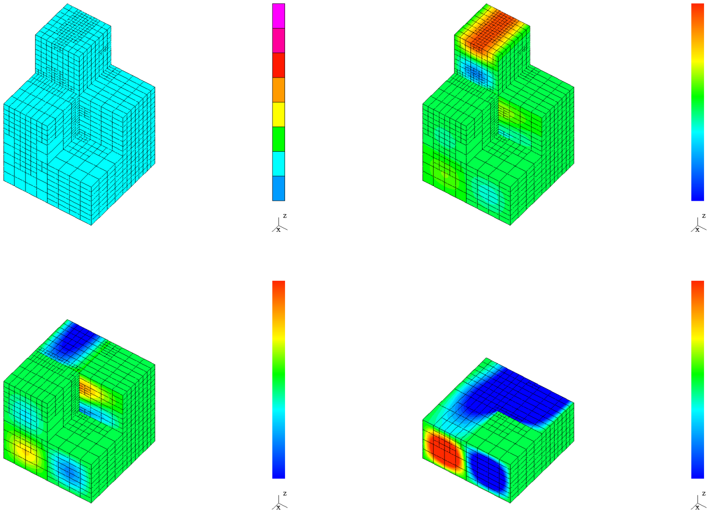
Mesh and real part of E_1 after four refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after six refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after eight refinements

Robust DPG Method: Controlling the Convergence (Trial) Norm

The simplest singular perturbation problem: reaction-dominated diffusion

L.D. and I. Harari, "Primal DPG Method for Reaction dominated Diffusion", in preparation.

The simplest singular perturbation problem: Reaction-dominated diffusion

$$\begin{cases} u &= 0 \quad \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x) u &= f \quad \text{in } \Omega \end{cases}$$

where $0 < c_0 \leq c(x) \leq c_1$.
Standard variational formulation:

$$\begin{cases} u \in H^1(\Omega) \\ \epsilon^2 (\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{cases}$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|^2_{\epsilon^k} := \epsilon^k \|\nabla u\|^2 + \|c^{1/2} u\|^2, \quad k = 2$$

Convergence in "balanced" norm

Fact: Under favorable regularity conditions, the solution is *uniformly* bounded in data f in a "balanced" norm :

$$\|u\|_\epsilon^2 := \epsilon \|\nabla u\|^2 + \|c^{1/2} u\|^2$$

i.e.

$$\|u\|_\epsilon \lesssim \|f\|_{\text{appropriate}}$$

Question: Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm ?

$$\|u - u_h\|_\epsilon + \|\hat{t} - \hat{t}_h\|_? \lesssim \inf_{w_h} \|u - w_h\|_\epsilon + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|_?$$

R. Lin and M. Stynes, "A balanced finite element method for singularly perturbed reaction-diffusion problems", *SIAM J. Numer. Anal.*, **50(5):** 2729–2743, 2012.

A bit of history: Optimal test functions of Barret and Morton

For each $w \in U_h$, determine the corresponding v_w that solves the auxiliary variational problem:

$$\left\{ \begin{array}{l} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\nabla \delta u, \nabla v_w) + (c\,\delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\nabla \delta u, w) + (c\,\delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{array} \right.$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, "balanced" norm:

$$\epsilon^2(\nabla(u - u_h), \nabla v_w) + (c\,(u - u_h), v_w) = 0 \quad \Longrightarrow \epsilon(\nabla(u - u_h), \nabla v_u) + (c\,(u - u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

- J.W. Barret and K. W. Morton, "Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems", *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).
- L. D. and J. T. Oden, "An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable", *Journal of Computational Physics*, **68(1):** 188–273, 1986.

Constructing optimal test norm

Theorem

Let v_u *be the Barret-Morton optimal test function corresponding to* u. *Let* $\|v_u\|_V$ *be a test norm such that*

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \textit{BAE estimate}$$

Proof:

$$\begin{aligned}
\|u\|_\epsilon^2 &= \epsilon(\nabla u, \nabla u) + (cu, u) = \epsilon^2(\nabla u, \nabla v_u) + (cu, v_u) \\
&= b((u,\hat{t}), v_u) \leq \frac{b((u,\hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V \\
&\leq \sup_v \frac{b((u,\hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u,\hat{t})\|_E \|v_u\|_V \\
&\lesssim \|(u,\hat{t})\|_E \|u\|_\epsilon
\end{aligned}$$

L. D., M. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems," *SIAM J. Num. Anal*, **51:** 2514–2537, 2013.

Constructing optimal test norm

The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2} v\|^2$$

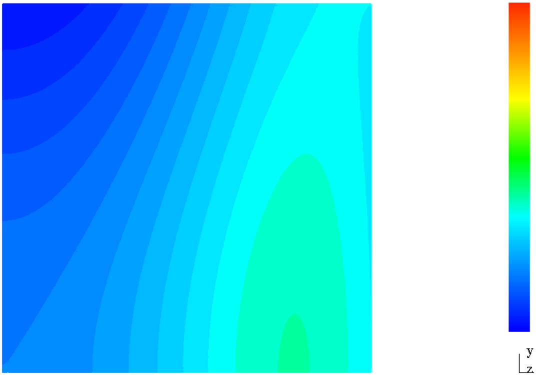
Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\{1, \frac{\epsilon^3}{h^2}\} \|c^{1/2} v\|^2$$

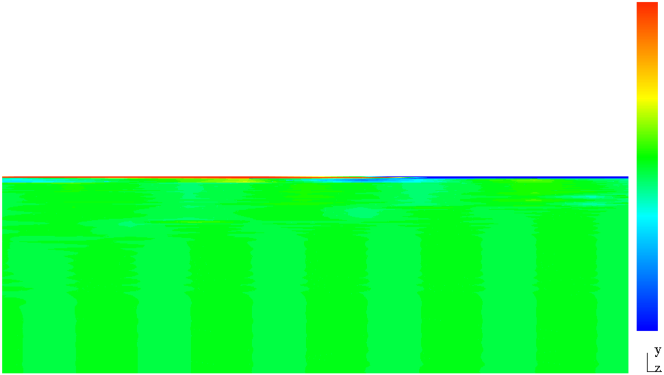
Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



The functions exhibits strong boundary layers invisible in this scale.

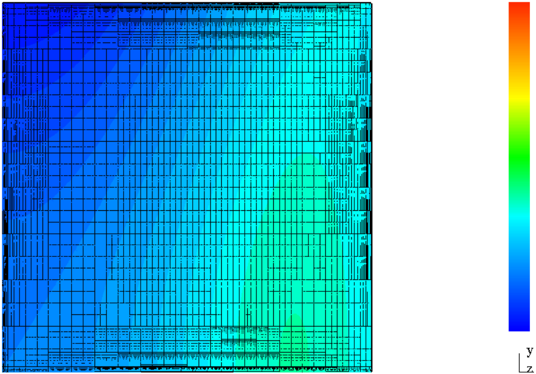
Range: (-0.6,0.6)

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



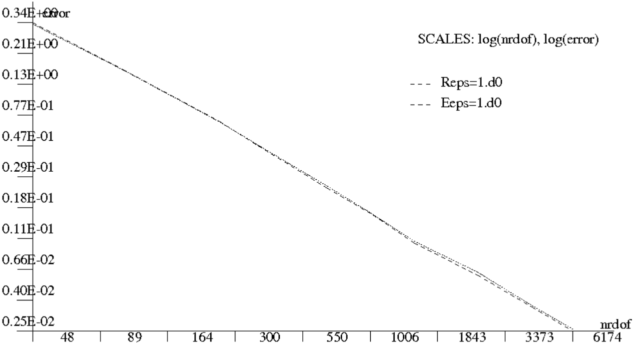
Zoom on the north boundary layer.

Optimal mesh for $\epsilon = 10^{-1}$



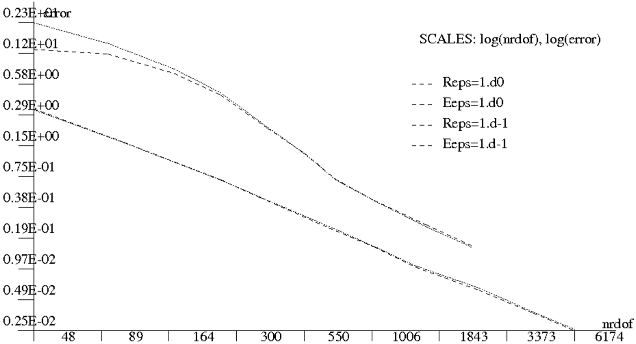
Optimal h-adaptive mesh and numerical solution for $\epsilon = 10^{-1}$

Lin/Stynes example, $\epsilon = 1$



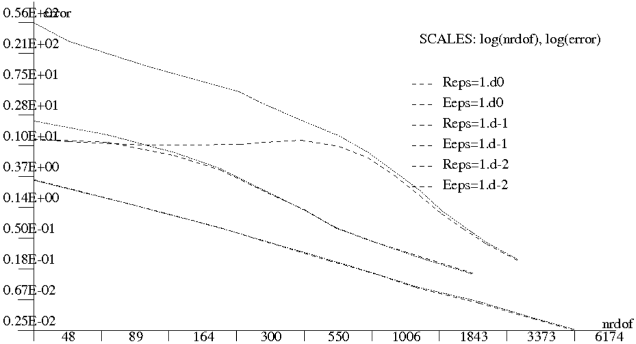
Residual and “balanced” error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}$



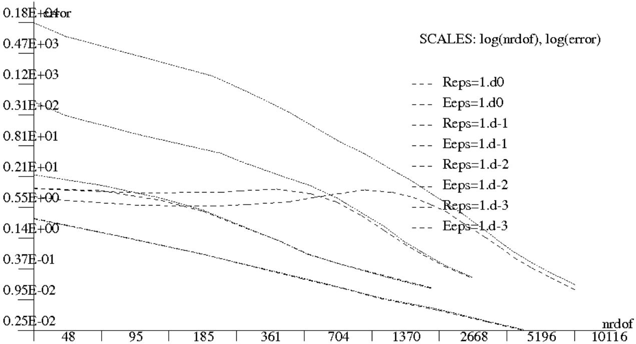
Residual and “balanced” error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.



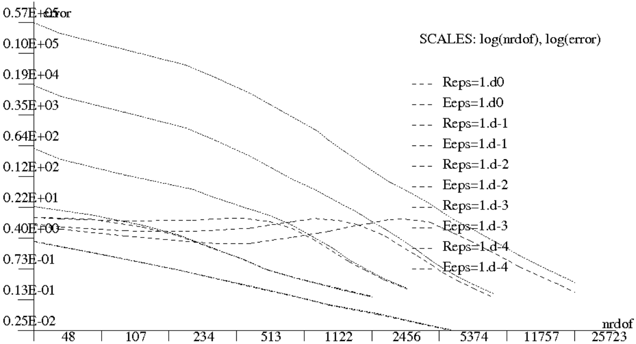
Residual and “balanced” error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$.



Residual and "balanced" error of u for h-adaptive solution, $p = 2$

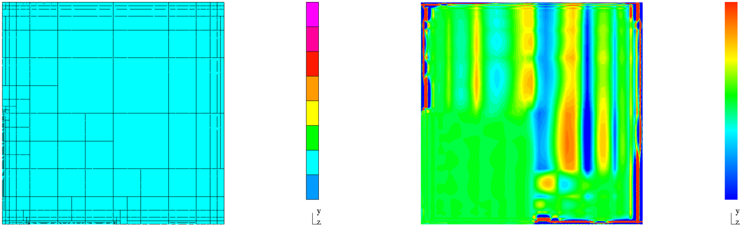
Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.



Residual and "balanced" error of u for h-adaptive solution, $p = 2$

Other tricks we can play: zooming on the solution

Question: Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0, 1)^2$?
Answer: Yes!



Optimal mesh and the corresponding pointwise error (range $(-0.001 - 0.001)$.

Ultraweak Variational Formulation

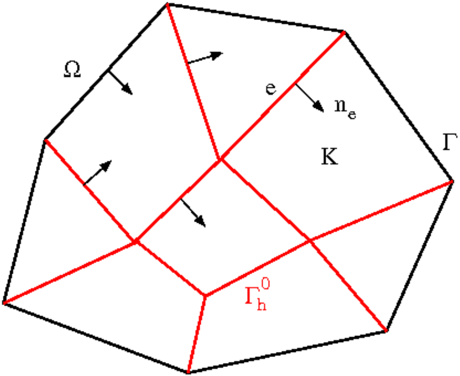
2D Convection-Dominated Diffusion (Confusion) Problem

$$\begin{cases} -\epsilon\Delta u + \mathrm{div}(\boldsymbol{\beta} u) & = f \quad \text{in } \Omega \\ u & = u_0 \quad \text{on } \Gamma \end{cases}$$

or, equivalently,

$$\begin{cases} \frac{1}{\epsilon}\boldsymbol{\sigma} - \boldsymbol{\nabla} u & = 0 \quad \text{in } \Omega \\ -\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) & = f \quad \text{in } \Omega \\ u & = u_0 \quad \text{on } \partial\Omega \end{cases}$$

Ultraweak (DPG) Variational Formulation



Elements: K
Edges: e
Skeleton: $\Gamma_h = \bigcup_K \partial K$
Internal skeleton: $\Gamma_h^0 = \Gamma_h - \partial\Omega$

Take an element K. Multiply the equations with test functions $\boldsymbol{\tau}, v$:

$$\begin{cases} \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \boldsymbol{\nabla} u \cdot \boldsymbol{\tau} & = 0 \\ -\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) v & = f v \end{cases}$$

DPG Method

Integrate over the element K:

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \boldsymbol{\nabla} u\cdot\boldsymbol{\tau} & = 0 \\ -\int_K \mathsf{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u)v & = fv \end{cases}$$

Integrate by parts (relax) *both* equations:

$$\left\{ \begin{array}{ll} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_K u \, \mathsf{div}\boldsymbol{\tau} - \int_{\partial K} u \, \tau_n & = 0 \\ \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{\nabla} v - \int_{\partial K} q \, \mathsf{sgn}(\boldsymbol{n}) \, v & = \int_K f v \end{array} \right.$$

where $q = (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{n}_e$ and

$$\mathsf{sgn}(\boldsymbol{n}) = \left\{ \begin{array}{rl} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{array} \right.$$

DPG Method

Declare traces and fluxes to be independent unknowns, common for adjacent elements:

$$\left\{ \begin{array}{rl} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_K u \, \mathsf{div}\boldsymbol{\tau} - \int_{\partial K} \hat{u} \, \tau_n & = 0 \\ - \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{\nabla} v + \int_{\partial K} \hat{q} \, \mathsf{sgn}(\boldsymbol{n}) v & = \int_K f v \end{array} \right.$$

DPG Method

Use BC to eliminate the known traces

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_K u \, \mathsf{div}\boldsymbol{\tau} - \int_{\partial K - \partial\Omega} \hat{u} \, \tau_n & = \int_{\partial K \cap \partial\Omega} u_0 \, \tau_n \\ -\int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{\nabla} v + \int_{\partial K} \hat{q} \, \mathsf{sgn}(\boldsymbol{n}) v & = \int_K f v \end{cases}$$

Abstract Notation

Integration by parts:

$$(Au, v) = (u, A_h^* v) - \langle \hat{u}, v \rangle_{\Gamma_h}$$

where (watch for overloaded symbols...)

$$\begin{aligned}
u &= (\boldsymbol{\sigma}, u) \\
v &= (\boldsymbol{\tau}, v) \\
Au &= (\tfrac{1}{\epsilon}\boldsymbol{\sigma} - \boldsymbol{\nabla} u, -\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u)) \\
A_h^* v &= (\tfrac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla}_h v, \mathrm{div}_h \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) \\
\langle \hat{u}, v \rangle_{\Gamma_h} &= \int_{\Gamma_h} (u[\tau_n] + \underbrace{\sigma_n - \beta_n u}_{q}[v]) \\
\hat{u} &= (\underbrace{\hat{u}}_{\text{trace}}, \underbrace{\hat{q}}_{\text{flux}}) \text{ with } \hat{u} = 0 \text{ on } \Gamma
\end{aligned}$$

DPG variational formulation:

$$\underbrace{(u, A_h^* v) - \langle \hat{u}, v \rangle_{\Gamma_h}}_{b((u,\hat{u}),v)} = \underbrace{(f, v) + \langle \widetilde{u_0}, v \rangle_{\Gamma}}_{l(v)}$$

Functional Setting for the Confusion Problem

General Functional setting:

- $u \in L^2(\Omega)$,
- broken graph space for v,

$$H_{A^*}(\Omega_h) := \{v \in L^2(\Omega_h) \; : \; A_h^* v \in L^2(\Omega_h\}$$

- trace space for $\hat{u}$ with minimum energy extension norm:

$$\|\hat{u}\|^2 = \inf_{u : u|_{\Gamma_h} = \hat{u}} (\|u\|^2 + \|Au\|^2)$$

Confusion problem: Group variables:
Solution $(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})$:

field variables: $u, \sigma_1, \sigma_2 \in L^2(\Omega_h)$
traces: $\hat{u} \in \tilde{H}^{1/2}(\Gamma_h^0)$
fluxes: $\hat{q} \in H^{-1/2}(\Gamma_h)$

Test function $(\boldsymbol{\tau}, v)$:

$$\boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, \Omega_h)$$
$$v \in H^1(\Omega_h)$$

The Point

- With broken test spaces, the inversion of Riesz operator is done element-wise.

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- We still can do it only approximately, using an enriched space and standard Bubnov-Galerkin method. If trial functions $u, \hat{u} \in \mathcal{P}^p$, we seek approximal optimal test functions by inverting the Riesz operator in an enriched space $\mathcal{P}^{p+\Delta p}$,

$$\left\{ \begin{array}{l} v_h \in \mathcal{P}^{p+\Delta p} \\ (v_h, \delta v)_V = (u, A^* \delta v) - \langle \hat{u}, \delta v \rangle \quad \forall v \in \mathcal{P}^{p+\Delta p} \end{array} \right.$$

  The error in approximating the optimal test functions is assumed to be negligible.

The Point

- With broken test spaces, the inversion of Riesz operator is done element-wise.
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  The error in approximating the optimal test functions is assumed to be negligible.
- As the determination of optimal test functions is done element-wise, the method fits into the standard FE technology.
  Standard FEM: *Input:* bilinear and linear form, trial and test shape functions,
  Output: element stiffness matrix and load vector,
  DPG: *Input:* bilinear and linear form, trial shape functions, test norm,
  Output: element stiffness matrix and load vector

Well Posedness

Theorem If the original operator A with homogenous BC is bounded below,

$$\|A\| \geq \gamma \|u\|$$

and the data u_0 comes from the trace space for the graph norm space, then the DPG formulation is well posed as well, with a mesh-independent inf-sup constant of order γ.

Corollary: If γ is independent of the singular perturbation parameter (ϵ for the confusion problem), then the DPG method is robust,

$$\|u - u_h\| + \|\hat{u} - \hat{u}_h\| \lesssim \inf_{w_h, \hat{w}_h} \{\|u - w_h\| + \|\hat{u} - \hat{w}_h\|\}$$

- L. D., J. Gopalakrishnan, "Analysis of the DPG Method for the Poisson Equation," *SIAM J. Num. Anal.*, **49**(5), 1788-1809, 2011.
- J. Bramwell, L.D.,J. Gopalakrishnan, and W. Qiu. "A Locking-free hp DPG Method for Linear Elasticity with Symmetric Stresses," *Num. Math.*, **122(4):** 671–707, 2012.
- L.D., J. Gopalakrishnan, I. Muga, and J. Zitelli. "Wavenumber Explicit Analysis for a DPG Method for the Multidimensional Helmholtz Equation," *CMAME*, **213-216**, 126-138, 2012.
- T. Bui-Thanh, L.D., O. Ghattas, "A Unified Discontinuous Petrov-Galerkin Method and its Analysis for Friedrichs' Systems,", *SIAM J. Num. Anal.*, **51(4):** 1933–1958, 2013. *ICES Report 2011/34*.
- N. Roberts, Tan Bui-Thanh, L.D., "The DPG Method for the Stokes Problem," *ICES Report 2012/22*, *CAMWA*, to appear.

Construction of an optimal test norm

Bad and good news

Bad news: the graph test norm may not be feasible
Good news: There is a systematic approach for determining alternate test norms

- L. D., M. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems," *SIAM J. Num. Anal*, **51:** 2514–2537, 2013.
- J. Chan, N. Heuer, T. Bui-Thanh, L.D., "Robust DPG Method for Convection-dominated Diffusion Problems II: Natural Inflow Condition," *ICES Report 2012/21*, *CAMWA*, in print.

Step 1: Decide what you want

We want the L^2 robustness in u:

$$\|u\| \lesssim \|(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})\|_E$$

($a \lesssim b$ means that there exists a constant C, independent of ϵ such that $a \leq Cb$). This implies

$$\begin{aligned} \|u - u_h\| &\lesssim \|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E \\ &= \underbrace{\inf_{(u_h, \boldsymbol{\sigma}_h, \hat{u}_h, \hat{q}_h)} \|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E}_{\text{Best Approximation Error (BAE)}} \\ &\leq C(\epsilon) h^p \end{aligned}$$

Step 2: Select a special test function...

$$
\begin{aligned}
b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau})) \quad &= (\boldsymbol{\sigma}, \tfrac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v)_{\Omega_h} + (u, \mathsf{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_{\Omega_h} \\
&\quad -\langle \hat{u}, \tau_n\rangle_{\Gamma_h^0} - \langle \hat{q}, v\rangle_{\Gamma_h}
\end{aligned}
$$

Choose a test function $(v,\boldsymbol{\tau})$ such that

$$
\left\{
\begin{array}{rl}
v \in H_0^1(\Omega),\ \boldsymbol{\tau} \in \boldsymbol{H}(\mathsf{div},\Omega) & \\
\frac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v & = 0 \\
\mathsf{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v & = u
\end{array}
\right.
$$

Then

$$
\begin{aligned}
\|u\|^2 \quad &= b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau})) = \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_V}\|(v,\boldsymbol{\tau})\|_V \\
&\le \sup_{(v,\boldsymbol{\tau})} \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_V}\|(v,\boldsymbol{\tau})\|_V = \|(u,\boldsymbol{\sigma},\hat{u},\hat{q})\|_E\,\|(v,\boldsymbol{\tau})\|_V
\end{aligned}
$$

… and request stability of the adjoint problem

Consequently, we need to select the test norm in such a way that

$$\|(v, \boldsymbol{\tau})\|_V \lesssim \|u\|$$

This gives,

$$\|u\|^2 \lesssim \|(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})\|_E \, \|u\|$$

Dividing by $\|u\|$, we get what we wanted.
The point: Construction of a robust DPG reduces to the classical stability analysis for the adjoint equation!

Step 3: Study the stability of the adjoint equation

Theorem (Generalization of Erickson-Johnson Theorem)

$$\left.\begin{array}{r} \|v\| \\ \|\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|_w, \sqrt{\epsilon}\|\boldsymbol{\nabla} v\| \\ \|\mathsf{div}\boldsymbol{\tau}\|_{w+\epsilon}, \frac{1}{\epsilon}\|\boldsymbol{\beta}\cdot\boldsymbol{\tau}\|_w, \frac{1}{\sqrt{\epsilon}}\|\boldsymbol{\tau}\| \end{array}\right\} \lesssim \|u\|$$

where $w = O(1)$ is a weight vanishing on the inflow boundary that satisfies some "mild" assumptions.

The terms on the left-hand side are our "Lego" blocks with which we can build different test norms.

L.D., N. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems", *ICES Report 2011/35*, submitted to *SIAM J. Num. Anal.*

Step 4: Construct test norm(s)

Graph norm:

$$\|(v,\boldsymbol{\tau})\|^2_{graph} := \|v\|^2 + \|\frac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v\|^2 + \|\mathsf{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|^2$$

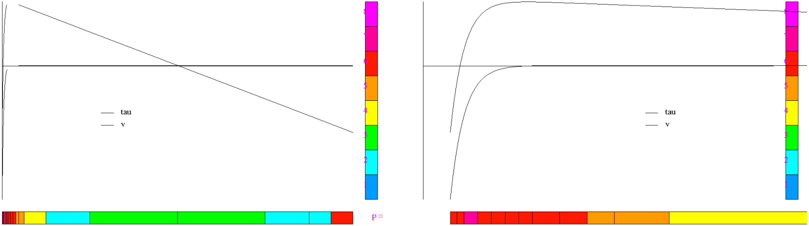
Mesh dependent weighted norm:

$$\begin{aligned}\|(v,\boldsymbol{\tau})\|^2_{w} := & \quad \min\{\tfrac{\epsilon}{h^2},1\}\|v\|^2 + \|\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|^2_w + \epsilon\|\boldsymbol{\nabla} v\|^2 \\ + & \quad \min\{\tfrac{1}{\epsilon},\tfrac{1}{h^2}\}\|\boldsymbol{\tau}\|^2_{w+\epsilon} + \|\mathsf{div}\boldsymbol{\tau}\|^2_{w+\epsilon}\end{aligned}$$

Remark: Both u-robust norms are also L^2-robust in $\boldsymbol{\sigma}$, as well as in traces and fluxes measured in minimum extension energy norms.

Pros and cons for both test norms

- The quasi-optimal test norm produces strong boundary layers that need to be resolved, also in 1D,



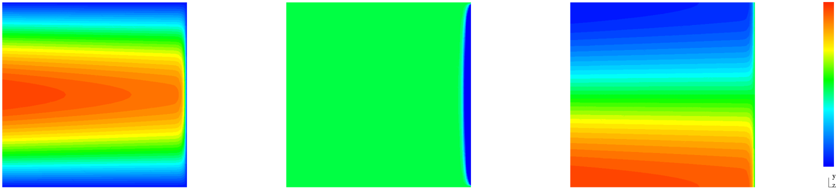
Left: τ and v components of the optimal test function corresponding to trial function $u = 1$ and element size $h = 0.25$, along with the optimal hp subelement mesh. Right: $10\times$ zoom on the left end of the element.
Determining optimal test functions is expensive.

- The weighted test norms produce no boundary layers. Solving for the optimal test functions is inexpensive (done with enriched space $\Delta p = 2$).
- Quasi-optimal test norm yields better estimates for the best approximation error measured in the energy norm.

2D: Model problem of Erickson and Johnson

$$\Omega = (0,1)^2, \quad \boldsymbol{\beta} = (1,0), f = 0, \qquad u_0 = \begin{cases} \sin \pi y & \text{on } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

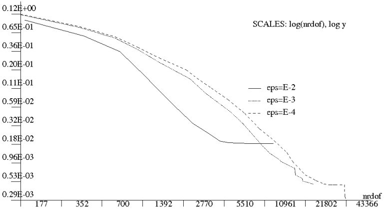
The problem can be solved analytically using separation of variables.

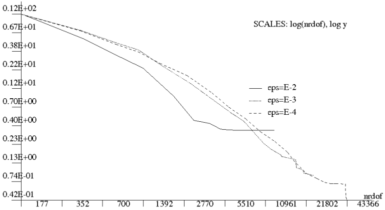


Velocity u and "stresses" σ_x, σ_y (using scale for σ_y) for $\epsilon = 0.01$.

2D: Weighted Norm

hp-**adaptivity:** $h_{min} = 2\epsilon, p_{max} = 5, w = x$.





$\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$. Left: convergence in energy error. Right: convergence in relative L^2-error for the field variables (in percent of their L^2-norm).

2D: Weighted Norm

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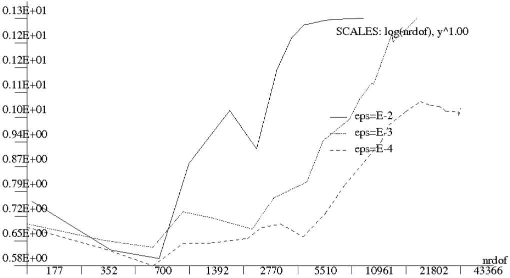
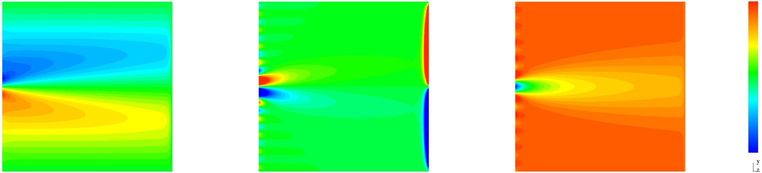


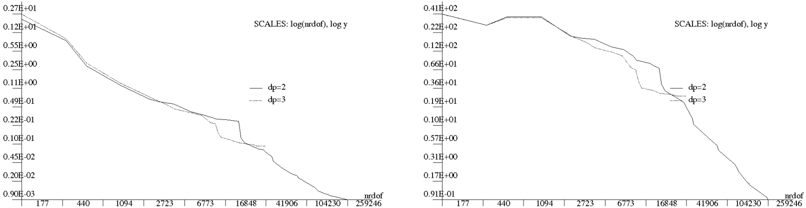
$\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$. Ratio of L^2 and energy norms.

hp-**adaptivity:** $h_{min} = 2\epsilon, p_{max} = 5, w = x$.



2D model problem with a "discontinuous" inflow data, $\epsilon = 0.01$. Velocity u and "stresses" σ_x, σ_y (using scale for σ_y).

2D: Example II, Weighted Norm, $\epsilon = 10^{-4}$



$\Delta p = 2, 3$. Left: convergence in energy error. Right: convergence in relative L^2-error for the field variables (in percent of their L^2-norm).

2D: Example II, Weighted Norm, $\epsilon = 10^{-4}$

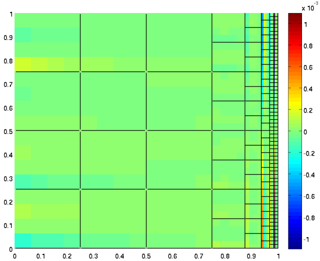
$\Delta p = 2, 3$. Ratio of L^2 and energy norms.

Good Boundary Conditions are Essential

For inflow boundary condition

$$\beta_n u - \sigma_n = u_0$$

and wall outflow boundary condition,



Mesh/pointwise error for $\epsilon = 1e-2$.

DPG delivers

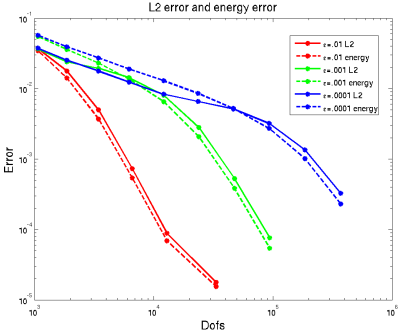
$$\|u\| + \|\sigma\| \lesssim \|(u, \sigma, \hat{u}, \hat{q})\|_E$$

using test norms *without* the weight, e.g.,

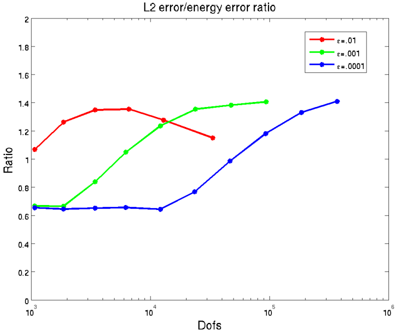
$$\|(v, \boldsymbol{\tau})\|^2 := \underbrace{\epsilon\|v\|^2 + \|\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|^2}_{\text{convection}} + \underbrace{\epsilon\|\boldsymbol{\nabla} v\|^2 + \|\boldsymbol{\tau}\|^2 + \|\mathrm{div}\boldsymbol{\tau}\|^2}_{\text{diffusion}}$$

J. Chan, N. Heuer, T. Bui-Thanh, L.D., "Robust DPG Method for Convection-dominated Diffusion Problems II: Natural Inflow Condition," *ICES Report 2012/21*, *CAMWA*, in print.

Confusion Revisited



(a) Convergence rates



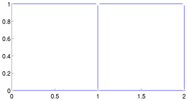
(b) L^2 and energy ratio

Extrapolation to Compressible Navier-Stokes Equations: Carter's flat plate problem

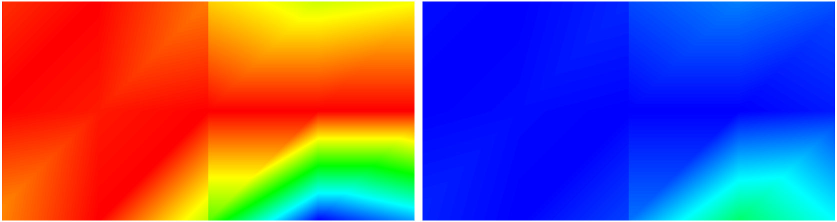
$$M_\infty = 3, \mathrm{Re}_L = 1000, \mathrm{Pr} = 0.72, \gamma = 1.4, \theta_\infty = 390^o[\mathrm{R}]$$

L.D., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and hp Adaptivity,", *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

Initial Mesh ($p = 2$):

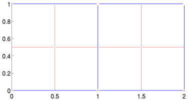


Horizontal velocity and temperature

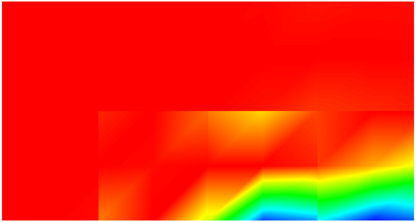


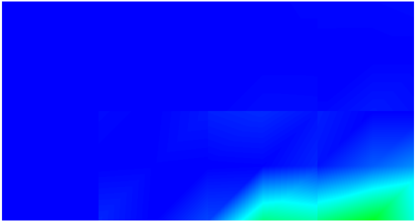
Extrapolation to Compresible NS Equations

Mesh 1:



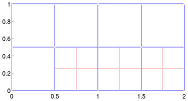
Horizontal velocity and temperature



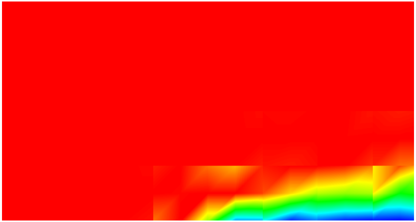


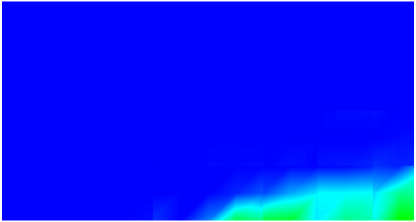
Extrapolation to Compresible NS Equations

Mesh 2:

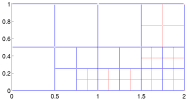


Horizontal velocity and temperature

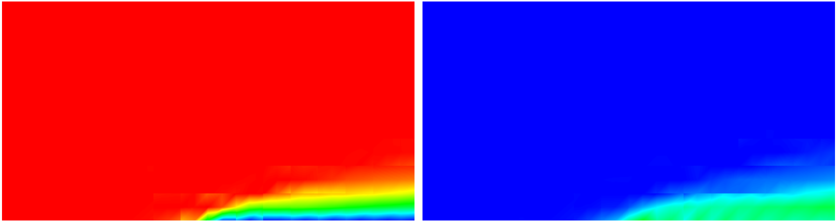




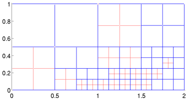
Mesh 3:



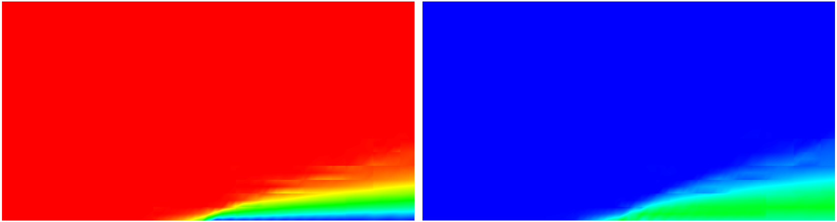
Horizontal velocity and temperature



Mesh 4:

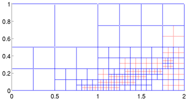


Horizontal velocity and temperature

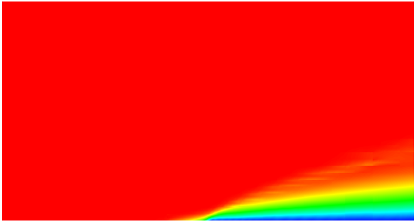


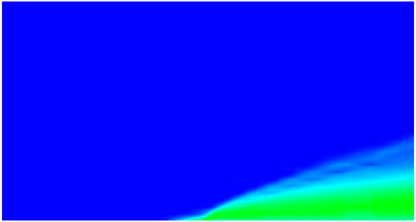
Extrapolation to Compresible NS Equations

Mesh 5:



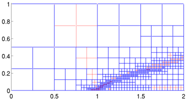
Horizontal velocity and temperature



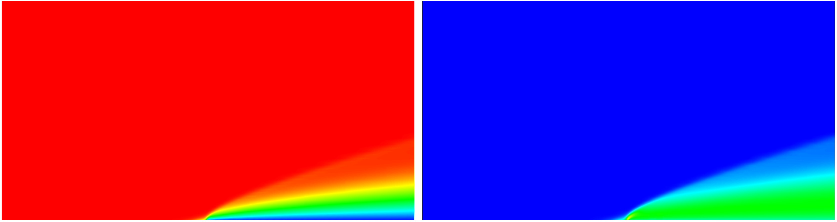


Extrapolation to Compresible NS Equations

Mesh 7:

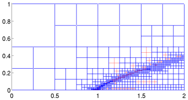


Horizontal velocity and temperature

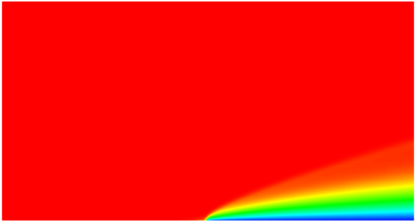


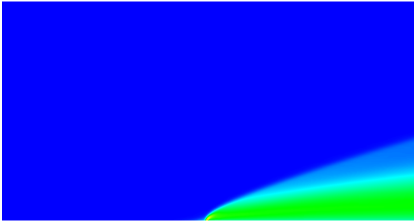
Extrapolation to Compresible NS Equations

Mesh 8:



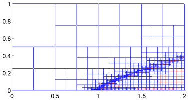
Horizontal velocity and temperature



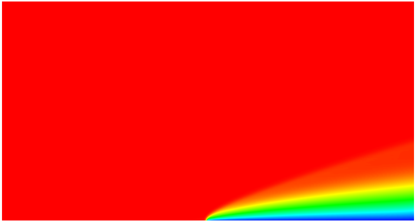


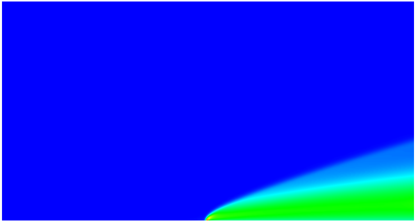
Extrapolation to Compresible NS Equations

Mesh 9:



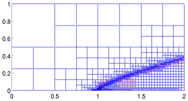
Horizontal velocity and temperature



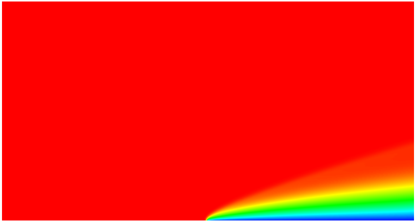


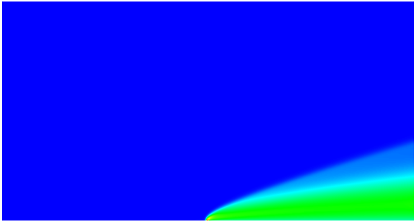
Extrapolation to Compresible NS Equations

Mesh 10:

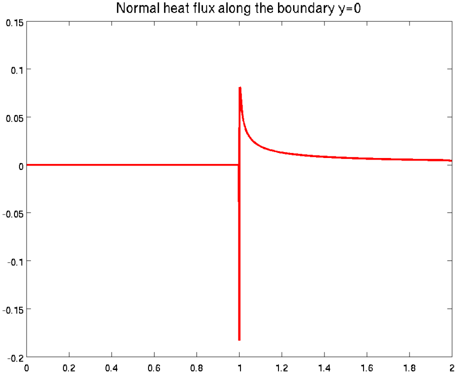


Horizontal velocity and temperature





Extrapolation to Compresible NS Equations



Heat flux along the plate

Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)

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- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Stokes and incompressible NS equations
- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials

Thank You !

Acknowledgments

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Air Force[#FA9550-09-1-0608]