A posteriori error estimates in FEEC for the de Rham complex

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joint work with

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Brief summary of FEEC

Goal of Finite Element Exterior Calculus: Systematically construct and analyze stable numerical methods for PDE of Hodge-Laplace type using differential complexes and related tools such as Hodge decompositions.

Characteristics in brief:

- *Related areas:* Maxwell's equations, elasticity, mixed FEM.
- *Forebears:* Hiptmair, Bossavit...
- *Main developers:* Arnold, Falk, Winther in [AFW '06, '10].
- Analysis begins on abstract realization of differential complexes (Hilbert complexes).
- Unified analysis of Hodge-Laplace problem for all slots in complex.

Previous work and goals

Literature relevant to a posteriori estimates for FEEC:

1. MFEM scalar Laplacian: [Braess-Verfürth '96], [Carstensen '97]...
2. Maxwell's equations: [Beck et. al. '00], [Schöberl '08]....
3. This talk: [Demlow-Hirani, FoCM, '14].

Our goals:

1. Give a "bird's eye view" of residual a posteriori techniques and estimates for differential forms.
   Translate, generalize ideas from individual de Rham "slots".
2. Develop a posteriori estimates for the Hodge Laplacian.
   Account for structure of PDE, including harmonic forms.

The de Rham complex

Definitions:

- $\Lambda^k(\Omega)$ is smooth k-forms on a Lipschitz domain $\Omega \subset \mathbb{R}^n$.
- *Exterior derivative* $d : \Lambda^k \to \Lambda^{k+1}$ $\quad$ (∇, curl, div....).
- $H\Lambda^k = \{v \in L_2\Lambda^k : dv \in L_2\Lambda^{k+1}\}$ $\quad$ (H^1, H(curl), H(div)....).
- de Rham complex:

$$0 \to H\Lambda^0 \xrightarrow{d^1} H\Lambda^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} L_2 \to 0.$$

- *Codifferential (adjoint)* $\delta : \Lambda^{k+1} \to \Lambda^k$ $\quad$ ($-$div, curl, $-\nabla$....).
- $d \circ d = \delta \circ \delta = 0$.
- tr$\,=\,$trace operator, $\star : \Lambda^k \to \Lambda^{n-k}$=Hodge star

Note: Can also consider essential boundary conditions.

Hodge decomposition

Hodge decomposition: $H\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k,\perp}$, where:

- $\mathfrak{B}^k = \mathrm{range}(d^{k-1})$.
- $\mathfrak{Z}^k$ is the nullspace of d^k.
- *Harmonic forms:* $\mathfrak{Z}^k = \mathfrak{B}^k \oplus \mathfrak{H}^k$ ($\dim(\mathfrak{H}^k)$ depends on topology).
- $\mathfrak{Z}^{k,\perp}$ is the range of δ_{k+1}.

Harmonic forms for 3D de Rham:

- $\mathfrak{H}^0$ is constants, $\mathfrak{H}^3 = \emptyset$.
- $k = 1, 2$: $\mathfrak{H}^k = \{p : \mathrm{curl}\, p = 0,\ \ \mathrm{div}\, p = 0\}$ with appropriate BC's.

Hodge Laplacian

Basic Hodge-Laplace PDE:

$$(\delta d + d\delta)u = f.$$

Mixed form: Find $(\sigma, u, p) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ with

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0 \qquad && (\sigma = \delta u) \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle v, p \rangle &= \langle f, v \rangle && ((\delta d + d\delta)u = f - p \perp \mathfrak{H}^k) \\ \langle u, q \rangle &= 0. && (u \perp \mathfrak{H}^k) \end{aligned}$$

for $(\tau, v, q) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$.

3D realizations (boundary conditions vary):

- $k = 0, 3$: $-\Delta u = f$ in Ω in primal, mixed forms.
- $k = 1, 2$: $(\operatorname{curl}\operatorname{curl} - \nabla \operatorname{div})u = f$ in Ω.

The discrete problem

Approximating subspaces: $\mathcal{T}_h$ is a regular simplicial mesh; corresponding spaces $V_h^k \subset H\Lambda^k$ (Lagrange, Nédélec, RT...) satisfy:

$$0 \to V_h^0 \xrightarrow{d^0} V_h^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V_h^n \to 0.$$

The discrete Hodge decomposition $V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp}$:

- $\mathfrak{B}_h^k = d(V_h^{k-1}) \subset \mathfrak{B}^k$.
- $\mathfrak{H}_h^k \subset \mathfrak{Z}^k$, but $\mathfrak{H}_h^k \not\subset \mathfrak{H}^k$. (But, $\dim(\mathfrak{H}_h^k) = \dim(\mathfrak{H}^k) < \infty$).
- $\mathfrak{Z}_h^{k\perp} \not\subset \mathfrak{Z}^{k,\perp}$.

AFW FEM: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$\begin{aligned} \langle \sigma_h, \tau_h \rangle - \langle d\tau_h, u_h \rangle &= 0, & \tau_h &\in V_h^{k-1}, \\ \langle d\sigma_h, v_h \rangle + \langle du_h, dv_h \rangle + \langle v_h, p_h \rangle &= \langle f, v_h \rangle, & v_h &\in V_h^k, \\ \langle u_h, q_h \rangle &= 0, & q_h &\in \mathfrak{H}_h^k. \end{aligned}$$

The “Harmonic Gap”

Goal: Measure the effect of $\mathfrak{H}_h^k \neq \mathfrak{H}^k$ on approximation quality.

Definitions: Given closed subspaces A, B of a Hilbert space W,

$$\sin \angle(A, B) = \sup_{x \in A, \|x\|=1} \|x - P_B x\|,$$

$$\operatorname{gap}(A, B) = \max(\sin \angle(A, B), \sin \angle(B, A)).$$

In our case: Must control $\operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k)$.

A priori analysis

Lemma 1 (AFW '10)**.** *Assume there is an $H\Lambda$-bounded, commuting cochain projection $\Pi_h : V^k \to V_h^k$, and let $e_u = u - u_h$, etc. Then*

$$\|e_\sigma\|_{H\Lambda^{k-1}} + \|e_u\|_{H\Lambda^k} + \|e_p\|_{H\Lambda^k} \lesssim \inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_{H\Lambda} + \inf_{v \in V_h^k} \|u - v\|_{H\Lambda} + \inf_{q \in V_h^k} \|p - q\|_{H\Lambda} + \left\{ \|P_{\mathfrak{H}_h^k} u\| \le \operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_{H\Lambda} \right\}.$$

Error is bounded by

- A best approximation term
- plus a harmonic nonconformity error (higher order...but can dominate error in some examples?).

Also: Analysis can be carried out entirely at Hilbert complex level.

Structure of a posteriori result

Theorem 1. *Let* $u_h^\perp = P_{\mathfrak{Z}_h^{k,\perp}} u_h$. *For* $0 \le k \le n$, *we have*

$$\|e_\sigma\|_{H\Lambda^{k-1}} + \|e_u\|_{H\Lambda^k} + \|e_p\|$$
$$\lesssim \Big(\sum_{K \in \mathcal{T}_h} \eta_{-1}(K)^2 + \eta_0(K)^2 + \eta_{\mathfrak{H}}(p_h)^2 \Big)^{1/2}$$
$$+ \Big\{ \|P_{\mathfrak{H}} u_h\| \lesssim \mu \Big(\sum_{K \in \mathcal{T}_h} \eta_{\mathfrak{H}}(K, u_h^\perp)^2 \Big)^{1/2} + \mu^2 \|u_h\| \Big\}.$$

Notes:

- Definitions of $\eta_{\mathfrak{H}}$, η_{-1}, η_0, $\mu \simeq \mathrm{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k)$ given later.
- Similar to a priori estimate, we have efficient and conforming residual terms + harmonic nonconformity term.
- Harmonic error should be higher order, but can't prove efficiency.
- Hilbert complex analysis not as helpful as a priori case.

Definition of $\mu \simeq \mathrm{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k)$

Lemma 2. *Given $q_i \in V_h^k$, let*

$$\eta_{\mathfrak{H}}(K, q_i) = h_K \|\delta q_i\|_{L_2(K)} + h_K^{1/2} \|[\![\mathrm{tr} \star q_i]\!]\|_{L_2(\partial K)}, \quad K \in \mathcal{T}_h.$$

Also, let $\{q_i\}_{i=1}^N$ be an orthonormal basis for $\mathfrak{H}_h^k$ and define

$$\mu_i = \Big(\sum_{K \in \mathcal{T}_h} \eta_{\mathfrak{H}}(K, q_i)^2\Big)^{1/2}.$$

Then

$$\mathrm{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \simeq \mu := \Big(\sum_{i=1}^N \mu_i^2\Big)^{1/2}.$$

Note: $\mathfrak{H}^k = \{p : dp = 0,\ \delta p = 0 \text{ in } \Omega,\ \mathrm{tr} \star p = 0 \text{ on } \partial\Omega\}.$

Definition of η_{-1}

Interpretation: Arises from testing 1st line in MFEM.

$$\sup_{\tau \in H\Lambda^{k-1}, \|\tau\|_{H\Lambda}=1} \langle \sigma - \sigma_h, \tau \rangle - \langle d\tau, u - u_h \rangle.$$

Definition: Given $K \in \mathcal{T}_h$ and $0 \leq k \leq n$, let

$$\eta_{-1}(K) = \begin{cases} 0 & \text{for } k = 0, \\ h_K \|\sigma_h - \delta u_h\|_K + h_K^{1/2} \|[\![\mathrm{tr} \star u_h]\!]\|_{\partial K} & \text{for } k = 1, \\ h_K (\|\delta \sigma_h\|_K + \|\sigma_h - \delta u_h\|_K) & \\ \quad + h_K^{1/2} (\|[\![\mathrm{tr} \star \sigma_h]\!]\|_{\partial K} + \|[\![\mathrm{tr} \star u_h]\!]\|_{\partial K}) & \text{for } 2 \leq k \leq n. \end{cases}$$

Efficiency: $\eta_{-1}(K) \lesssim \|e_u\|_{L_2\Lambda^k(\omega_K)} + \|e_\sigma\|_{L_2\Lambda^{k-1}(\omega_K)}$.

Definition of η_0

Interpretation: Arises from testing second line in MFEM.

$$\sup_{v \in H\Lambda^k, \|v\|_{H\Lambda=1}} \langle d(\sigma - \sigma_h), v\rangle + \langle d(u - u_h), dv\rangle + \langle (p - p_h), v\rangle.$$

Definition: Given $K \in \mathcal{T}_h$ and $0 \le k \le n$, let

$$\eta_0(K) = \begin{cases} h_K \|f - p_h - \delta du_h\|_K + h_K^{1/2} \|[\![\operatorname{tr} \star du_h]\!]\|_{\partial K} & \text{for } k = 0, \\ \|f - d\sigma_h\|_K & \text{for } k = n, \\ h_K(\|f - d\sigma_h - p_h - \delta du_h\|_K + \|\delta(f - d\sigma_h - p_h)\|_K) & \\ \quad + h_K^{1/2}(\|[\![\operatorname{tr} \star du_h]\!]\|_{\partial K} + \|[\![\operatorname{tr} \star (f - d\sigma_h - p_h)]\!]\|_{\partial K}), & \\ & 1 \le k \le n-1. \end{cases}$$

- Efficiency holds up to data oscillation.
- Note: $f = d\sigma + p + \delta du$, and residual is $f - d\sigma_h - p_h - \delta du_h$.
- More regularity of f is needed than $f \in L_2\Lambda^k$.

A "Hodge imbalance" in our norms

Question: $h_K\|\delta(f - d\sigma_h - p_h)\|_K$, $h_K^{1/2}\|[\![\mathrm{tr} \star (f - d\sigma_h - p_h)]\!]\|_{\partial K}$ require more regularity than $f \in L_2$. *Why is this necessary?*

- *Residual:* $\mathcal{R} = d(\sigma - \sigma_h) + \delta d(u - u_h) + (p - p_h)$.
- $d\sigma + p$ is directly approximated in L_2 by $d\sigma_h + p_h$
- δdu is only weakly approximated (in H^{-1}).
- *Must Hodge decompose f to construct error indicators with correct "strength" for each variable.*
- The above indicators Hodge decompose f weakly by killing δdu.
- *Literature:* A term involving $\mathrm{div}\, f$ arises in time-harmonic Maxwell's equations if $\mathrm{div}\, f \neq 0$.

Example 1: $k = 0$

- $d\delta + \delta d = -\Delta$ with Neumann BC's.
- Assumption: Standard compatibility condition $\int_\Omega f = 0$ holds ($\iff f \perp \mathfrak{H}^0 = \mathbb{R}$, and $p = p_h = 0$).
- The AFW mixed method is a standard primal FEM.
- Estimates reduce to standard ones:

$$\begin{aligned}
&\|u - u_h\|_{H^1(\Omega)} \\
&\qquad \lesssim \Big(\sum_{T\in\mathcal{T}_h} h_K^2 \|f - p_h - \delta d u_h\|_K + h_K \|[\![\mathrm{tr} \star d u_h]\!]\|_{\partial K}^2\Big)^{1/2} \\
&\qquad = \Big(\sum_{T\in\mathcal{T}_h} h_K^2 \|f + \Delta u_h\|_K^2 + h_K \|[\![\nabla u_h]\!]\|_{\partial K}^2\Big)^{1/2}.
\end{aligned}$$

Example 2: $k = n = 3$

- $d\delta + \delta d = -\Delta$ with Dirichlet BC's.
- AFW gives standard mixed method with $\sigma = -\nabla u$ and norm $H(\mathrm{div}) \times L_2$ *(not so interesting in practice...)*.
- A posteriori estimates:

$$\eta_{-1} = h_K(\|\operatorname{curl}\sigma_h\|_K + \|\sigma_h + \nabla u_h\|_K) \\ + h_K^{1/2}(\|[\![u_h]\!]\|_{\partial K} + \|[\![\sigma_{h,t}]\!]\|_{\partial K}),$$

$$\eta_0 = \|f - \operatorname{div}\sigma_h\|_K,$$

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_{L_2} \simeq \Big(\sum_{K\in\mathcal{T}_h} \eta_{-1}(K)^2 + \eta_0(K)^2\Big)^{1/2}.$$

- Similar to [Carstensen '97], but has not appeared previously in the literature.
- [Ca '97] assumes convexity of Ω; no restriction here.

Example 3: $n = 3$, $k = 1$ (Vector Laplacian)

- $\delta d + d\delta = (\operatorname{curl}\operatorname{curl} - \nabla \operatorname{div})$; $u \cdot n = 0$, $\operatorname{curl} u \times n = 0$ on $\partial\Omega$.
- Error indicators: Let $\{q_i\}_{i=1}^N$ be an orthonormal basis for $\mathfrak{H}_h^1$.

$$\eta_{-1} = h_K \|\sigma_h + \operatorname{div} u_h\| + h_K^{1/2} \| [\![u_h \cdot n]\!] \|_{\partial K},$$

$$\eta_0 = h_K (\| f - \nabla \sigma_h - p_h - \operatorname{curl}\operatorname{curl} u_h \|_K + \| \operatorname{curl}(f - d\sigma_h - p_h)\|_K) \\ + h_K^{1/2} (\| [\![\operatorname{curl} u_h \times n]\!] \|_{\partial K} + \| [\![(f - \nabla \sigma_h - p_h) \cdot n]\!] \|_{\partial K}),$$

$$\eta_{\mathfrak{H}}(K, q) = h_K \| \operatorname{div} q \|_K + h_K^{1/2} \| [\![q \cdot n]\!] \|_{\partial K}.$$

- Final estimate is exactly as in Theorem 1.
- First a posteriori estimates for the vector Laplacian.
- It seems the effect of harmonic forms on a posteriori estimates has not been studied before.

Tool 1 for proof: Regular decompositions

Need: Decomposition of $v \in H\Lambda^k$ as $v = d\varphi + z$, where z, ϕ are smooth enough to give an "h" in interpolation estimates.

Previous literature: Regular decompositions are a well-known tool for Maxwell's equations ([Hiptmair '02], [Pasciak-Zhao '02]).

Generalization to arbitrary n, k:

Lemma 3. *Given $v \in H\Lambda^k$, there are $\varphi \in H^1\Lambda^{k-1}$ and $z \in H^1\Lambda^k$ such that $v = d\varphi + z$, and $\|\varphi\|_{H^1} + \|z\|_{H^1} \lesssim \|v\|_H$.*

Proof uses: [Mitrea-Mitrea-Monniaux '08] for stable solution of relevant BVP, [M.-M.-Shaw '08] for bounded extension operator.

Tool 2: Interpolation operators

Desirable properties of $\Pi_h : L_2\Lambda^k \to V_h^k$:

- Commutes with d, locally bounded, projection.

We prove: Only local boundedness and commutativity.

Lemma 4. *For $0 \le k \le n$, there exists $\Pi_h : L_2\Lambda^k \to V_h^k$ such that $d\Pi_h = \Pi_h d$, and for $K \in \mathcal{T}_h$ and $z \in H^1\Lambda^k$,*

$$\|z - \Pi_h z\|_{L_2(K)} \lesssim h_K |z|_{H^1(\omega_K)}.$$

We "average" the approaches of:

- [Schöberl '01, '08] constructed Π_h for the 3D de Rham complex.
- [Christiansen-Winther '08]: Projecting, commuting, *globally* bounded Π_h.

Note: Supplanted by recent work of [Falk-Winther]?

Thoughts on AFEM convergence in FEEC

Literature:

- [Zhong et. al '12] prove optimality of AFEM for time-harmonic Maxwell's equations.
- [Chen-Holst-Xu '09] prove optimality of AFEM for controlling $\|\sigma - \sigma_h\|_{L_2}$ in case $k = n$, Ω simply connected.
- [Holst-Mihalik-Szypowski] recently extended MFEM results to arbitrary domain topology in FEEC notation.

Difficulties in proving AFEM convergence for arbitrary k, natural variational norm:

- Lack of orthogonality (inf-sup).
- Harmonic errors (mess up a priori optimality).

Convergence of AFEM for $\mathrm{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k)$

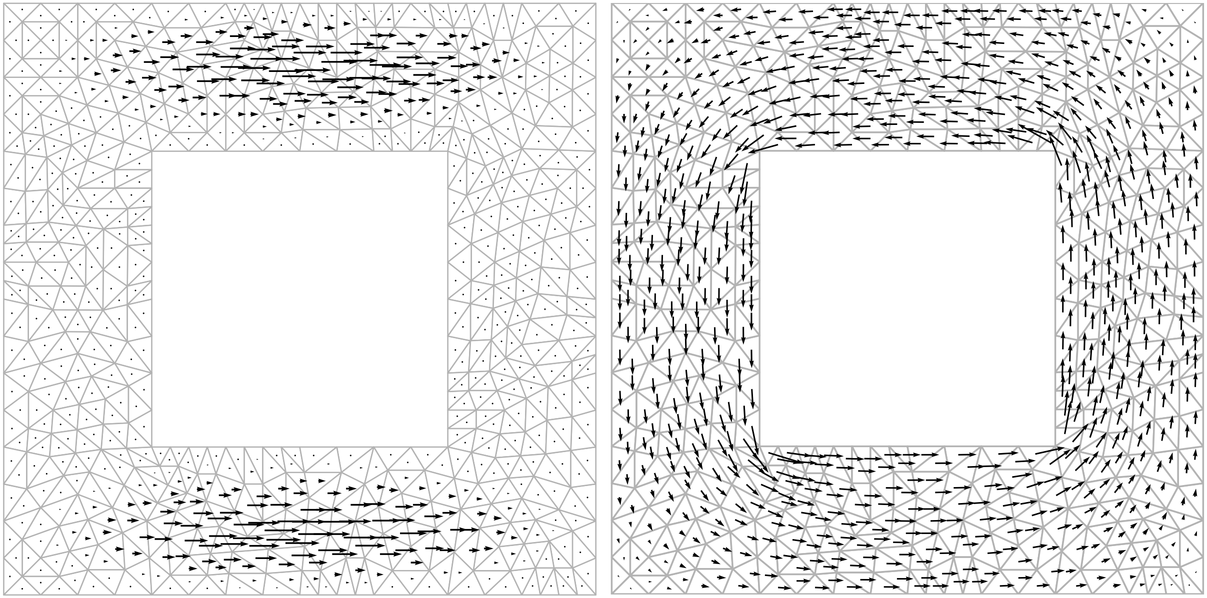
Based on work in progress, can prove:

Lemma 5. *Assume that* $\dim(\mathfrak{H}^k) = \dim(\mathfrak{H}_h^k) = 1$. *Then a standard AFEM based on Dörfler marking for controlling* $\mathrm{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k)$ *using the above estimates and estimators is contractive.*

Notes:

- $\dim(\mathfrak{H}^k) = 1$ shouldn't be essential.
- Computation of harmonic forms is a "miniature eigenvalue problem" (we know the eigenvalue, only need the eigenvectors.).
- AFEM convergence results for eigenvalues are harder to prove, BUT existing results require mesh fineness condition (we don't).

A 2D Example



Left: A smooth vector field $u \in \mathfrak{B}^1 \oplus \mathfrak{Z}^{1,\perp}$ satisfying natural BC's.
Right: A (discrete) harmonic vector field q.

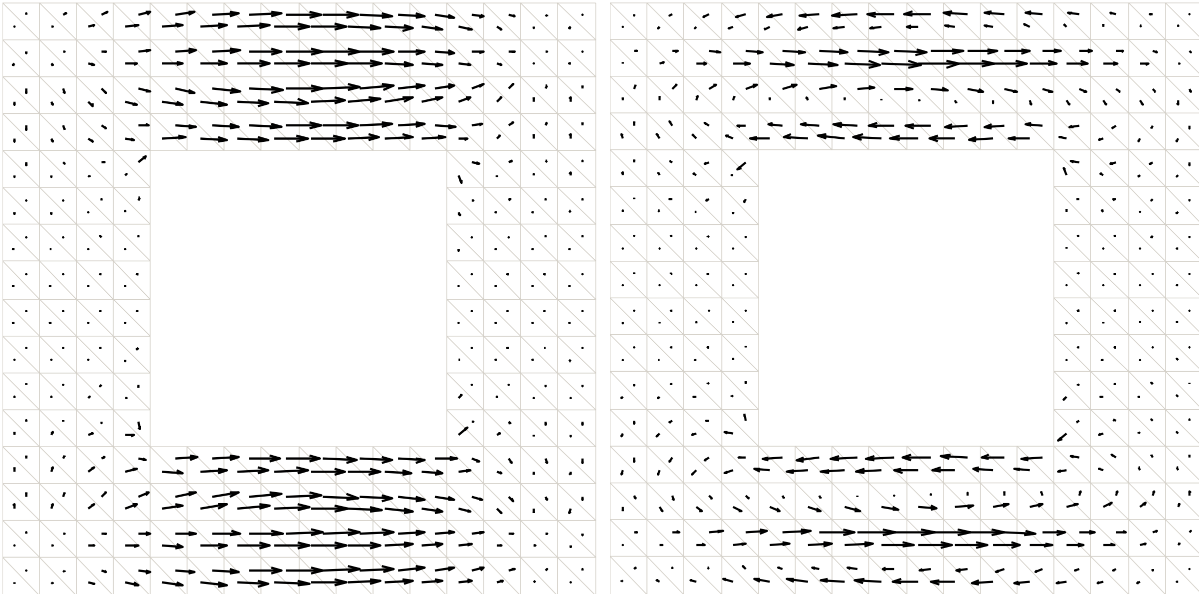
Example (cont.)

Computation of Hodge decomposition: Use FEM to solve $\Delta z = \operatorname{div} u$ on a *uniform* sequence of meshes. Then $P_{\mathfrak{B}^1} u = \nabla z$.

Expectation: $\|\nabla(z - z_h)\| \sim h^{\frac{2}{3}}$ if $P_{\mathfrak{B}^1} u = \nabla z$ has corner singularities.

Iteration	Energy error	EOC	Iteration	Energy error	EOC
1	60.06	.72	6	3.77	.78
2	36.46	.78	7	2.20	.75
3	21.19	.84	8	1.31	.73
4	11.82	.84	9	.79	.71
5	6.61	.81	10	.48	

Example (cont.)



Left: $P_{\mathfrak{B}^1} u$.

Right: $P_{\mathfrak{Z}^{1,\perp}} u$.

Effects on error bounds

Recall the harmonic nonconformity error:

$$\left\{ \|P_{\mathfrak{H}_h^k} u\| \le \operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_{H\Lambda} \right\}.$$

Expectation: $\operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \sim h^{2/3}$, so $\|P_{\mathfrak{H}_h^k} u\| \sim h^{4/3}$.

Iteration	$\|P_{\mathfrak{H}_h^k} u\|$	EOC	Iteration	Energy error	EOC
1	0.014		4	0.0014	1.237
2	0.0076	0.927	5	0.00059	1.275
3	0.0034	1.155	6	0.00024	1.297

- $\|P_{\mathfrak{H}_h} u\| \le \|u - u_h\|_{L_2}$, so $\|u - u_h\|_{L_2} \ge Ch^{4/3}$ also.
- Used lowest-order element, but higher order shouldn't affect rates.
- A (well-founded) conjecture: Mixed approximation to the Hodge Laplacian may converge suboptimally even if u, σ, p are smooth.

Comments and Conclusions

What we've accomplished:

- Generalized and unified tools and techniques for a posteriori error estimation for the de Rham complex.
- Explained the "Hodge imbalance" in residuals.
- Gave a posteriori upper bounds for harmonic forms and their effect on errors in approximating Hodge Laplace problems.

To do:

- Clarify whether our a posteriori estimation of the harmonic nonconformity error is efficient.
- Applications?

Credit: *Kaushik Kalyanaraman of UIUC performed some of the computations in the final section.*