A posteriori error estimators for weighted norms. Adaptivity for point sources and local errors

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Outline

1. Problem
2. Finite element discretization
3. Some results in weighted spaces on simplices
4. A posteriori error estimates
5. Numerical experiments
6. Local estimation

Problem

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Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \boldsymbol{b} \cdot \nabla u + cu = \delta_{x_0} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where

- $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded polygonal/polyhedral domain with Lipschitz boundary.
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- x_0: inner point of Ω
- δ_{x_0}: Dirac delta distribution supported at x_0.
- $\mathcal{A} \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ piecewise-$W^{1,\infty}$ and uniformly symmetric positive definite over Ω.
- $\boldsymbol{b} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$ with $c - \frac{1}{2}\operatorname{div}(\boldsymbol{b}) \geq 0$.

- Usual test and ansatz space: $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

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- For $n = 2$, *Araya-Behrens-Rodríguez (2007)*:
  - Test space: $W_0^{1,p'}(\Omega) \subset C(\Omega)$, for some $p' > 2$.
  - Ansatz space: $W_0^{1,p}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ ($\Rightarrow p < 2$).

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- For $n = 2$, *Gaspoz-M-Veeser (2014, in prep.)*:
  - Test space: $H_0^{1+s}(\Omega) \subset C(\Omega)$ if $s > 0$.
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Goals:
- Not modify the integrability power nor the differentiability order.
- Obtain results also valid for $n = 3$.

We use weighted spaces -*D'Angelo & Quarteroni (2008,2012)*-

Weighted spaces

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$$\frac{1}{n^2 - (2\beta)^2} \leq \underbrace{\sup_{\substack{B=B(y,r) \\ y\in\mathbb{R}^n,\, r>0}} \left(\frac{1}{|B|} \int_B \mathrm{d}_{x_0}^{2\beta} \right) \left(\frac{1}{|B|} \int_B \mathrm{d}_{x_0}^{-2\beta} \right)}_{A_2 \text{ constant}} \leq \frac{C_n}{n^2 - (2\beta)^2},$$

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- $$-\frac{n}{2} < \beta < \frac{n}{2} \qquad \Longleftrightarrow \qquad d_{x_0}^{2\beta} \in A_2.$$

Weighted spaces

- For $-\frac{n}{2} < \beta < \frac{n}{2}$,

$$L^2_\beta(\Omega) := \{u \text{ measurable } : \|u\|_{L^2_\beta(\Omega)} < \infty\},$$

where

$$\|u\|_{L^2_\beta(\Omega)} := \|u\|_{L^2(\Omega, \mathrm{d}_{x_0}^{2\beta})} = \left(\int_\Omega |u(x)|^2 \, \mathrm{d}_{x_0}(x)^{2\beta} dx\right)^{\frac{1}{2}}$$

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- Weighted Sobolev space:

$$H^1_\beta(\Omega) = \{u \text{ weakly differenciable } : \|u\|_{H^1_\beta(\Omega)} < \infty\},$$

where

$$\|u\|_{H^1_\beta(\Omega)} := \|u\|_{L^2_\beta(\Omega)} + \|\nabla u\|_{L^2_\beta(\Omega)}$$

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- D'Angelo and Quarteroni (2008) + a weighted Hardy's inequality:

Lemma

If $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$,

$$\delta_{x_0} \in \left(H^1_{-\alpha}(\Omega)\right)'.$$

- Define

$$W_\beta := \{u \in H^1_\beta(\Omega) : u_{|\partial\Omega} = 0\}, \qquad \|u\|_{W_\beta} := \|\nabla u\|_{L^2_\beta(\Omega)}$$

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- The norm in W_β is equivalent to the inherited norm $\|u\|_{H^1_\beta(\Omega)}$. The equivalence constant blows up when $|\beta|$ approaches $\frac{n}{2}$.

Variational formulation

- Let $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$.

Weak form

$$u \in W_\alpha : \qquad a(u,v) = \delta_{x_0}(v), \quad \forall\, v \in W_{-\alpha},$$

where

$$a(u,v) = \int_\Omega \mathcal{A}\nabla u \quad \cdot \nabla v \quad + \boldsymbol{b} \cdot \nabla u \quad v \quad + c\, u \quad v \quad ,$$

is well-defined and bounded in $W_\alpha \times W_{-\alpha}$ due to Hölder inequality.

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where

$$a(u,v) = \int_\Omega \mathcal{A}\nabla u \mathrm{d}_{x_0}^\alpha \cdot \nabla v \frac{1}{\mathrm{d}_{x_0}^\alpha} + \boldsymbol{b} \cdot \nabla u \mathrm{d}_{x_0}^\alpha\, v \frac{1}{\mathrm{d}_{x_0}^\alpha} + c\, u \mathrm{d}_{x_0}^\alpha\, v \frac{1}{\mathrm{d}_{x_0}^\alpha},$$

is well-defined and bounded in $W_\alpha \times W_{-\alpha}$ due to Hölder inequality.

Existence and uniqueness of the weak solution

General problem

Given $F \in (W_{-\alpha})'$, find $u \in W_\alpha$ such that

$$a(u, v) = F(v), \quad \forall\, v \in W_{-\alpha}.$$

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- D'Angelo (2012):

$$\inf_{u \in W_\alpha} \sup_{v \in W_{-\alpha}} \frac{\int_\Omega \nabla u \cdot \nabla v}{\|u\|_{W_\alpha} \|v\|_{W_{-\alpha}}} \geq \frac{1}{2} \quad , \quad \inf_{v \in W_{-\alpha}} \sup_{u \in W_\alpha} \frac{\int_\Omega \nabla u \cdot \nabla v}{\|u\|_{W_\alpha} \|v\|_{W_{-\alpha}}} \geq \frac{1}{2} \quad ,$$

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where γ_1 is the smallest eigenvalue of $\mathcal{A}$.

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  where γ_1 is the smallest eigenvalue of $\mathcal{A}$.

- $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ continuous and coercive.

Existence and uniqueness of the weak solution

$$\alpha \in \mathbb{I} := \begin{cases} (0,1) & \text{if} \quad n=2 \quad \text{and} \quad \boldsymbol{b} \in W^{1,\infty}(\Omega;\mathbb{R}^2),\ c \in L^\infty(\Omega) \\ (\frac{1}{2},1) & \text{if} \quad n=3 \quad \text{and} \quad \boldsymbol{b} \in W^{1,\infty}(\Omega;\mathbb{R}^3),\ c \in L^\infty(\Omega) \\ (\frac{1}{2},\frac{3}{2}) & \text{if} \quad n=3 \quad \text{and} \quad \boldsymbol{b}=0,\ c=0 \end{cases}$$

Well-posedness and stability

There exists an unique solution u of the problem and there holds that

$$\|u\|_{W_\alpha} \leq C_* \|F\|_{(W_{-\alpha})'}.$$

- Case $\boldsymbol{b}=c=0$: $\quad C_* = 2/\gamma_1$.
- Otherwise: $\quad C_* = C_*(\Omega, \mathcal{A}, \boldsymbol{b}, c, \alpha) \to \infty$ when $\alpha \to 1$.

An Inf-Sup condition

Existence and uniqueness of the weak solution

There exists a unique solution of

$$u \in W_\alpha : \qquad a(u,v) = \delta_{x_0}(v), \quad \forall\, v \in W_{-\alpha},$$

which satisfies

$$\|u\|_{W_\alpha} \leq C_* \|\delta_{x_0}\|_{(W_{-\alpha})'}.$$

An Inf-Sup condition

$$\inf_{u \in W_\alpha} \sup_{v \in W_{-\alpha}} \frac{a(u,v)}{\|u\|_{W_\alpha} \|v\|_{W_{-\alpha}}} = \frac{1}{C_*}.$$

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Galerkin discretization

- $\mathcal{T}$ conforming triangulation of Ω.
- $$\kappa := \sup_{T \in \mathcal{T}} \frac{\operatorname{diam}(T)}{\rho_T} \quad \text{(mesh regularity)}$$
- Lagrange finite elements of degree $\ell \in \mathbb{N}$:

$$\mathbb{V}_{\mathcal{T}}^{\ell} := \{V \in H_0^1(\Omega) \mid V_{|_T} \in \mathcal{P}_\ell(T),\ \forall\, T \in \mathcal{T}\}$$

Galerkin discretization

Discrete problem

$$\text{Find } U \in \mathbb{V}_{\mathcal{T}}^{\ell} : \qquad a(U, V) = \delta_{x_0}(V), \quad \forall\, V \in \mathbb{V}_{\mathcal{T}}^{\ell}.$$

- The discrete problem has a unique solution for each mesh and

$$\|U\|_{W_\alpha} \leq C \|\delta_{x_0}\|_{(W_{-\alpha})'},$$

where $C = C(\Omega, \mathcal{A}, \boldsymbol{b}, c, \kappa, \ell, \alpha) \to \infty$ as $\alpha \to$ right endpoint of $\mathbb{I}$.

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Local Poincaré inequality

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Let $\beta \in (-\frac{n}{2}, \frac{n}{2})$. There exists $C_P = C_P(\beta, \kappa) > 0$ such that

$$\|v - v_T\|_{L^2_\beta(T)} \leq C_P h_T \|\nabla v\|_{L^2_\beta(T)}, \quad \forall T \in \mathcal{T}, \ \forall v \in H^1_\beta(\Omega)$$

where $v_T := \frac{1}{|T|} \int_T v$.
The constant C_P blows up when $|\beta|$ approaches $\frac{n}{2}$.

- $h_T := |T|^{\frac{1}{n}} \simeq \operatorname{diam}(T)$.

Let $0 < \gamma < n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function.

Fractional Integral

$$T_\gamma(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy$$

Fractional Maximal Function

$$f_\gamma^*(x) := \sup_{B=B_x} \frac{1}{|B|^{1-\gamma/n}} \int_B |f(y)|\, dy$$

Lemma (Muckenhoupt and Wheeden (1974))

Let $0 < \gamma < n$, $w \in A_\infty = \cup_{q \geq 1} A_q$, and $1 < p < \infty$. Then,

$$\left(\int_{\mathbb{R}^n} |T_\gamma(f)|^p w \right)^{\frac{1}{p}} \leq c \left(\int_{\mathbb{R}^n} |f_\gamma^*|^p w \right)^{\frac{1}{p}},$$

for all measurable functions f.

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Lemma (Fabes, Kenig and Serapioni (1982))

Let $w \in A_p$, for some p, $1 < p < \infty$. Then, there exists a constant $c > 0$, depending only on the A_p constant of w, such that

$$\left(\int_{\mathbb{R}^n} |f_1^*|^p w\right)^{\frac{1}{p}} \leq cR \left(\int_{B_R} |f|^p w\right)^{\frac{1}{p}},$$

for all ball B_R of radius $R > 0$, and for all f measurable and supported in B_R.

Local Poincaré inequality

Proof. Let $v \in C^1(\bar{\Omega})$. Since T is convex,

$$|v(x) - v_T| \leq \frac{\operatorname{diam}(T)^n}{n\,|T|} \underbrace{\int_T \frac{|\nabla v(z)|}{|x-z|^{n-1}}\, dz}_{=T_1(|\nabla v|\chi_T)(x)}, \qquad \text{a.e. } x \in T.$$

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Since $\mathrm{d}_{x_0}^{2\beta} \in A_2 \subset A_\infty$, due to the lemmas stated above it follows that

$$\|T_1(f)\|_{L^2_\beta(\mathbb{R}^n)} \leq cR\|f\|_{L^2_\beta(B_R)} = cR\|\nabla v\|_{L^2_\beta(T)}, \tag{2}$$

for balls $B_R \supset T$.

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for balls $B_R \supset T$. Taking a ball with $R \lesssim h_T$ and considering (1) and (2), we obtain the result for smooth functions v.
The assertion of the theorem follows by density arguments. q.e.d.

Interpolation estimates

- $\mathcal{P} : H^1(\Omega) \to \mathbb{V}^1_{\mathcal{T}}$ Clément or Scott-Zhang interpolation operator.

Classical Interpolation Estimates

$$\begin{aligned}
\|v - \mathcal{P}v\|_{L^2(T)} &\lesssim h_T \|\nabla v\|_{L^2(S_T)} , && \forall T \in \mathcal{T}, \\
\|\nabla(v - \mathcal{P}v)\|_{L^2(T)} &\lesssim \|\nabla v\|_{L^2(S_T)} , && \forall T \in \mathcal{T}.
\end{aligned}$$

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$$\|\nabla(v - \mathcal{P}v)\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(S_T)}, \qquad \forall T \in \mathcal{T}.$$

- $\mathcal{P}$ is well defined in $H^1_{-\alpha}(\Omega)$, since $H^1_{-\alpha}(\Omega) \subset H^1(\Omega)$, for $\alpha > 0$.

Weighted Interpolation Estimates

$$\|v - \mathcal{P}v\|_{L^2_{-\alpha}(T)} \leq C_I h_T \|\nabla v\|_{L^2_{-\alpha}(S_T)}, \qquad \forall T \in \mathcal{T},$$
$$\|\nabla(v - \mathcal{P}v)\|_{L^2_{-\alpha}(T)} \leq C_I \|\nabla v\|_{L^2_{-\alpha}(S_T)}, \qquad \forall T \in \mathcal{T}.$$

Here, $C_I = C_I(\kappa, \alpha) \to \infty$ as $\alpha \to \frac{n}{2}$.

A local bound for δ_{x_0}

A precise bound of δ_{x_0}

Let $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$ and $T \in \mathcal{T}$ such that $x_0 \in T$. Then

$$|\delta_{x_0}(v)| \lesssim h_T^{\alpha - \frac{n}{2}} \|v\|_{L^2_{-\alpha}(T)} + C_\alpha h_T^{\alpha + 1 - \frac{n}{2}} \|\nabla v\|_{L^2_{-\alpha}(T)}, \qquad \forall\, v \in H^1_{-\alpha}(T),$$

where $C_\alpha := \frac{\alpha^{\frac{\alpha-1}{2}}}{(\alpha+1)^{\frac{\alpha+1}{2}}}$ if $n = 2$ and $C_\alpha := \frac{(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2\alpha+2)^{\frac{\alpha+1}{3}}}$ if $n = 3$.

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where $C_\alpha := \frac{\alpha^{\frac{\alpha-1}{2}}}{(\alpha+1)^{\frac{\alpha+1}{2}}}$ if $n = 2$ and $C_\alpha := \frac{(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2\alpha+2)^{\frac{\alpha+1}{3}}}$ if $n = 3$.

C_α blows up as α approaches $\frac{n}{2} - 1 \;\Leftarrow\; \delta_{x_0} \in \left(H^1_{-\alpha}(\Omega)\right)'$, for $\alpha > \frac{n}{2} - 1$

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A posteriori error estimates

$U \in \mathbb{V}_{\mathcal{T}}^{\ell}$: solution of discrete problem.

- The *element residual* R

$$R_{|_T} = -\nabla \cdot [\mathcal{A}\nabla U] + \mathbf{b} \cdot \nabla U + cU, \qquad \forall T \in \mathcal{T}$$

- The *jump residual* J

$$J_{|_S} = \begin{cases} \frac{1}{2}\left[(\mathcal{A}\nabla U)_{|_{T_1}} \cdot \vec{n}_1 + (\mathcal{A}\nabla U)_{|_{T_2}} \cdot \vec{n}_2\right] & \text{if } S \in \mathcal{E}_\Omega \\ 0 & \text{if } S \in \mathcal{E}_{\partial\Omega} \end{cases}$$

A posteriori error estimates

A posteriori local error estimators

$$\eta_T^2 := \begin{cases} h_T^2 D_T^{2\alpha} \|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha} \|J\|_{L^2(\partial T)}^2 + h_T^{2\alpha+2-n}, & \text{if } x_0 \in T \\ h_T^2 D_T^{2\alpha} \|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha} \|J\|_{L^2(\partial T)}^2, & \text{if } x_0 \notin T \end{cases}$$

where $D_T := \max_{x \in T} |x - x_0|$.

A posteriori error estimates

A posteriori local error estimators

$$\eta_T^2 := \begin{cases} h_T^2 D_T^{2\alpha} \|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha} \|J\|_{L^2(\partial T)}^2 + h_T^{2\alpha+2-n}, & \text{if } x_0 \in T \\ h_T^2 D_T^{2\alpha} \|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha} \|J\|_{L^2(\partial T)}^2, & \text{if } x_0 \notin T \end{cases}$$

where $D_T := \max_{x \in T} |x - x_0|$.

Global error estimator

$$\eta := \left(\sum_{T \in \mathcal{T}} \eta_T^2 \right)^{\frac{1}{2}}$$

Reliability of the globar error estimator

Global upper bound

- $\alpha \in \mathbb{I}$.
- $u \in W_\alpha$ solution of continuous problem.
- $U \in \mathbb{V}^\ell_{\mathcal{T}}$ solution of discrete problem.

There exists $C_{\mathcal{U}} = C_{\mathcal{U}}(\mathrm{diam}(\Omega), \kappa, \alpha) > 0$ such that

$$\|U - u\|_{H^1_\alpha(\Omega)} \leq C_* C_{\mathcal{U}}\, \eta,$$

where C_* is the continuous inf-sup constant. The constant $C_* C_{\mathcal{U}}$ blows up when α approaches an endpoint of $\mathbb{I}$.

A remark about the test functions

Previous test functions:

- $W_0^{1,p'}(\Omega) \subset \mathcal{C}(\Omega)$.
- $H_0^{1+s}(\Omega) \subset \mathcal{C}(\Omega)$.

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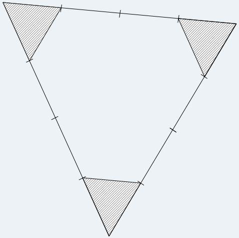
We are not able to use Lagrange interpolation. Instead, we resort to Clément or Scott-Zhang operator. $\rightsquigarrow$ No need to define a new operator.

Local efficiency

Local lower bound

- $\alpha \in \mathbb{I}$.
- $u \in W_\alpha$ solution of continuous problem.
- $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ solution of discrete problem.



There exists $C_{\mathcal{L}} = C_{\mathcal{L}}(\kappa, \alpha) > 0$ such that

$$C_{\mathcal{L}} \eta_T \leq C_a \|U - u\|_{H^1_\alpha(S_T)} + \operatorname{osc}_T, \qquad \forall\, T \in \mathcal{T}.$$

The constant $C_{\mathcal{L}}$ goes to zero if α approaches $\frac{n}{2}$.

Here, $C_a := \max\{\gamma_2, \|\boldsymbol{b}\|_{L^\infty}, \|c\|_{L^\infty}\}$, with γ_2 the biggest eigenvalue of $\mathcal{A}$.

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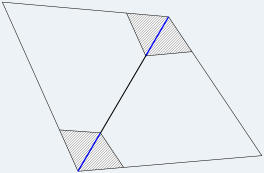

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- The weight only weakens the norm around x_0, but behaves as the usual H^1 norm in subsets at a positive distance to x_0. The H^1 error over such sets converges to zero.
- Our estimates are valid in two and three dimensions, whereas the results from previous works cannot be immediately extended to the three dimensional case.

Outline

1. Problem
2. Finite element discretization
3. Some results in weighted spaces on simplices
4. A posteriori error estimates
5. Numerical experiments
6. Local estimation

Adaptive algorithm

SOLVE $\longrightarrow$ ESTIMATE $\longrightarrow$ MARK $\longrightarrow$ REFINE

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REFINE: Perform two bisections to each marked element, and refine some extra elements in order to keep conformity of the mesh.

A problem with two singularities

Poisson problem in L-shaped domain

$$\begin{cases} -\Delta u = \delta_{x_0} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ and $x_0 = (0.5, 0.5)$.

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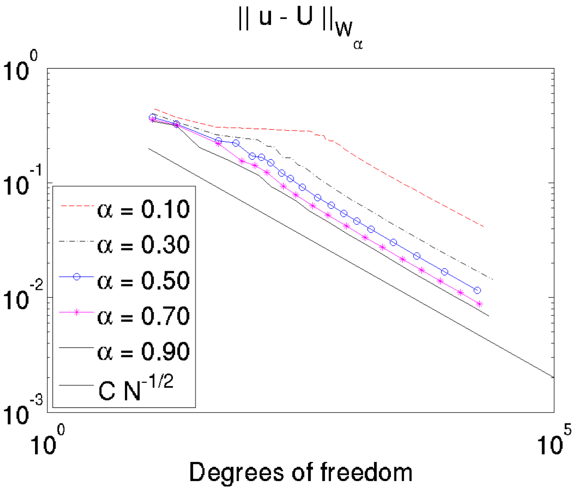
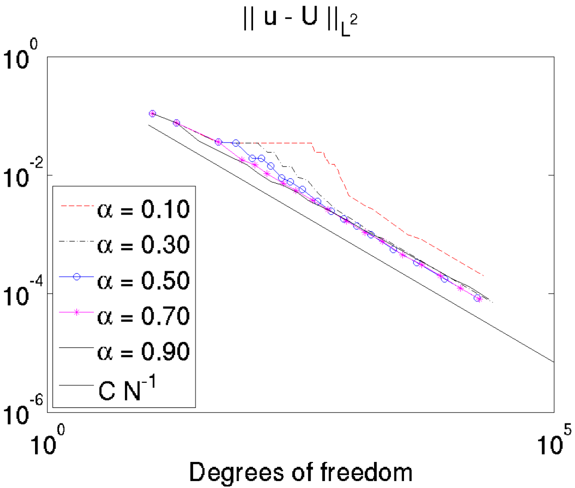
where $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ and $x_0 = (0.5, 0.5)$.

- Exact solution $u(x) = -\frac{1}{2\pi} \log |x - (0.5, 0.5)| + |x|^{2/3} \sin(2\theta/3)$.
- Goals:
  - Test the behavior of the adaptive method guided by the a posteriori estimators η_T for different values of α.
  - Compare the behavior of adaptive algorithms guided by different error estimators.

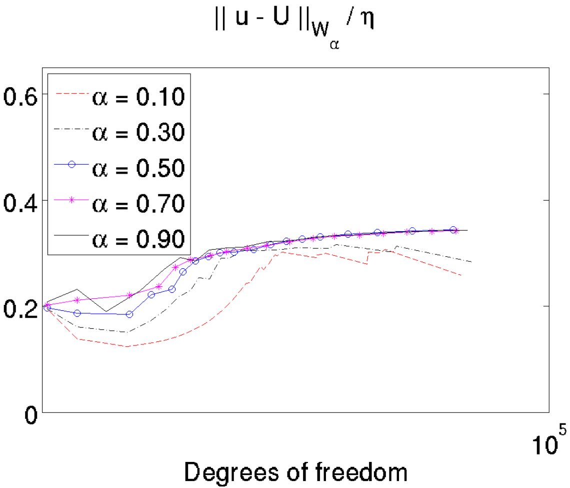
Exact errors



Exact errors

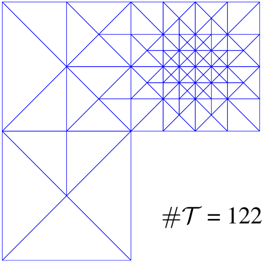


Effectivity indices

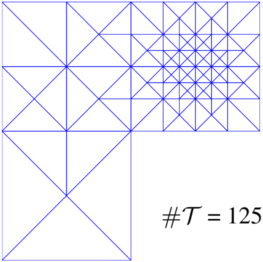


Meshes after 4 iterations

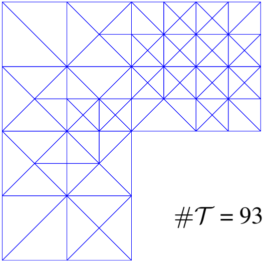
$\alpha = 0.25$



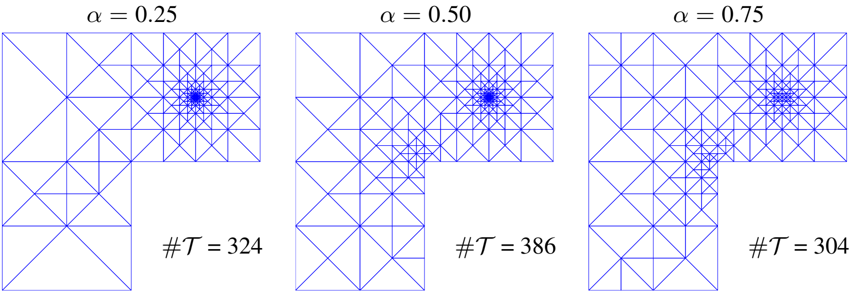
$\alpha = 0.50$



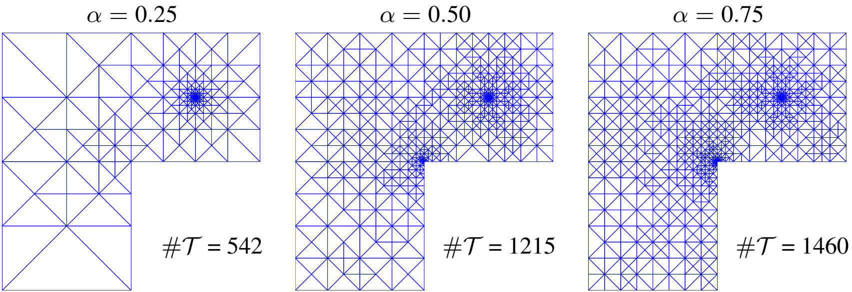
$\alpha = 0.75$



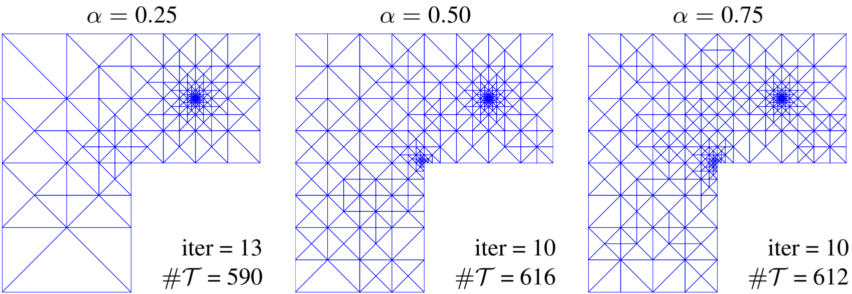
Meshes after 8 iterations



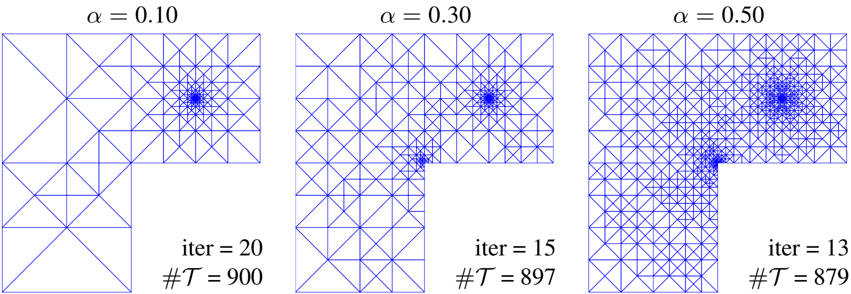
Meshes after 12 iterations



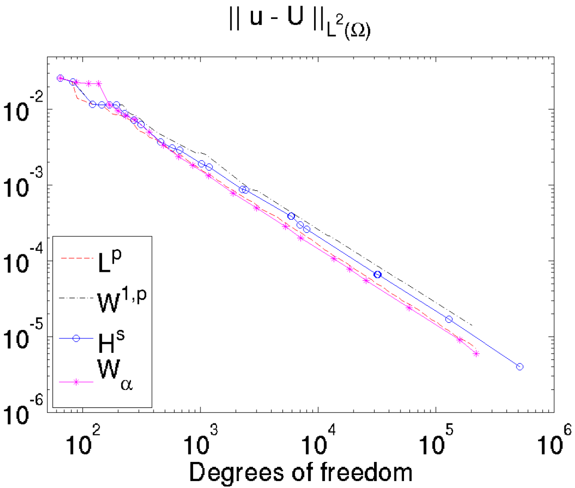
Meshes with similar number of elements



Meshes with similar number of elements (α small)



Comparison with algorithms guided by other error estimators



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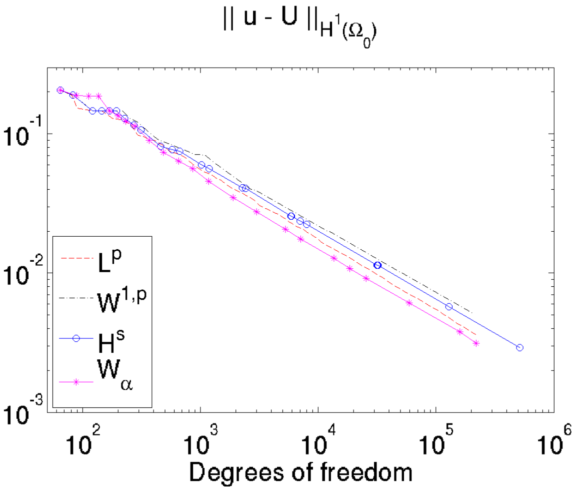

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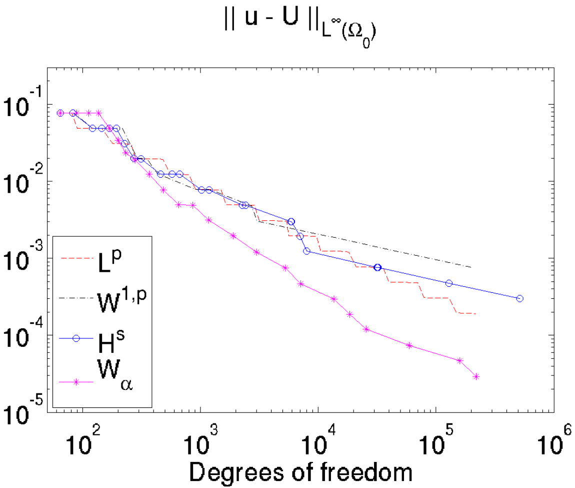

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- Liao and Nochetto (2003)

  $$\|u - U\|^2_{H^1(\Omega_0)} \lesssim \sum_{T \subset \Omega_1} \eta^2_{H^1}(T) + \text{a posteriori estimators for } \|u - U\|_{L^2(\Omega, \omega)}$$

  $\omega(x)$ is a weight that blows up in re-entrant corners.
  ($\omega \equiv 1$ if Ω is convex or smooth)

- Demlow (2010)

  $$\|u - U\|^2_{H^1(\Omega_0)} \lesssim \sum_{T \subset \Omega_1} \eta^2_{H^1}(T) + \text{a posteriori estimators for } \|u - U\|_{L^p(\Omega)}$$

  for some $p > 2$.
  ($p = 2$ if Ω is convex or smooth)

Local estimation. New simple idea

Let

$$\varphi(x) = \varphi_0(\mathrm{dist}(x, \Omega_0))$$

with $\varphi_0 > 0$ a decreasing function such that $\varphi_0(0) = 1$

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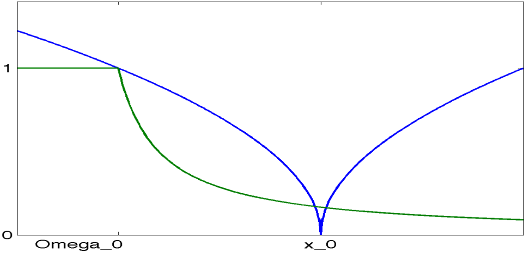
$$\omega(x) = \min\left\{\varphi(x), \left(\frac{|x - x_0|}{\mathrm{dist}(x_0, \Omega_0)}\right)^{2\alpha}\right\}$$



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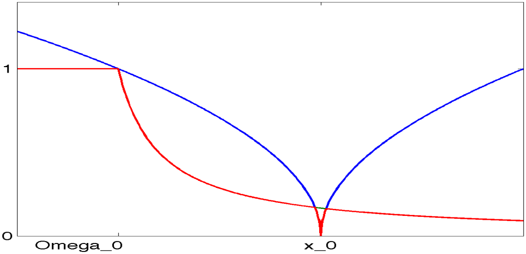

$$\|u - U\|^2_{H^1(\Omega_0)} \leq \|u - U\|^2_{H^1(\Omega,\omega)}$$

Local estimation. New simple idea

Then

$$u \in H_0^1(\Omega,\omega): \qquad a(u,v) = \delta_{x_0}(v), \quad \forall\, v \in H_0^1(\Omega,\omega^{-1})$$

A posteriori estimation

$$\sum_T \eta_\omega^2(T) - \mathrm{osc} \lesssim \|u - U\|^2_{H^1(\Omega,\omega)} \lesssim \sum_T \eta_\omega^2(T)$$

With

$$\eta_\omega(T)^2 := \begin{cases} h_T^2 \omega_T^{2\alpha} \|R\|^2_{L^2(T)} + h_T \omega_T^{2\alpha} \|J\|^2_{L^2(\partial T)} + h_T^{2\alpha+2-n}, & \text{if } x_0 \in T \\ h_T^2 \omega_T^{2\alpha} \|R\|^2_{L^2(T)} + h_T \omega_T^{2\alpha} \|J\|^2_{L^2(\partial T)}, & \text{if } x_0 \notin T \end{cases}$$

and $\omega_T = \sup_{x \in S_T} \omega(x)$

Numerical experiments

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where:

- $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$
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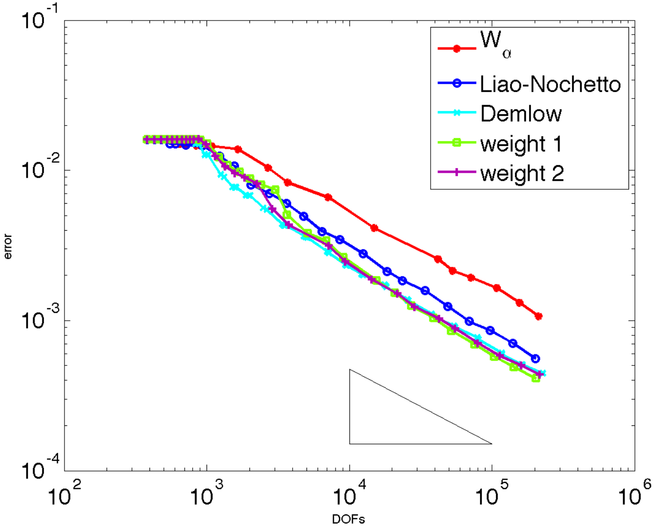
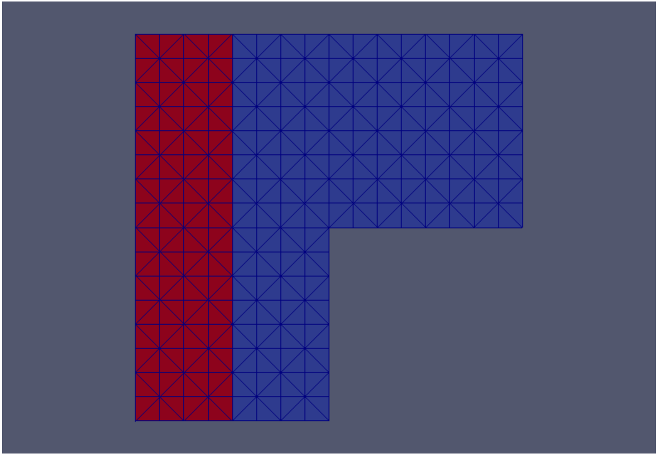
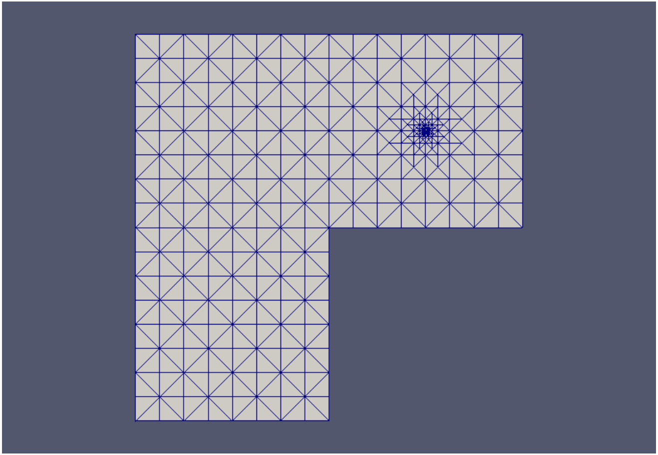
Exact errors $\|u - U\|_{H^1(\Omega_0)}$



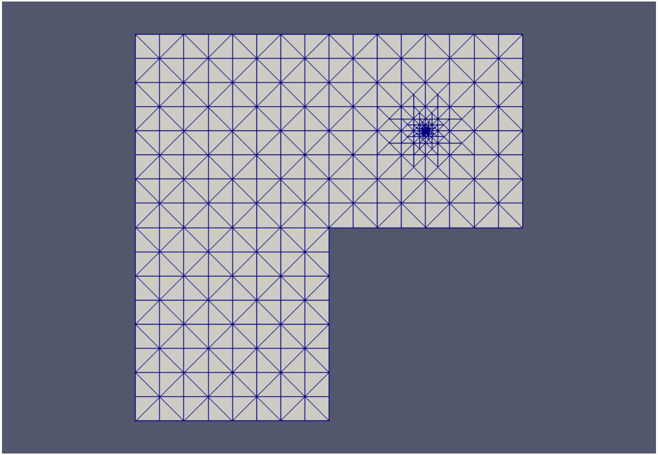
Initial mesh and Ω_0. 225 DOFs



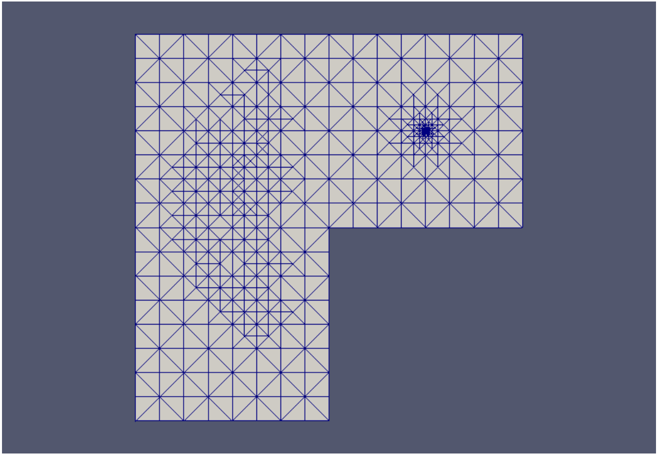
Iteration 4. 321 DOFs



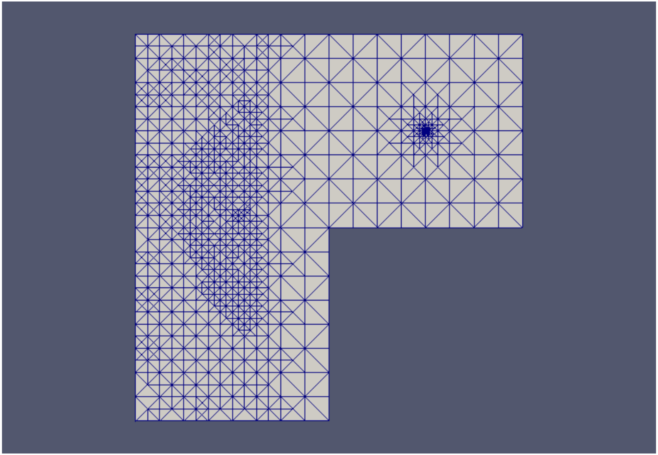
Iteration 8. 417 DOFs



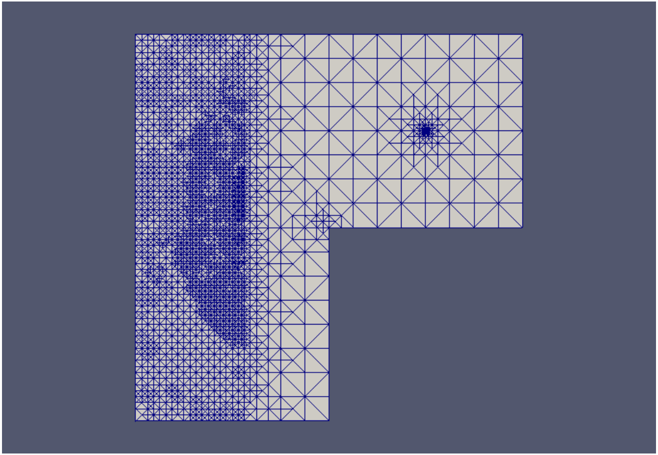
Iteration 12. 643 DOFs



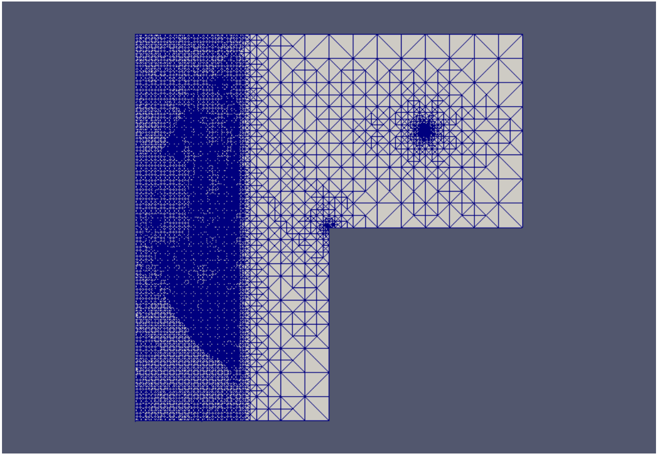
Iteration 16. 1251 DOFs



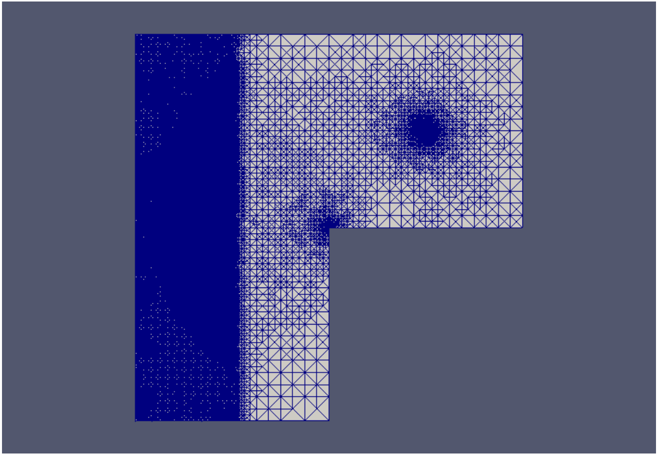
Iteration 20. 3523 DOFs



Iteration 24. 13790 DOFs



Iteration 28. 52386 DOFs



Numerical experiments

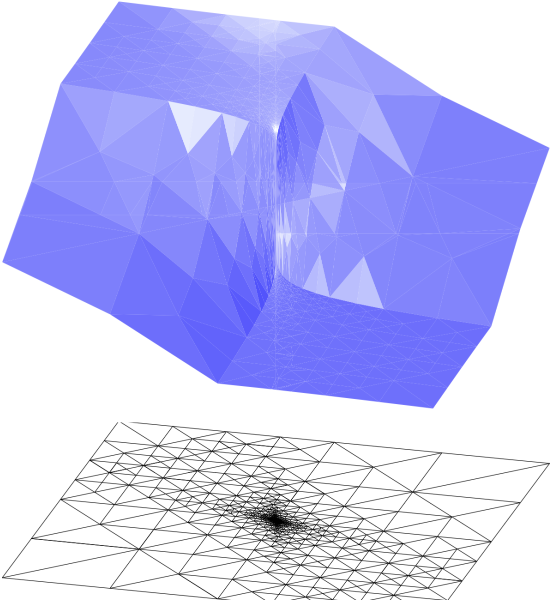
Poisson problem with discontinuous coefficients

$$\begin{cases} -\nabla \cdot (a \nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

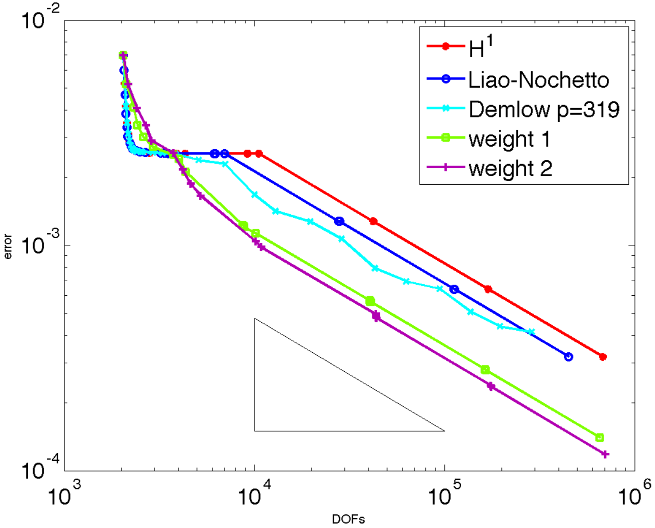
where:

- $\Omega = (-1, 1)^2$
- $a(x_1, x_2) = \begin{cases} 25, & \text{if } x_1 x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$
- $\Omega_0 = (-1, 1) \times (-1, -0.75)$
- Exact solution $u(x) \cong |x|^{1.07}$.

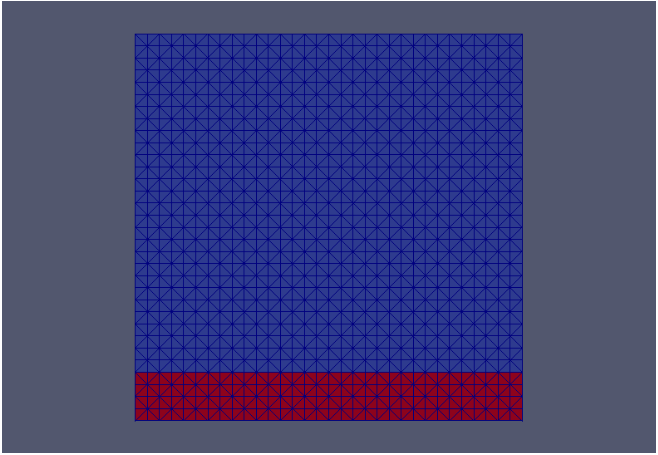
Solution of the discontinuous coefficient example



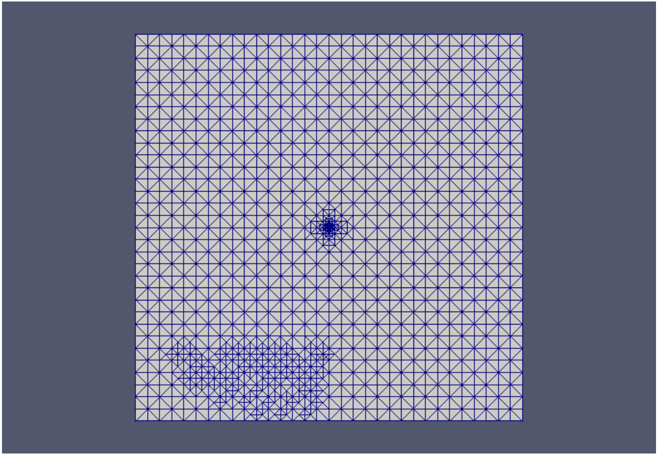
Exact errors $\|u - U\|_{H^1(\Omega_0)}$



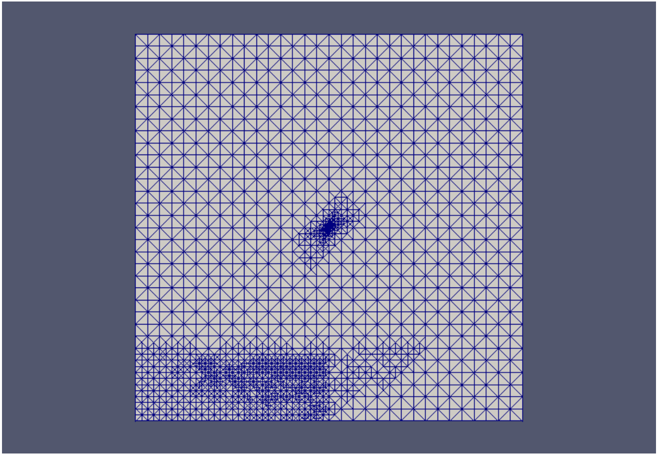
Initial mesh and Ω_0. 1089 DOFs



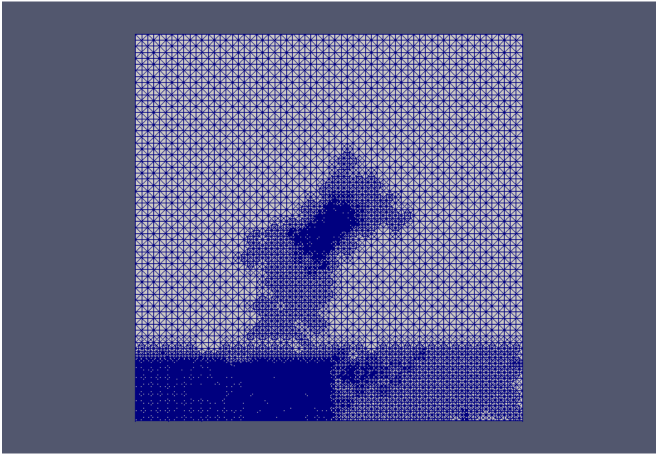
Iteration 4. 1373 DOFs



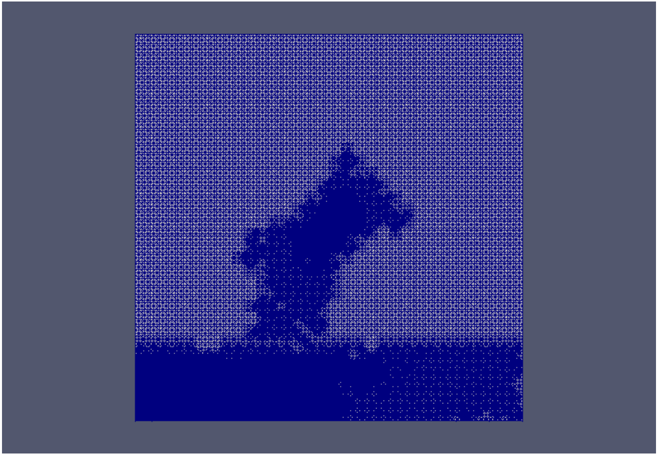
Iteration 8. 2266 DOFs



Iteration 12. 20559 DOFs



Iteration 16. 82653 DOFs



Numerical experiments

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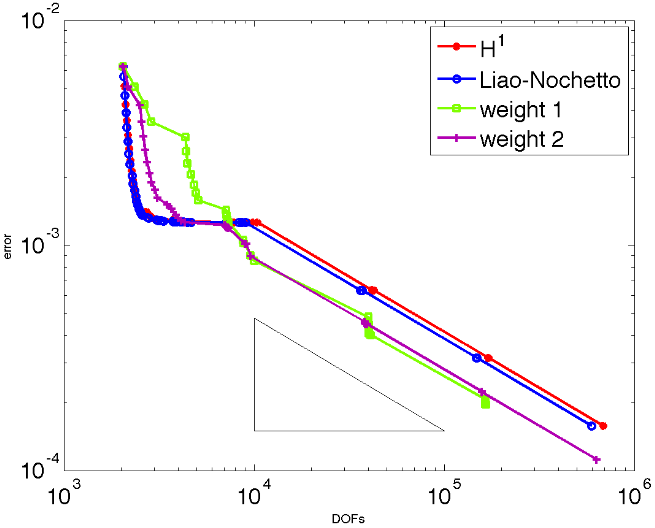
where:

- $\Omega = (-1, 1)^2$
- $a(x_1, x_2) = \begin{cases} 121, & \text{if } x_1 x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$
- $\Omega_0 = (-1, 1) \times (-1, -0.75)$
- Exact solution $u(x) \cong |x|^{1.007}$.

Exact errors $\|u - U\|_{H^1(\Omega_0)}$

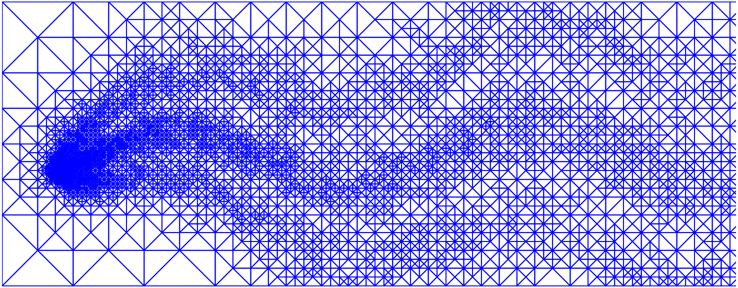


Example 2

Diffusion-advection-reaction equation

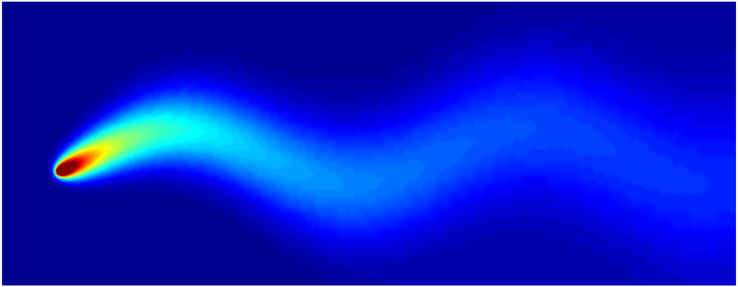
$$\begin{cases} -0.02\Delta u + \begin{bmatrix} 2 \\ \sin(5x_1) \end{bmatrix} \cdot \nabla u + 0.1u = \delta_{(0.2,0.4)} & \text{in } \Omega = (0,3) \times (0,1) \\ u = 0 & \text{on } \partial\Omega \cap \{x_1 < 3\} \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \cap \{x_1 = 3\} \end{cases}$$

Solution of the diffusion-advection-reaction equation



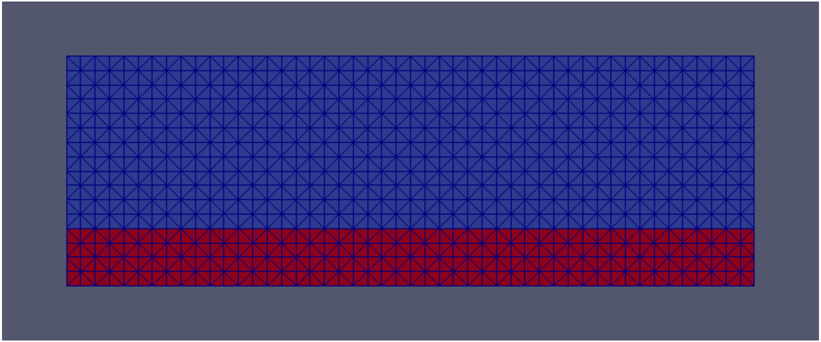
- Final mesh obtained by the adaptive loop and the W_α norm after 20 iterations on a mesh with 22256 elements and 11212 DOFs.

Solution of the diffusion-advection-reaction equation

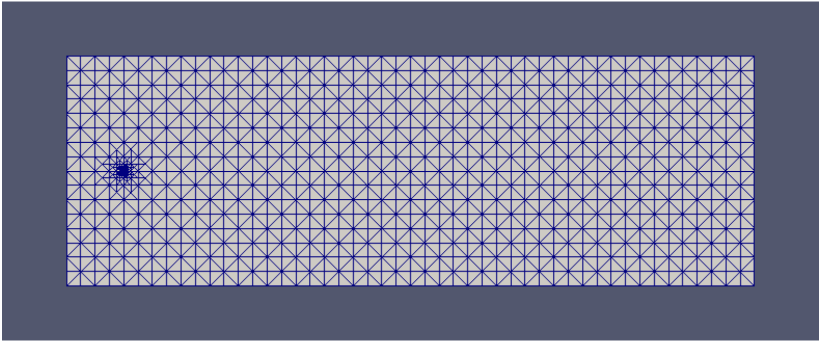


- Final solution obtained by the adaptive loop and the W_α norm after 20 iterations on a mesh with 22256 elements and 11212 DOFs.

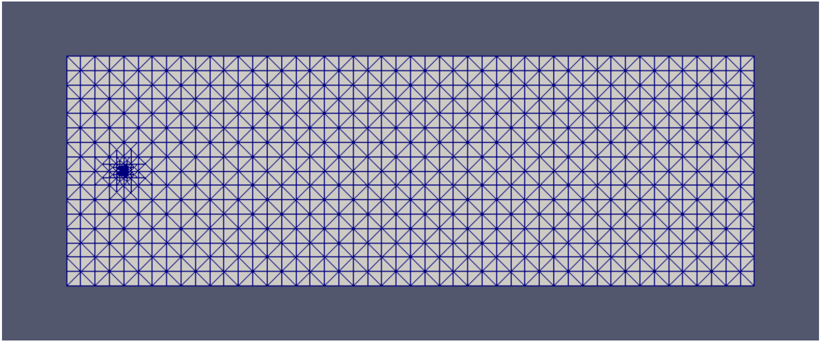
Initial mesh and Ω_0. 833 DOFs



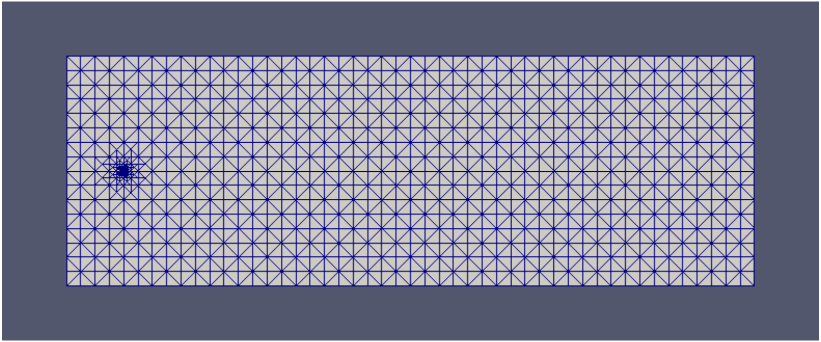
Iteration 4. 929 DOFs



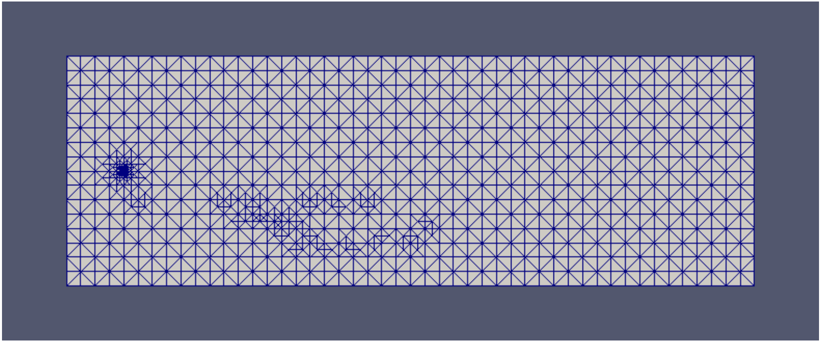
Iteration 8. 1025 DOFs



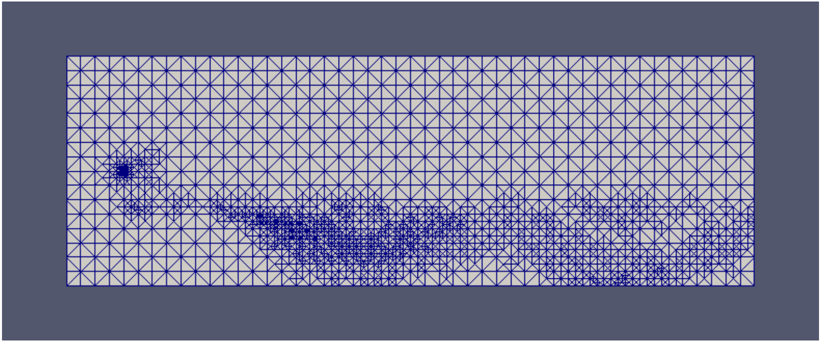
Iteration 12. 1121 DOFs



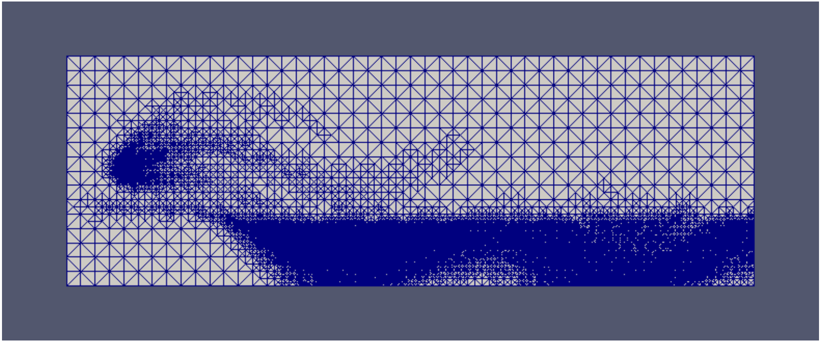
Iteration 16. 1316 DOFs



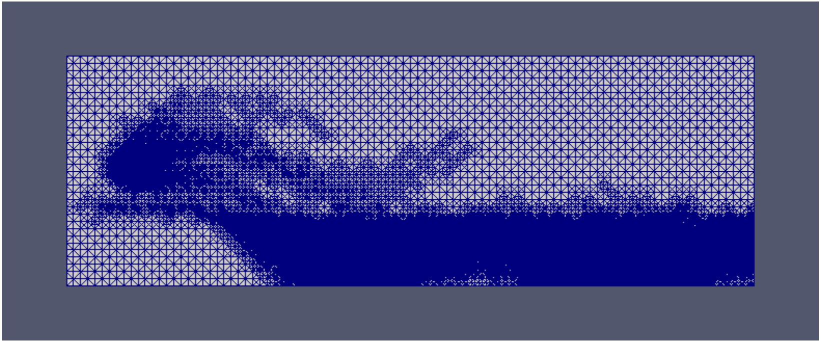
Iteration 20. 2473 DOFs



Iteration 24. 31623 DOFs



Iteration 25. 126181 DOFs



Estimator