# The normal subsemigroups of the monoid of injective maps

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$M$ infinite set

$S(M) = \mathrm{Sym}(M) = \{$all permutations $g : M \to M\}$

$\mathrm{Inj}(M) = \{ f : M \to M \mid f \text{ injective} \}$

with composition of functions: a monoid

$U \subseteq \mathrm{Inj}(M)$ is <u>normal</u>

$:\Longleftrightarrow$ $U^g \subseteq U$ for all $g \in S(M)$.

$f \in \mathrm{Inj}(M) \rightsquigarrow |f| = |\{ x \in M \mid xf \neq x \}|$

$S^\nu(M) := \{ g \in S(M) \mid |g| < \nu \}$ $\quad (\nu \leq |M|^+)$.

<u>Thm. 1</u> (Schreier + Ulam 1933, Baer 1934).

The normal subgroups of $S(M)$ are

$S^\nu(M)$ $(\nu \leq |M|^+)$, $\mathrm{Alt}(M)$ and $\{1\}$.

Thm. 2 (Mesyan 2010, 2012).

$M$ countable.

Description of all normal subsemigroups of $\mathrm{Inj}(M)$.

There are $2^{2^{\aleph_0}}$ many.

$\mathrm{Inj}(M)$ has 3 maximal normal subsemigroups.

Goal:

Description of all normal subsemigroups of $\mathrm{Inj}(M)$ for uncountable $M$.

Cor. $M$ infinite, $|M| = \aleph_\lambda$

$$\kappa(M) := |\{\nu \mid \nu \text{ cardinal}, 0 \leq \nu \leq |M|\}|$$

$\mathrm{Inj}(M)$ has $2^{\kappa(M)^{\aleph_0}}$ normal subsemigroups.

$\mathrm{Inj}(M)$ has $|\lambda| + 3$ maximal normal subsemigroups.

E.g. $|M| = \aleph_0$ or $\aleph_1$ $\Rightarrow$ $\kappa(M) = \aleph_0$

$$2^{\kappa(M)^{\aleph_0}} = 2^{2^{\aleph_0}}.$$

New phenomenon for $f \in \mathrm{Inj}(M) \setminus S(M)$:

$$\exists\, x \in M \setminus Mf$$

Then $x \mapsto xf \mapsto xf^2 \mapsto \ldots$

infinite "forward orbit"

$\Rightarrow |M \setminus Mf| \leq |f|$

<u>Def.</u> $\bar{f} : \mathbb{N}_\infty \to$ cardinals $\leq |M|$

$n \mapsto |\{U \subseteq M \mid U = Uf,\ |U| = n\}|$

$U$ closed orbit of $f$

$\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$

Well-known:

<u>Fact 1</u> $f, g \in S(M)$

$$g \in f^{S(M)} \iff \bar{f} = \bar{g}$$

<u>Fact 2</u> (Mesyan) $f, g \in \mathrm{Inj}(M)$

$$g \in f^{S(M)} \iff |M \setminus Mf| = |M \setminus Mg| \text{ and } \bar{f} = \bar{g}$$

<u>Pf.</u> "$\Rightarrow$" clear.

"$\Leftarrow$" $h : M \setminus Mf \to M \setminus Mg$

(n-orbits of $f$) $\to$ (n-orbits of $g$)

extends to $h \in S(M)$ with $f^h = g$.

More difficult:

<u>Thm. 3</u> (a) (D. + Göbel 1979) $f, h \in S(M)$, $|f| \geq \aleph_0$

$h \in (f^{S(M)})^4 \iff |h| \leq |f|$. [$\Rightarrow$ Baer]

(b) (Moran 1985, D+G.) $f, h \in S(M)$ almost of type 0

$h \in (f^{S(M)})^2 \iff |h| \leq |f|$.

$g \in S(M)$ <u>almost of type 0</u>

$:\iff \tilde{g}(n) = 0$ or $\tilde{g}(n) \geq \aleph_0$ for all $n \in \mathbb{N}_\infty$.

<u>Observation</u> (Mesyan) $f, g \in \mathrm{Inj}(M)$

$\Rightarrow |M \setminus Mfg| = |M \setminus Mf| + |M \setminus Mg|$.

<u>Thm. 4</u> (Mesyan 2010). $M$ countable

$f, g, h \in \mathrm{Inj}(M) \setminus S(M)$

$h \in f^{S(M)} \cdot g^{S(M)} \iff |M \setminus Mh| = |M \setminus Mf| + |M \setminus Mg|$.

Thm. 3 (b) + Thm. 4 $\Rightarrow$

<u>Cor.</u> $f, h \in \mathrm{Inj}(M)$

$|M \setminus Mh| = |M \setminus Mf| \geq \aleph_0$

$h \in (f^{S(M)})^2 \iff |h| \leq |f|$.

Obvious normal subsemigroups of $\mathrm{Inj}^-(M)$:

for $\aleph_0 \le \mu, \nu \le |M|^+$ :

$$\mathrm{Inj}^{-\nu}(M) := \{ f \in \mathrm{Inj}^-(M) \mid |f| < \nu \} \lhd \mathrm{Inj}^-(M)$$

$$\mathrm{Inj}^-_\mu(M) := \{ f \in \mathrm{Inj}^-(M) \mid |M \setminus Mf| = \mu \} \lhd \mathrm{Inj}^-(M)$$

$$\mathrm{Inj}^{-\nu}_\mu(M) := \mathrm{Inj}^-_\mu(M) \cap \mathrm{Inj}^{-\nu}(M) \lhd \mathrm{Inj}^-(M) \quad (\mu < \nu)$$

$$\mathrm{Inj}^-_{fin}(M) := \{ f \in \mathrm{Inj}^-(M) \setminus S(M) \mid M \setminus Mf \text{ finite} \} \lhd \mathrm{Inj}^-(M)$$

Baer-Levi semigroup

Maltcev, Mitchell, Ruškuc (2009): $\neg$ Bergman property

$G \lhd \mathrm{Inj}^-(M)$ $\curvearrowright$

$G_{grp} := G \cap S(M)$ ... is a group:

$$g \in G_{grp} \Rightarrow \bar{g} = g^{-1}$$

$$\Rightarrow g^{-1} \in g^{S(M)} \subseteq G \text{ by } G \lhd \mathrm{Inj}^-(M)$$

$G_{fin} := G \cap \mathrm{Inj}^-_{fin}(M)$

$G_\mu := G \cap \mathrm{Inj}^-_\mu(M)$

Thm. 5. Let $|M| = \kappa \geq \aleph_0$ and $G \lhd \mathrm{Inj}^{-}(M)$. Then

$$G = G_{grp} \mathbin{\dot\cup} G_{fin} \mathbin{\dot\cup} \dot\bigcup_{\aleph_0 \leq \mu \leq \kappa} G_\mu \qquad \text{(clear).}$$

a) $G_{grp} \lhd S(M)$ or $G_{grp} = \emptyset$.

b) $G_\mu = \mathrm{Inj}_\mu^{-\nu}(M)$ for some $\mu < \nu \leq \kappa^+$ or $G_\mu = \emptyset$.

c) $G_\mu = \mathrm{Inj}_\mu^{-\nu}(M)$, $G_{\mu'} \neq \emptyset$, $\mu < \mu' < \nu$

$\Rightarrow G_{\mu'} = \mathrm{Inj}_{\mu'}^{-\nu'}(M)$ for some $\nu' \geq \nu$

d) $G_{fin} \nsubseteq \mathrm{Inj}^{-\nu}(M)$, $G_{\mu'} \neq \emptyset$, $\aleph_0 \leq \mu' \leq \nu$

$\Rightarrow \mathrm{Inj}_{\mu'}^{-\nu^+}(M) \subseteq G_{\mu'}$.

All these combinations with some normal $G_{fin} \subseteq \mathrm{Inj}_{fin}^{-}(M)$ lead to $G \lhd \mathrm{Inj}^{-}(M)$.

$\rightsquigarrow$ determine $G_{fin} = \{g \in G \mid M \neq Mg,\ M \setminus Mg \text{ finite}\}$

finitely many forward orbits

$x \mapsto xg \mapsto xg^2 \mapsto \ldots$

$G \triangleleft \mathrm{Inj}^{-}(M)$

$N(G) := \{ |M \setminus Mf| \mid f \in G_{fin} \}$

$|M \setminus Mfg| = |M \setminus Mf| + |M \setminus Mg| \Longrightarrow$

$N(G) \subseteq (\mathbb{N}, +)$ subsemigroup

Conversely:

$N \subseteq (\mathbb{N}, +)$ subsemigroup $\rightsquigarrow$

$\mathrm{Inj}^{-}_N(M) := \{ g \in \mathrm{Inj}^{-}(M) \mid |M \setminus Mg| \in N \} \triangleleft \mathrm{Inj}^{-}(M)$

... used to describe $G_{fin}$ if $M$ is countable

(involved!)

$M$ uncountable: we need more.

Moran (1985) described all conjugacy classes

$C_1, C_2, C_3$ in $S(M)/S^k(M)$ $(k = |M| \geq \aleph_1)$ with

$C_1 \subseteq C_2 \cdot C_3$.

$\rightsquigarrow$ reduces to $f \in S(M)$ with

$\bar{f}(n) = 0$ or $\bar{f}(n) = |M|$ for each $n \in \mathbb{N}_\infty$.

$f$ "<u>simple</u>".

<u>Thm. 6</u>. (Moran 1985) $M$ uncountable,

$f, g, h \in S(M)$ simple.

$h \notin f^{S(M)} \cdot g^{S(M)} \iff$

$\{\bar{f}, \bar{g}, \bar{h}\}$ realizes

- one of 3 possibilities involving only 1-, 2-, 3-orbits
- one possibility involving 1-, 2-, $U$-orbits, for some set $U \subseteq \mathbb{N}$ of odd numbers.

Depends only on $\bar{f} : \mathbb{N}_\infty \to$ cardinals $\leq |M|$.

We need a characterization of $h \in f^{S(M)} \cdot g^{S(M)}$

for <u>almost simple</u> $f, g, h \in S(M)$,

i.e., $\bar{f}(n) = 0$ or $\bar{f}(n) \geq \aleph_1$ for each $n \in \mathbb{N}_\infty$.

Let $\aleph_1 \leq \alpha \leq |M|$, $M_\alpha = \alpha$.

$f_\alpha \in S(M_\alpha)$ is <u>$\alpha$-simplification</u> of $f \in S(M)$

$$:\Longleftrightarrow \quad \overline{f_\alpha}(n) = \begin{cases} \alpha & \text{if } \bar{f}(n) \geq \alpha \\ 0 & \text{if } \bar{f}(n) < \alpha \end{cases}$$

$\Rightarrow$ $f_\alpha \in S(M_\alpha)$ is simple.

<u>Thm. 7.</u> $M$ uncountable

$f, g, h \in S(M)$ almost simple.

$h \in f^{S(M)} \cdot g^{S(M)} \Longleftrightarrow$

$h_\alpha \in f_\alpha^{S(M_\alpha)} \cdot g_\alpha^{S(M_\alpha)}$ for some (equiv., each)

$\alpha$-simplification $f_\alpha, g_\alpha, h_\alpha$ of $f, g, h$,

for each $\alpha$ with $\aleph_1 \leq \alpha \leq |M|$ and $\mathrm{cf}(\alpha) \neq \omega$.

$\leadsto$ condition only on $\overline{f_\alpha}, \overline{g_\alpha}, \overline{h_\alpha}$ simple.

Let $f \in \mathrm{Inj}(M)$, $M \setminus Mf$ countable

$f' \in S(M)$ is a <u>type</u> of $f$

$$:\Leftrightarrow \quad \overline{f'}(n) = \begin{cases} \overline{f}(n) & \text{if } \overline{f}(n) \geq \aleph_1 \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N}_\infty).$$

$\rightsquigarrow$ $f'$ is almost simple.

<u>Cor.</u> $M$ uncountable

$f, g, h \in \mathrm{Inj}(M) \setminus S(M)$

$M \setminus Mf$, $M \setminus Mg$, $M \setminus Mh$ countable.

$h \in f^{S(M)} \cdot g^{S(M)} \quad \Leftrightarrow$

$|M \setminus Mh| = |M \setminus Mf| + |M \setminus Mg|$ and

$h' \in f'^{\,S(M)} \cdot g'^{\,S(M)}$ for some (equiv., all)

types $f', g', h'$ of $f, g, h$.

$\rightsquigarrow$ condition on almost simple $f', g', h'$.

$\mathcal{T}_M := \{ \bar{f} \mid f \in S(M),\ f \text{ almost simple} \}$

$(\bar{f}, \bar{g}, \bar{h}) \in R_M \ :\Longleftrightarrow\ h \in f^{S(M)} \cdot g^{S(M)}$

$N \subseteq (\mathbb{N}, +)$ subsemigroup

**Def.** $P \subseteq N \times \mathcal{T}_M$ <u>$N$-type set</u>

$:\Longleftrightarrow \ \forall n \in N \ \exists T \in \mathcal{T}_M : \ (n, T) \in P$

$\forall (n_1, T_1), (n_2, T_2) \in P,\ T \in \mathcal{T}_M :$

$(T, T_1, T_2) \in R_M \Rightarrow (n_1 + n_2, T) \in P.$

**Fact.** 1) $G \lhd \mathrm{Inj}^-(M)$

$\Rightarrow P(G) := \{ (|M \setminus Mf|, \bar{f'}) \mid f \in G_{fin} \}$

is an $N(G)$-type set.

2) $N \subseteq (\mathbb{N}, +)$,

$P$ an $N$-type set

$\Rightarrow \mathrm{Inj}^-_P(M) := \{ f \in \mathrm{Inj}^-(M) \mid (|M \setminus Mf|, \bar{f'}) \in P \}$

$\lhd \mathrm{Inj}^-(M)$

## Characterization of $G_{fin}$ : 4 cases:

1) $G$ contains a permutation with infinite support

2) $G_{grp} = G \cap S(M) = \{1\}$

3) $G_{grp} = S^{\aleph_0}(M)$

4) $G_{grp} = \mathrm{Alt}(M)$

### Case 1:

**Thm. 8.** $M$ uncountable

$G \lhd \mathrm{Inj}^{-}(M)$ as above

$G_{fin} \neq \emptyset$

$\Rightarrow \exists N \leq (\mathbb{N},+) \; \exists P \subseteq N \times \mathcal{T}_M$ $N$-type set:

$$G_{fin} = \mathrm{Inj}^{-}_{P}(M).$$

Moreover:

- $G_{fin} \subseteq \mathrm{Inj}^{-\aleph_1}(M) \Rightarrow G_{fin} = \mathrm{Inj}^{-\aleph_1}_{N}(M)$,
  i.e. $P = N \times \{\bar{1}\}$
- $G_{grp} = S(M) \Rightarrow G_{fin} = \mathrm{Inj}_N(M)$,
  i.e. $P = N \times \mathcal{T}_M$.

Cases 2-4 similar, more technical.

<u>Thm. 9.</u> $M$ infinite.

Every injection $f \in \mathrm{Inj}(M)$ is a product $f = g \cdot h$ of

- a permutation $g \in S(M)$
- an injection $h \in \mathrm{Inj}(M)$

both having only infinitely many infinite orbits.

<u>Cor.</u> (Ore 1951) $M$ infinite.

Each permutation $f \in S(M)$ is a commutator:

$$f = [g, k].$$

<u>Proof.</u> Thm. 9 $\Rightarrow$ $f = g \cdot h = g \cdot (g^{-1})^k$. $\square$

only infinite orbits
infinitely many

## Proof of Thm. 9.

W.l.o.g. $|M| = \aleph_0$, $M = \mathbb{Z} \times \mathbb{Z}$.

Let $f \in \mathrm{Inj}(M)$.

$\Rightarrow \exists f' \in \mathrm{Inj}(M)$:

$$|M \setminus Mf| = |M \setminus Mf'|$$

$$\bar{f} = \bar{f'}$$

$$\forall (i,j) \in M: (i,j)f' \in \mathbb{Z} \times \{j-1, j, j+1\}$$

i.e., $f'$ moves each point $x \in \mathbb{Z} \times \mathbb{Z}$ at most 1 unit up or down.

$\Rightarrow f \in f'^{S(M)}$.

Def. $g: M \to M$

$$(i,j) \mapsto (i, j+2)$$

$\leadsto$ $g \in S(M)$, has only infinite orbits, inf. many

$\Rightarrow h := g \cdot f' \in \mathrm{Inj}(M)$

$$\forall (i,j) \in M: (i,j)h \in \mathbb{Z} \times \{j+1, j+2, j+3\}$$

i.e., $h$ moves each point at least 1 unit up.

$\Rightarrow h$ has only infinite orbits, inf. many

$\Rightarrow f' = g^{-1} \cdot h$ as claimed. $\square$