Embedding in 2-generated semigroups using transformations

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- Any finite semigroup embeds in a 2-generator semigroup (BH Neumann, 1960)

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Proof.

Embed S via S^1 in $T_X = T_n$, where $n = |S^1|$. Write $S^1 = \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$, where $\alpha_0 = \iota$, the identity mapping in T_n. We embed S in $T \leq PT_Z$ where $Z = X \times \{0, 1, 2, \cdots, n\}$, where we also put $\alpha_n = \alpha_0$.

$$(x, i)\cdot\alpha = (x\cdot\alpha_i, 0) \ (0 \leq i \leq n), \ \ (x, i)\cdot\beta = (x, i+1) \ (0 \leq i \leq n-1).$$

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Put $\lambda = \beta^n \alpha$; then $\lambda = \iota|_{X \times \{0\}}$. Let $\gamma_i = \lambda \beta^i \alpha \in T$; then

$$(x, 0) \cdot \gamma_i = (x, 0) \cdot \lambda \beta^i \alpha = (x, 0) \cdot \beta^i \alpha = (x, i) \cdot \alpha = (x \cdot \alpha_i, 0),$$

and so $\alpha_i \mapsto \gamma_i$ is a monomorphism of S^1 into T.

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Also Margolis shows that any (finite) n-generated semigroup embeds in a (finite) semigroup generated by $n+1$ idempotents.
Hence any finite semigroup S embeds in a finite semigroup generated by 3 idempotents.
Any semigroup (finite or not) generated by 2 idempotents has at most 6 idempotents and no 3-element chain. (Benzaken and Mayr) characterised all such semigoups.

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Definition

The MC sequence of non-negative integers begins $m_0 = 0$ and $m_i > m_{i-1}$ is least such that there are no repeated differences between any pairs in the sequence.

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The construction for 2-generator semigroups has one principal generator, α, containing copies of all mappings in $S \leq PT_X$; dom α and ran α consist of n copies of X; the second generator β moves us around that cycle. The domain intervals are sparsely placed so that products with multiple factors of α are defined for one interval at most. The MC property ensures that unwanted products do not arise - the main subsemigroup of T is a Rees-matrix semigroup over S with identity matrix.

First Construction

- The Ingredients

$S^1 = \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\} \leq PT_X, \ \alpha_0 = \iota;$
$Z = X \times \{0, 1, 2, \cdots, m_{2n-1}\}$, put $m = 1 + m_{2n-1}$.

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$$(x, i) \cdot \beta = (x, i + 1)\ (0 \leq i \leq m_{2n-1})$$

The principal generator α satisfies $\alpha^2 = 0$ and acts only on the *intervals* $X \times \{m_{n+j}\}$:

$$(x, m_{n+j}) \cdot \alpha = (x \cdot \alpha_j, m_j)\ (0 \leq j \leq n - 1)$$

- Structure of $T = \langle \alpha, \beta \rangle$:

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and $D_\alpha > T_1 \cong (S \times B)/I$, where B is an $m \times m$ combinatorial Brandt semigroup and I is the ideal $S \times \{0\}$ of $S \times B$.

$$T_1 = \{\lambda(\alpha_i, j, k) : 0 \le i \le n-1,\ 0 \le j, k \le m-1\},$$
$$(x, j) \cdot \lambda(\alpha_i, j, k) = (x \cdot \alpha_i, k).$$

$$E(T) = \bigcup_{i=1}^{m} E_i \cup (0, \iota) \text{ where } E_i = \{\lambda(e, i, i) : e \in E(S),\ 0 \le i \le m-1\}.$$

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Corollary

Let S be a finite monoid with $E(S) \le S$. Then S may be embedded in a finite monoid $T = \langle \alpha, \beta \rangle$ as above such that $E(T)$ is a submonoid satisfying the same semigroup identities as $E(S)$.

Second Construction: orthodox semigroups

The next construction looks to preserve regularity as well as the idempotent structure. Here β is again a cycle but α now satisfies $\alpha = \alpha^3$. We now work with $m_i = 2^i$ and $m = 1 + 2^{n-1}$ as we need a sequence where the MC property to hold for sums and differences of more than two of its members. All additions in what follows are now modulo m.

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$$(x, m_t) \cdot \alpha = (x \cdot \alpha_{t \pm n}, m_{t \pm n}) \ (0 \le t \le 2n - 1)$$

subscript signs are $+$ or $-$ according as $0 \le t \le n - 1$ or $n \le t \le 2n - 1$.

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$$T_1 = \{\lambda_{i,j,k}\} \cup \{0\} \ (0 \le i \le n-1, \ 0 \le j, k \le m-1\}$$

$$E(T) = E \cup F \cup \{\iota, 0\} \text{ where}$$

$$E = \{\lambda(e, i, i) : e \in E(S), \ 0 \le i \le m-1\},$$

$$F = \{\beta^j \alpha^2 \beta^{-j} : 0 \le j \le m-1\}$$

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(b) Any finite orthodox monoid S^1 may be embedded as a semigroup in a finite 2-generated orthodox monoid T whose subband of idempotents satisfies the same semigroup identities as S^1.

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(b) Any finite orthodox monoid S^1 may be embedded as a semigroup in a finite 2-generated orthodox monoid T whose subband of idempotents satisfies the same semigroup identities as S^1.

Corollary

(McAlister, Stephen and Vernitski) Every finite inverse semigroup may be embedded is a finite 2-generated semigroup that is an inverse semigroup.

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