Fricke S-duality and BPS-state counting

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"New Moonshines, Mock Modular Forms and String Theory"
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Talk based on:
[arXiv:1504.07260] (w/ R. Volpato)
[arXiv:1312.0622] (w/ R.Volpato)
[arXiv:1302.5425] (w/ M. Gaberdiel, & R.Volpato)
[arXiv:1211.7074] (w/ M. Gaberdiel, H. Ronellenfitsch, R.Volpato)

Moonshine conjecture (Conway-Norton): The McKay-Thompson series are modular-invariant under some genus zero $\Gamma_g \subset SL(2,\mathbb{R})$

$$\Gamma_g \text{ genus zero} \longleftrightarrow \Gamma_g\backslash\mathbb{H} \sim$$

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- $\Gamma_0(N) \subset SL(2,\mathbb{Z})$ — subgroup that preserves the twisted boundary conditions in the CFT
- $W_N : \tau \to -\dfrac{1}{N\tau}$ — **Fricke involution**

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This is a big remaining mystery of monstrous moonshine!

Is there any situation in string theory where we have discrete "**S-duality symmetries**"

$$\Gamma \subset SL(2,\mathbb{R})$$

but which lie ***outside*** of $SL(2,\mathbb{Z})$?

In our earlier work we defined a class of functions on the Siegel upper half plane:

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for each commuting pair $g, h \in M_{24}$ $\Phi_{g,h}$

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Theorem (D.P.-Volpato):

The functions $\Phi_{g,h}$ *are Siegel modular forms for certain (para-modular) subgroups* $\Gamma^{(2)}_{g,h} \subset Sp(4, \mathbb{R})$

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What is the physical interpretation of these Siegel modular forms?

For $(g,h)=(1,1)$ we obtain $\qquad q=e^{2\pi i\tau}, y=e^{2\pi i z}$

$$\Phi_{1,1}=\Phi_{10}=pqy\prod_{(m,n,\ell)>0}(1-p^m q^n y^\ell)^{c(mn,\ell)}$$

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$$\chi(K3;\tau,z)=\sum_{n\geq 0,\ell\in\mathbb{Z}} c(n,\ell)q^n y^\ell$$

K3 elliptic genus

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The 'Igusa cusp form' Φ_{10} is the generating function of 1/4 BPS-states in N=4 string theory: [Dijkgraaf, Verlinde, Verlinde][Shih, Strominger, Yin]

$$\frac{1}{\Phi_{10}}=\sum_{Q^2/2,P^2/2,P\cdot Q\in\mathbb{Z}} B_6(P,Q)p^{Q^2/2}q^{P^2/2}y^{P\cdot Q}$$

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electric-magnetic charges

$$(Q,P)\in\Gamma^{6,22}\oplus\Gamma^{6,22}$$

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where the coefficients encode the **sixth helicity supertrace**: [Kiritsis]

$$B_6(P,Q):=\frac{1}{6!}\mathrm{Tr}_{\mathcal{H}_{P,Q}}\left((-1)^J(2J)^6\right)$$

J = helicity

Conjecture (Cheng, Govindarajan, DP-Volpato):

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The curious modular property $N = \mathcal{O}(g)$

$$\Phi_{g,h}(\sigma,\tau,z) = \Phi_{g,h'}\left(\tau/N, N\sigma, z\right)$$

then suggests a new 'electric-magnetic duality' in CHL-models:

$$\begin{pmatrix} Q \\ P \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{\sqrt{N}} P \\ -\sqrt{N} Q \end{pmatrix}$$

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In this talk I will show that this is a consequence of a novel **Fricke S-duality** in CHL-models!

1. Fricke S-duality in CHL-models

$\mathcal{N} = 4$ string theory

$$\boxed{\mathrm{Het}/T^6} \longleftrightarrow \boxed{\mathrm{IIA}/K3 \times T^2} \longleftrightarrow \boxed{\mathrm{IIB}/K3 \times T^2}$$

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- Gauge group is generically $U(1)^{28}$
- Electric-magnetic charges $(P, Q) \in \Gamma = \Gamma^{6,22} \oplus \Gamma^{6,22}$
- Duality group $SL(2, \mathbb{Z}) \times O(6, 22; \mathbb{Z})$
- Moduli space

$$SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{Z}) / SO(2) \times O(6, 22; \mathbb{Z}) \backslash O(6, 22; \mathbb{R}) / (O(6) \times O(22))$$

CHL-models

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- δ a shift of order N along T^2
- $g \in O(\Gamma^{4,20})$ a symmetry of the K3 non-linear sigma model:
  - which has **order** N
  - **preserves all spacetime supersymmetries**
  - **exists at generic points** where the gauge group is $U(1)^{28}$

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This construction yields a class of 4d $\mathcal{N} = 4$ string theories

[Chaudhuri, Hockney, Lykken][Chaudhuri, Lowe]

Dualities of CHL-models

$$\boxed{\mathrm{Het}/\frac{T^4\times T^2}{\langle(g,\delta)\rangle}} \longleftrightarrow \boxed{\mathrm{IIA}/\frac{K3\times T^2}{\langle(g,\delta)\rangle}} \longleftrightarrow \boxed{\mathrm{IIB}/\frac{K3\times T^2}{\langle(g,\delta)\rangle}}$$

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- At least 3 moduli in each sector:

Heterotic: $S_{\rm het}$ $T_{\rm het}$ $U_{\rm het}$

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axio-dilaton

Kähler mod. of T^2

Cplx. str. mod. of T^2

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- At least 3 moduli in each sector:

Heterotic: S_{het} T_{het} U_{het}

IIA: S_{IIA} T_{IIA} U_{IIA}

$$\begin{tikzcd}
S_{\mathrm{het}} \arrow[dr, leftrightarrow] & T_{\mathrm{het}} \arrow[dl, leftrightarrow] & U_{\mathrm{het}} \arrow[d, leftrightarrow] \\
S_{\mathrm{IIA}} & T_{\mathrm{IIA}} & U_{\mathrm{IIA}}
\end{tikzcd}$$

string-string duality

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\text{Heterotic:} & S_{\mathrm{het}} \arrow[dr, leftrightarrow] & T_{\mathrm{het}} \arrow[dl, leftrightarrow] & U_{\mathrm{het}} \arrow[d, leftrightarrow] \\
\text{IIA:} & S_{\mathrm{IIA}} & T_{\mathrm{IIA}} \arrow[r, phantom, "\textit{string-string duality}"] & U_{\mathrm{IIA}}
\end{tikzcd}$$

- The S-duality group $SL(2,\mathbb{Z})$ is broken to

$$\Gamma_1(N)=\left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in SL(2,\mathbb{Z})\,\middle|\, a\equiv 1 \bmod N, c\equiv 0 \bmod N\right\}$$

Classification of CHL-models

All symmetries of K3 sigma models have been classified by Gaberdiel, Hohenegger, Volpato:

- Each $g \in O(\Gamma^{4,20})$ that preserves the sigma model corresponds to an element of the Conway group Co_0

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This implies that inequivalent CHL-models are characterised by the eigenvalues of g in the defining 24-dimensional reps of Co_0

Frame shape: $g \leftrightarrow \prod_{a|N} a^{m(a)}$ where $\sum_{a|N} am(a) = 24$

Ex: $g = \text{identity} \leftrightarrow 1^{24}$ *product of 24 identity permutations*

The orbifold groups $\langle(\delta, g)\rangle$ are defined up to $O(\Gamma^{6,22})$-conjugation

$$\boxed{O(\Gamma^{6,22})\text{-classes } \langle(\delta, g)\rangle}$$

$$\downarrow$$

$$\boxed{\text{conjugacy class } [g] \in Co_0}$$

$$\downarrow$$

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Frame shape $\prod_{a|N} a^{m(a)}$

***Inequivalent CHL-models** associated to pairs (δ, g) are **classified by the frame shape of** $[g] \in Co_0$*

Fricke T-duality

$$\mathrm{IIA}/\frac{K3 \times T^2}{\langle(\delta, g)\rangle}$$

$$T^2 = S^1 \times \tilde{S}^1$$

special circle on which δ acts by an order N shift

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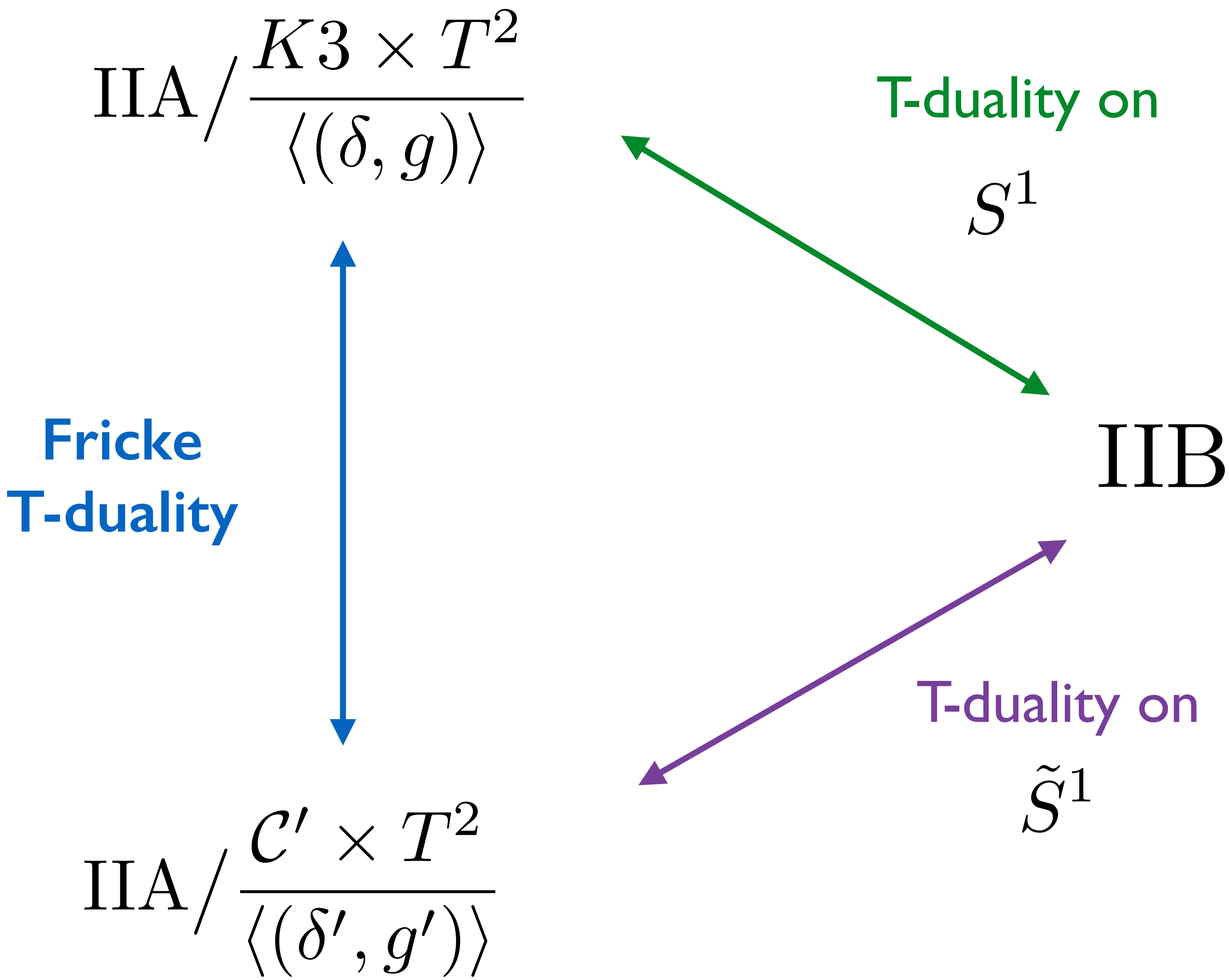
$$\mathrm{IIA}/\frac{\mathcal{C}'\times T^2}{\langle(\delta',g')\rangle}$$

Fricke T-duality



Fricke T-duality

$$\mathrm{IIA}/\frac{K3\times T^2}{\langle(\delta,g)\rangle}$$

Fricke T-duality $\updownarrow$

$$\mathrm{IIA}/\frac{\mathcal{C}'\times T^2}{\langle(\delta',g')\rangle}$$

acts on the moduli by:

$$T_{\mathrm{IIA}}\to-\frac{1}{NT_{\mathrm{IIA}}}$$

$$U_{\mathrm{IIA}}\to-\frac{1}{NU_{\mathrm{IIA}}}$$

$$S_{\mathrm{IIA}}\to S_{\mathrm{IIA}}$$

(similar to earlier work by Vafa in the non-compact setting)

3 possible cases

The image of the Fricke T-duality is a **non-linear sigma model** $\mathcal{C}'$

Compute the **Witten index** (Euler characteristic):

$$I^{\mathcal{C}'} = \mathrm{Tr}_{\mathcal{C}'}(-1)^{F_L+F_R}$$

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$$I^{\mathcal{C}'} = \mathrm{Tr}_{\mathcal{C}'}(-1)^{F_L+F_R} = \sum_{a|N} m(a/N)a$$

Non-linear sigma model on K3: $I^{\mathcal{C}'} = 24$

Non-linear sigma model on T^4: $I^{\mathcal{C}'} = 0$

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We find 3 possibilities

- $I^{\mathcal{C}'} = 24$ and (g, g') have the **same** Frame shape
- $I^{\mathcal{C}'} = 24$ and (g, g') have **different** Frame shapes
- $I^{\mathcal{C}'} = 0$

Case I

Frame shape is **balanced:** $m(a) = m(N/a)$

The CHL-model is self-dual under Fricke T-duality: $T_{\mathrm{IIA}} \to -\frac{1}{NT_{\mathrm{IIA}}}$

In the heterotic picture this yields a new **Fricke S-duality**

$$S_{\mathrm{het}} \to -\frac{1}{NS_{\mathrm{het}}}$$

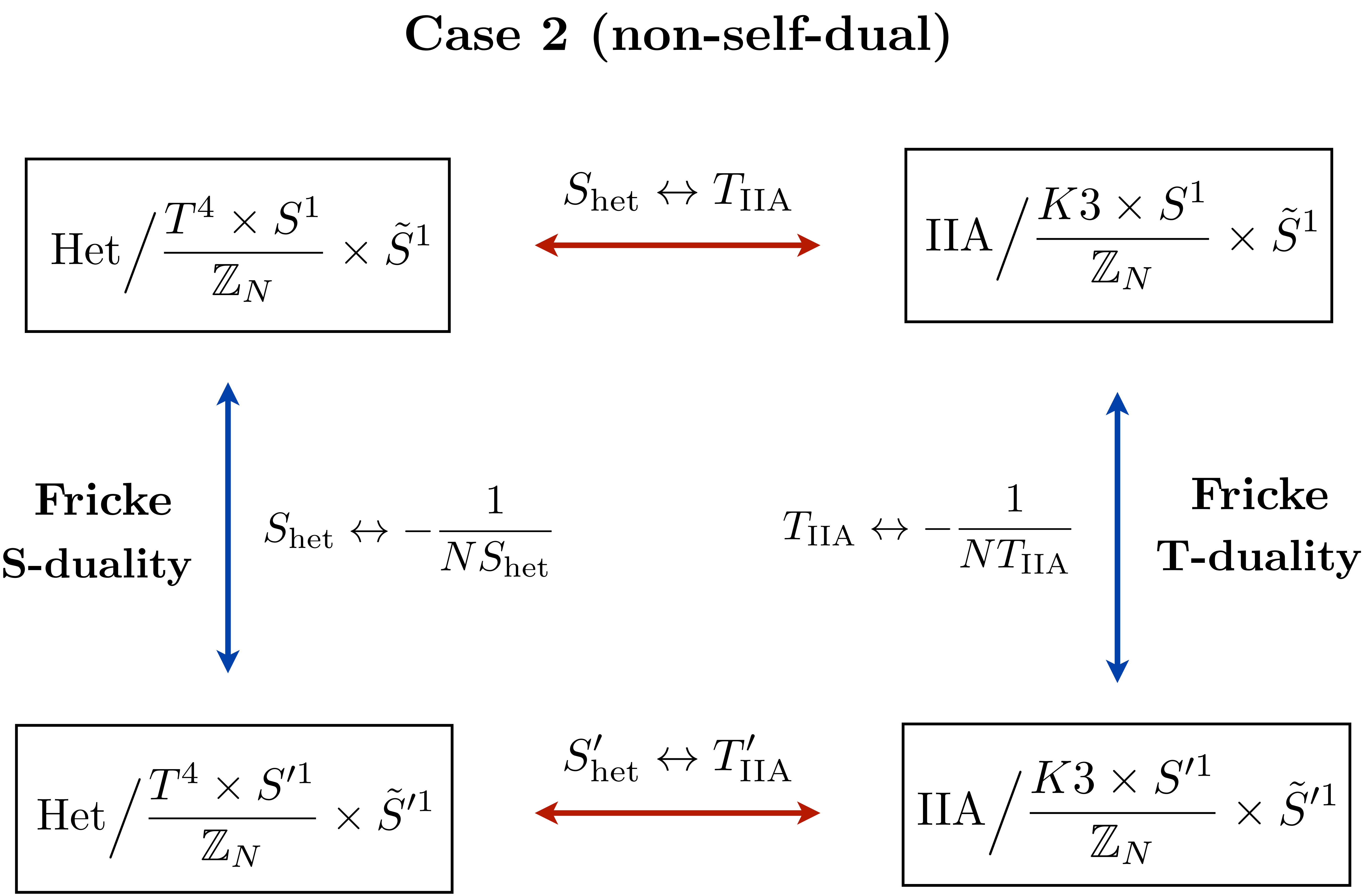
This is a new symmetry of CHL-models which lies outside of the $SL(2,\mathbb{Z})$ -symmetry of the parent theory Het/T^6 !

Case 1 (self-dual)

$$\boxed{\mathrm{Het}\Big/\frac{T^4\times S^1}{\mathbb{Z}_N}\times\tilde{S}^1} \xleftrightarrow{\;S_{\mathrm{het}}\,\leftrightarrow\, T_{\mathrm{IIA}}\;} \boxed{\mathrm{IIA}\Big/\frac{K3\times S^1}{\mathbb{Z}_N}\times\tilde{S}^1}$$

Fricke S-duality $\quad S_{\mathrm{het}}\leftrightarrow-\dfrac{1}{NS_{\mathrm{het}}}$

Fricke T-duality $\quad T_{\mathrm{IIA}}\leftrightarrow-\dfrac{1}{NT_{\mathrm{IIA}}}$


Case 2 (non-self-dual)
$\mathrm{Het} \Big/ \frac{T^4 \times S^1}{\mathbb{Z}_N} \times \tilde{S}^1$
$S_{\text{het}} \leftrightarrow T_{\text{IIA}}$
$\mathrm{IIA} \Big/ \frac{K3 \times S^1}{\mathbb{Z}_N} \times \tilde{S}^1$
Fricke S-duality
$S_{\text{het}} \leftrightarrow -\frac{1}{N S_{\text{het}}}$
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Fricke T-duality
$\mathrm{Het} \Big/ \frac{T^4 \times S'^1}{\mathbb{Z}_N} \times \tilde{S}'^1$
$S'_{\text{het}} \leftrightarrow T'_{\text{IIA}}$
$\mathrm{IIA} \Big/ \frac{K3 \times S'^1}{\mathbb{Z}_N} \times \tilde{S}'^1$

Case 3 (non-self-dual)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{Het}\Big/\dfrac{T^4\times S^1}{\mathbb{Z}_N}\times \tilde{S}^1} \arrow[r, leftrightarrow, "S_{\text{het}} \leftrightarrow T_{\text{IIA}}"] \arrow[d, leftrightarrow, swap, "\substack{\textbf{Fricke}\\ \textbf{S-duality}}"] \arrow[d, leftrightarrow, "S_{\text{het}} \leftrightarrow -\dfrac{1}{NS_{\text{het}}}"] & \boxed{\text{IIA}\Big/\dfrac{K3\times S^1}{\mathbb{Z}_N}\times \tilde{S}^1} \arrow[d, leftrightarrow, swap, "T_{\text{IIA}} \leftrightarrow -\dfrac{1}{NT_{\text{IIA}}}"] \arrow[d, leftrightarrow, "\substack{\textbf{Fricke}\\ \textbf{T-duality}}"] \\
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\end{tikzcd}
$$

Electric-magnetic duality and N-modularity

Consider now the **self-dual case.** The full **S-duality group** is

$$\Gamma_g = \left\langle \Gamma_0(N), \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix} \right\rangle$$

This acts by:

axiodilaton $$S_{\rm het} \to \frac{aS_{\rm het} + b}{cS_{\rm het} + d}$$

electric-magnetic charges $$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

Restricting to the Fricke part we find

electric-magnetic charges $$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{N}} Q \\ -\sqrt{N} P \end{pmatrix}$$

As a consequence the charge lattices $\Gamma = \Gamma_e \oplus \Gamma_m$ must satisfy

$$\Gamma_m \cong \sqrt{N} \Gamma_e$$

But we also have $\Gamma_m \cong \Gamma_e^*$ which yields

$$\Gamma_e^* \cong \sqrt{N} \Gamma_e \qquad N\text{-modular}$$

This is a non-trivial prediction of Fricke S-duality!

3. BPS-state counting

Counting of Dabholkar-Harvey states

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These can be taken to have **purely electric charges** $Q \in \Gamma^{6,22}$

The degeneracy $\Omega(Q)$ of such states is captured by

$$\frac{1}{\Delta(\tau)} = \frac{1}{\eta(\tau)^{24}} = \sum_{n\in\mathbb{Z}} d(n) q^n$$

$$\Omega(Q) = d(Q^2/2)$$

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In the type IIB picture these correspond to certain bound states of D0-D4-NS5-branes on $K3 \times T^2$ with momentum along the torus.

These are 'small black holes' with zero classical entropy:

$$\log \Omega(Q) \sim 4\pi\sqrt{Q^2}$$

Topological BPS-couplings

In general, 1/2 BPS-states in $\mathcal{N} = 4$ theories are counted by the **4th helicity supertrace**: [Kiritsis]

$$B_4 = \mathrm{Tr}(-1)^F J^4 \qquad\qquad J = \text{helicity}$$

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$$B_4 = \mathrm{Tr}(-1)^F J^4 \qquad\qquad J = \text{helicity}$$

In $\mathrm{IIA}/K3\times T^2$ this determines the **topological 1-loop amplitude**:

$$F_1 = \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \frac{d^2\tau}{\tau_2} B_4(T,U)$$

$$= \log(T_2^{24}|\Delta(T)|^4) + \log(U_2^{24}|\Delta(U)|^4) + \mathrm{const}$$

[Harvey, Moore]

$$F_1 = \log(T_2^{24}|\Delta(T)|^4) + \log(U_2^{24}|\Delta(U)|^4) + \mathrm{const}$$

Notice that the discriminant Δ now appears as a function of the **spacetime** moduli (T, U) rather than the worldsheet τ

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Notice that the discriminant Δ now appears as a function of the **spacetime** moduli (T, U) rather than the worldsheet τ

This is not a coincidence but follows from the OSV-conjecture:

$$Z_{CFT} = Z_{BH} = |Z_{top}|^2$$

which requires a particular identification of worldsheet and spacetime variables. In the case at hand we indeed have:

$$Z_{CFT}(\tau) = \frac{1}{\Delta(\tau)} \qquad Z_{BH}(T) = e^{F_1^{hol}(T)} = e^{-\log\,\Delta(T)}$$

which coincide provided we identify $\tau = T$

[Ooguri, Strominger, Vafa][Dabholkar][Dabholkar, Denef, Moore, Pioline]

BPS-counting in CHL-models

In general the **n:th helicity supertraces** can be calculated via: [Kiritsis]

$$B_n = \left(\frac{1}{2\pi i} \frac{\partial}{\partial v} + \frac{1}{2\pi i} \frac{\partial}{\partial \bar{v}} \right)^n Z(v, \bar{v}) \Big|_{v=\bar{v}=0}$$

where the **generating function** is defined by

$$Z(v, \bar{v}) = \mathrm{Tr}(-1)^F e^{2\pi i v J_3^R} e^{2\pi i \bar{v} J_3^L} q^{L_0} \bar{q}^{\bar{L}_0}$$

BPS-counting in CHL-models

For the type IIA CHL-model $\mathrm{IIA}/\dfrac{K3\times T^2}{\langle(\delta,g)\rangle}$ with Frame shape

$$g \leftrightarrow \prod_{a|N} a^{m(a)}$$

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Siegel-Narain theta function for the lattice $\Gamma^{2,2}$

The associated 1/2 BPS-saturated topological amplitude is

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and we obtain

$$F_1^{[g]}(T) = \log \prod_{a|N} \left(aT_2|\eta(aT)|^4\right)^{m(a)} = \log\left(T_2^{24}|\eta_g(T)|^4\right)$$

where the eta-product is defined by

$$\eta_g(T) = \prod_{a|N} \eta(aT)^{m(a)}$$

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This generalizes earlier results in the literature. For example:

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2^{12}	$F_1^{[g]} = 12\log(T_2\|\eta(T)^2\theta_4(T)^2\|)$

Matches with [Antoniadis, Gava, Narain, Taylor][Dabholkar, Denef, Moore, Pioline]

Fricke duality of BPS-couplings

Whenever the type IIA CHL-model is self-dual we expect that the BPS-coupling to be invariant under **Fricke T-duality**

$$T \longrightarrow -\frac{1}{NT}$$

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$$F_1^{[g]}(-1/NT) = \log \prod_{a|N} \left(aT_2|\eta(aT)|^4\right)^{m(N/a)}$$

This is invariant if the Frame shape is **balanced:** $m(N/a) = m(a)$

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This is precisely the case for the self-dual models!

By heterotic-type II duality the corresponding heterotic coupling is invariant under **Fricke S-duality**

$$S \longrightarrow -\frac{1}{NS}$$

Summary

- Uncovered novel Fricke dualities in a large class of CHL-models
- Demonstrated consistency with heterotic-type II duality
- Checked the prediction of N-modularity of charge lattices
- Demonstrated that 1/2 BPS-couplings are Fricke invariant
- Physical interpretation of the modular properties of Siegel modular forms arising in Mathieu moonshine

Outlook

- Do Fricke dualities exist also in models with less susy?
  - connection with Fricke symmetries observed in topological strings?

    [Alim, Scheidegger, Yau, Zhou]

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Conjecture:

$$\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} B_6^{[g]} = \log\left((\det\Im\Omega)^{w_{g,e}}|\Phi_{g,e}(T,U,V)|^2\right)$$

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- Do the Siegel modular forms $\Phi_{g,e}$ count **reduced** Gromov-Witten invariants on $(K3 \times T^2)/\langle(\delta, g)\rangle$?

  This would generalise a recent conjecture of [Oberdieck, Pandharipande] corresponding to the case $g = e$

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- Can we make a similar "CHL-version" of the monster CFT to shed light on the elusive genus zero property of moonshine?

  [Paquette, D.P., Volpato] (in progress)

See Roberto's talk tomorrow!

Thank you!