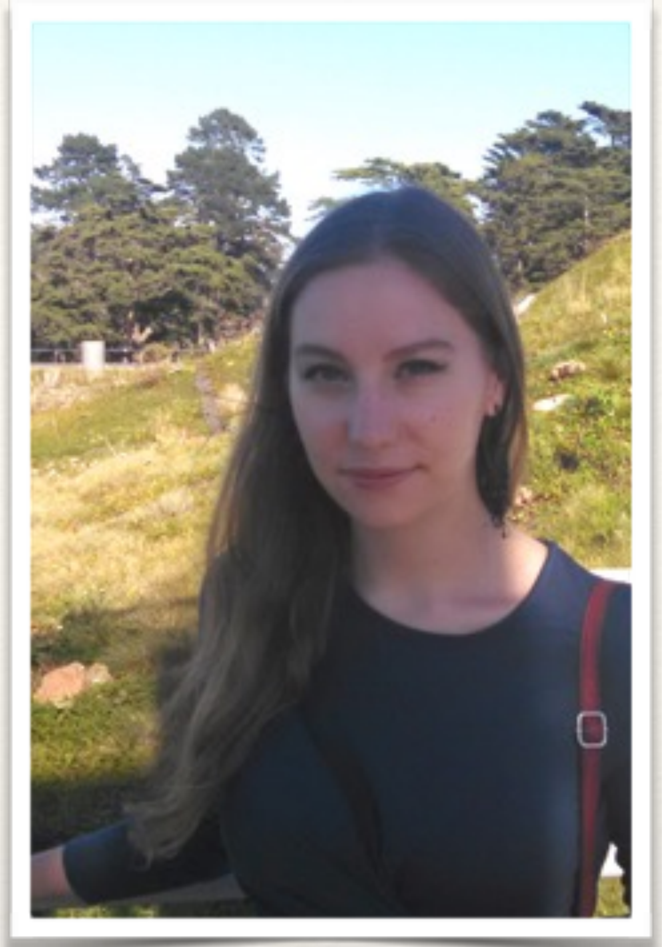
LMS-EPSRC Durham Symposia, 12 August 2015

Monstrous Heterosis

Roberto Volpato
SLAC & SITP Stanford

New Moonshines, Mock Modular Forms and String Theory *work in progress with*



Natalie Paquette

Daniel Persson

Motivation

- Monstrous Moonshine: mysterious relation

  [McKay,Thompson; Conway, Norton]

  McKay-Thompson $T_g(\tau)$ $\longleftrightarrow$ Monster group $\mathbb{M}$

- "Physics" interpretation:
  - 2-D holomorphic CFT $V^\natural$ with $c = 24$ such that
    - $\mathbb{M}$ is group of automorphisms of $V^\natural$
    - $T_g(\tau)$ is a "twined" partition function

    [Frenkel, Lepowsky, Meurman]
  - Explains invariance of $T_g(\tau)$ under subgr. of $SL_2(\mathbb{Z})$

Motivation

- $T_g(\tau)$ is Hauptmodul for genus zero group $\Gamma_g \subset SL_2(\mathbb{R})$
- In many cases, $\Gamma_g \not\subset SL_2(\mathbb{Z})$ but contains Atkin-Lehner involutions, such as

$$\tau \to -\frac{1}{N\tau}$$

- Proved using (chiral!) bosonic string theory + generalized (Borcherds-)Kac-Moody algebras

[Borcherds; Scott's talks]

- Physical meaning of A-L involutions?
- Physical meaning of Monstrous Lie algebra?

Ideas!

- Atkin-Lehner involutions appear naturally in CHL models as T-dualities [Persson, R.V. (Daniel's talk)]
- BPS indices are invariant under T-dualities
- Idea: find some CHL models whose BPS indices equal McKay-Thompson series
- We provide a physical interpretation of Γ_g
- Hauptmodul property? Monstrous Lie algebra?

Outline

- Monstrous Heterotic Model
- Supersymmetric index
- Heterotic CHL models and moonshine groups
- Conclusions and open questions

Monstrous Heterotic Model

The Model

- Heterotic compactification on $V^\natural \otimes \bar{V}^{s\natural}$
- $\bar{V}^{s\natural}$ is anti-holomorphic SVOA with $c = 12$ and

  [Jonh Duncan's talk]
  - no NS states of conformal weight $1/2$
  - 24 Ramond ground states of conformal weight $1/2$
- 2-D theory with (0,24) space-time SUSY with algebra

$$\{Q^i, Q^j\} = 2\delta^{ij}(k_R^0 - k_R^1)$$

where k_R^μ, $\mu = 0, 1$, are right-moving momenta

[Green, Kutasov; Bergman, Distler, Varadarajan; …]

BPS states

$$\{Q^i, Q^j\} = 2\delta^{ij}(k_R^0 - k_R^1)$$

- Compactify on a circle of radius R

$$k_L^1 = \frac{1}{\sqrt{2}}(\frac{m}{R} - wR) \qquad k_R^1 = \frac{1}{\sqrt{2}}(\frac{m}{R} + wR) \qquad k_L^0 = k_R^0 = E$$

where $m, w \in \mathbb{Z}$

- "BPS condition" $E = k_R^1$
- BPS + physical state condition

$$(\text{state in } V^{\natural}_{mw+1}) \otimes (\text{state in } \bar{V}^{s\natural}_{1/2}) \otimes |k^\mu\rangle$$

BPS states

- BPS + physical state condition

$$(\text{state in } V^{\natural}_{mw+1}) \otimes (\text{state in } \bar{V}^{s\natural}_{1/2}) \otimes |k^{\mu}\rangle$$

- The only states in $\bar{V}^{s\natural}_{1/2}$ are Ramond
  $\Rightarrow$ all space-time fermions (same chirality)

- For each momentum-winding $m,\ w$ there are $24c(mw)$ fermionic BPS states of energy

$$E = \frac{1}{\sqrt{2}}\left(\frac{m}{R} + wR\right)$$

(recall: $J(\tau) = \sum_n c(n) q^n$)

Supersymmetric index

Supersymmetric index

- Second-quantized space of states $\mathcal{H}$
- Refined supersymmetric index

$$Z(R, \beta, b, v) = \mathrm{Tr}_{\mathcal{H}}((-1)^F e^{-\beta H} e^{2\pi i b W} e^{2\pi i v M})$$

  - non-vanishing contributions only from BPS states
  - independent of string coupling constant

Second-quantized strings

- Construct a "BPS Fock space" (free theory limit) $\mathcal{H}_{BPS}$
  - single-particle BPS state $a \longrightarrow$ fermionic operator η_a
  - a ground state $|0\rangle_R$ with $\eta_a|0\rangle_R = 0$ for $E(a) < 0$
  - Space $\mathcal{H}_{BPS}$ acting on $|0\rangle_R$ by η_a for $E(a) > 0$
  - Possible non-zero ground momentum and winding

$$M|0\rangle_R = m_0|0\rangle_R \qquad\qquad W|0\rangle_R = w_0|0\rangle_R$$

BPS index

- Can restrict the trace to this BPS space

$$Z(R,\beta,b,v) = \mathrm{Tr}_{\mathcal{H}_{BPS}}((-1)^F e^{-\beta H} e^{2\pi i b W} e^{2\pi i v M})$$

where the following relation holds

$$H = \frac{1}{\sqrt{2}}\left(\frac{M}{R} + WR\right)$$

- Define $\quad T = b + i\frac{\beta R}{2\sqrt{2}\pi} \quad$ and $\quad U = v + i\frac{\beta}{2\sqrt{2}\pi R}$

$$Z(T,U) = \mathrm{Tr}_{\mathcal{H}_{BPS}}((-1)^F e^{2\pi i T W} e^{2\pi i U M})$$

BPS index

$$Z(T,U) = \mathrm{Tr}_{\mathcal{H}_{BPS}}((-1)^F e^{2\pi i T W} e^{2\pi i U M})$$

- Easy to compute ($R > 1$)

$$Z(T,U)^{\frac{1}{24}} = e^{2\pi i (T w_0 + U m_0)} \prod_{m,w} (1 - e^{2\pi i w T} e^{2\pi i m U})^{c(mw)}$$

product over $m, w > 0$ or $(m,w) = (-1,1)$

- Same form as denominator of Monster Lie algebra! (for suitable (m_0, w_0))
- Is there any Lie algebra involved?

A Lie algebra

- Let $\mathcal{V}_a$ be vertex operator of (single-string) BPS state a
- SUSY variation is either zero or BRST exact

$$\{Q_i, \mathcal{V}_a\} = [\mathcal{Q}_{BRST}, \mathcal{U}_a]$$

- Recall: *massless* BRST exact states generate algebra of gauge symmetries
- $\mathcal{U}_a$ are not massless, but generate a Lie algebra $\mathfrak{g}$

$$[\mathcal{U}_a, \mathcal{U}_b] = f^c{}_{ab}\mathcal{U}_c$$

- $\mathcal{U}_a$ has the form

$$(\text{state in } V^\natural_{mw+1}) \otimes |k_L^\mu\rangle \otimes |k_R^\mu\rangle$$

A Lie algebra

- $\mathcal{U}_a$ are not massless, but generate a Lie algebra $\mathfrak{g}$

$$[\mathcal{U}_a, \mathcal{U}_b] = f^c{}_{ab}\mathcal{U}_c$$

- $\mathcal{U}_a$ has the form

$$(\text{state in } V^\natural_{mw+1}) \otimes |k^\mu_L\rangle \otimes |k^\mu_R\rangle$$

- $\mathfrak{g}$ is Monster Lie algebra!!

- $\mathfrak{g}$ has a linear action on space of BPS states

$$\mathcal{U}_a(\mathcal{V}_b) = f^c{}_{ab}\mathcal{V}_c$$

Algebra vs BPS states

- Single particle BPS states $\cong \mathfrak{g}$
- Positive energy BPS states $\cong \mathfrak{g}_+$
- Fock space $\mathcal{H} \cong \bigwedge \mathfrak{g}_+$
- Momentum-winding or $k_L^\mu \longrightarrow$ roots
- Ground state mom-wind (m_0, w_0)
  $\longrightarrow$ $1/2$ sum over positive roots (regularized)
- We can show that (m_0, w_0) is Weyl vector
- "Additive" side of Weyl-Kac-Borcherds denom. formula

Algebra Homology

- $\mathcal{H}^j \cong \bigwedge^j \mathfrak{g}_+$ space of j- particles states
- Define nilpotent operators

$$d : \mathcal{H}^j \to \mathcal{H}^{j-1} \qquad\qquad d^\dagger : \mathcal{H}^j \to \mathcal{H}^{j+1}$$

- $Z(T,U)$ gets contributions only from ker of $\{d, d^\dagger\}$
- Physical meaning of d and $d^\dagger$ not clear…

[Garland, Lepowsky; Jurisich]

Algebra homology

Theorem(?)

1. Regularized ground state winding-momentum

$$(m_0(s), w_0(s)) := \tfrac{1}{2} \textstyle\sum_{m,w} (m,w) c(mw) e^{-sE}$$

   converges to analytic function for $\Re s > s_0$

2. Analytic continuation $(m_0(0), w_0(0))$ is Weyl vector

3. Anticommutator of $d, d^\dagger$ equals quadratic Casimir

$$\{d, d^\dagger\} \sim 2(M - m_0)(W - w_0) - 2m_0 w_0$$

Denominator identity

- For $R > 1$
  - Weyl vector $(m_0, w_0) = (0, 1)$
  - $\{d, d^\dagger\} = 2M(W-1)$
  - Positive energy states $w \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$
- Contribution from $W = 1$ states is $-J(U)$
- Contribution from $M = 0$ states is $J(T)$

$$Z(T,U)^{1/24} = e^{-2\pi i T} \prod_{w>0,m} (1 - e^{2\pi i T w} e^{2\pi i U m})^{c(mw)} = J(T) - J(U)$$

Path integral

- $Z(T,U)$ given by path integral on Euclidean $\mathbb{T}^2$ with Kaehler modulus T and cplx structure U

- $Z(T,U)$ independent of string coupling $\longrightarrow$ 1-loop exact

$$Z(T,U) = \exp(-S_{1-loop}(T,U))$$

- 1-loop string amplitude (naive!)

$$S^{\pm}_{\text{1-loop}} = \int_{\mathcal{F}} \frac{d\tau^2}{2\tau_2} \left(\mathrm{Tr}_{NS}(q^{L_0}\bar{q}^{\bar{L}_0}\, \tfrac{1-(-1)^{\bar{F}}}{2}) - \mathrm{Tr}_{R}(q^{L_0}\bar{q}^{\bar{L}_0}\, \tfrac{1\pm(-1)^{\bar{F}}}{2}) \right)$$

- GSO projection not quite correct for R ground states…

GSO projection

- $S^{+}_{1-loop}(T,U)$ introduces contributions from R ground states with wrong space-time chirality
- Wrong contributions make the path integral invariant under space-time parity transformation
- Under parity transformation

$$Z(T,U) \to \overline{Z(T,U)}$$

Expected:

$$\exp(-S^{+}_{1-loop}) = |Z(T,U)|^2$$

1-loop integral

- Evaluating the traces gives

$$S^{+}_{\text{1-loop}}(T,U) = \frac{1}{2}\int_{\mathcal{F}} \frac{d\tau^2}{\tau_2}(-24)J(\tau)\Theta(T,U;\tau)$$

where

- -24 from $\bar{V}^{s\natural}$
- $J(\tau)$ from $V^{\natural}$
- $\Theta(T,U,\tau) = \sum_{m_i,w_i} q^{\frac{k_L^2}{2}}\bar{q}^{\frac{k_R^2}{2}}$ from winding-mom. along $\mathbb{T}^2$

- This is theta lift of $J(\tau)$!

[Harvey, Moore; Borcherds]

Summary

3 ways of computing $Z(T,U)$

1. Second quantized Fock space

$$Z(T,U)^{1/24} = e^{-2\pi i T} \prod_{w>0,m} (1 - e^{2\pi i T w} e^{2\pi i U m})^{c(mw)}$$

2. 1-loop string vacuum amplitude on Euclidean target $\mathbb{T}^2$

$$|Z(T,U)|^2 = \exp\left(-\frac{1}{2}\int_{\mathcal{F}} \frac{d\tau^2}{\tau_2}(-24)J(\tau)\Theta(T,U;\tau)\right)$$

3. Weyl-Kac-Borcherds denominator formula

$$Z(T,U)^{1/24} = J(T) - J(U)$$

Monstrous CHL models

CHL models

- Consider Monstrous Heterotic model on circle
- Take orbifold by (δ, g), where
  - δ is shift along circle of $1/N$ period
  - $g \in Aut(V^\natural) = \mathbb{M}$ of order N
- All previous constructions generalize:
  - can construct 2nd quantized BPS space
  - can define index $Z_{g,e}(T,U)$
  - Lie algebra from null states

CHL index

1. Second quantized Fock space

$$Z_{g,e}(T,U)^{1/24} = e^{-2\pi i T} \prod (1 - e^{2\pi i U \frac{m}{N}} e^{2\pi i T w})^{\hat{c}_{w,m}(\frac{mw}{N})}$$

where $\hat{c}_{r,s}$ are coefficients of $\frac{1}{N}\sum_{k=1}^{N} e^{-\frac{2\pi i s k}{N}} T_{g^r,g^k}$

2. 1-loop string amplitude on Euclidean target $\mathbb{T}^2$

$$|Z_{g,e}(T,U)|^2 = \exp\Big(-\int_{\mathcal{F}} \frac{d^2\tau}{2\tau_2} \frac{-24}{N} \sum_{r,s=1}^{N} \Theta_{r,s} T_{g^r,g^s}\Big)$$

3. Denominator formula

[Carnahan]

$$Z_{g,e}(T,U)^{1/24} = T_{e,g}(T) - T_{g,e}(U)$$

T-duality

- Euclidean CHL model on $\mathbb{T}^2$ has w-m lattice

$$(m_1, w_1, m_2, w_2) \in L_N = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \tfrac{1}{N}\mathbb{Z}$$

(more complicated depending on level matching)

- $\mathrm{Aut}(L)$ is subgroup of $SO^+(2,2) \cong SL_2(\mathbb{R})_T \times SL_2(\mathbb{R})_U$

$$\mathrm{Aut}(L) = \Gamma_0(N) \times \Gamma_0(N) + (W_e, W_e)$$

where W_e are Atkin-Lehner involutions

- Projection $\mathrm{Aut}(L) \to SL_2(\mathbb{R})_{T,U}$ contains Γ_g
- $\mathrm{Aut}(L)$ is group of T-dualities

[Persson, R.V.; Daniel's talk]

T-duality

- In general, $\mathrm{Aut}(L)$ maps to a *different* CHL model
- $\mathrm{Aut}_0(L) \subset \mathrm{Aut}(L)$ is group of self-dualities
- $Z_{g,e}(T,U)$ invariant under $\mathrm{Aut}_0(L)$
  $\longrightarrow$ $T_{e,g}(T)$ invariant under image $\mathrm{Aut}_0(L) \to SL_2(\mathbb{R})_T$

Conjecture: Image $\mathrm{Aut}_0(L) \to SL_2(\mathbb{R})_T$ is Γ_g

- If true, group Γ_g is a T-duality group!
- To be done: show that Γ_g is not accidentally larger

Genus zero

- Cusps for $(T, U) \in \mathbb{H} \times \mathbb{H}$ are decompactification limits
- Decomp. limits are heterotic on orbifold $V^\natural / \langle g^e \rangle \times \bar{V}^{s\natural}$
- At each cusp, $Z_{g,e}(T, U)$ is un/bounded iff decomp. limit has/hasn't massless states ($V^\natural / \langle g^e \rangle$ has/hasn't currents)

Conjecture: If decomp. limit (cusp) has no currents, it is related to $R \to \infty$ cusp by a self-duality in $\mathrm{Aut}_0(L_N)$

- If true, then $T_{e,g}$ has only one single pole on $\overline{\mathbb{H}/\Gamma}_g$
  $\longrightarrow$ Γ_g has genus zero and $T_{e,g}$ is Hauptmodul

[Tuite]

Conclusions

- Denominator formula for (twisted) Monster Lie algebra is BPS index in second quantized heterotic (CHL) model
- Algebra realized in terms of BRST exact states in string theory
- Moonshine group Γ_g is subgroup of T-duality group of CHL model on $\mathbb{T}^2$ (maybe equal self-duality group)
- Order of $T_{e,g}$ at cusps related to nature of CHL models in decompactification limits

Open questions

- Physical interpretation of many ingredients ($d, d^\dagger$, decomp. limits,…) not clear
- Genus zero as Rademacher summability? [Duncan, Frenkel]
- Unfolding 1-loop integral as sum over BPS states?
- Generalized Moonshine? [Norton; Hoehn; Carnahan]
- BPS algebra as described by Harvey-Moore?
- Same construction starting from models with currents?
- Type IIA duals?