Mathematical Modeling of Complex Microlasers

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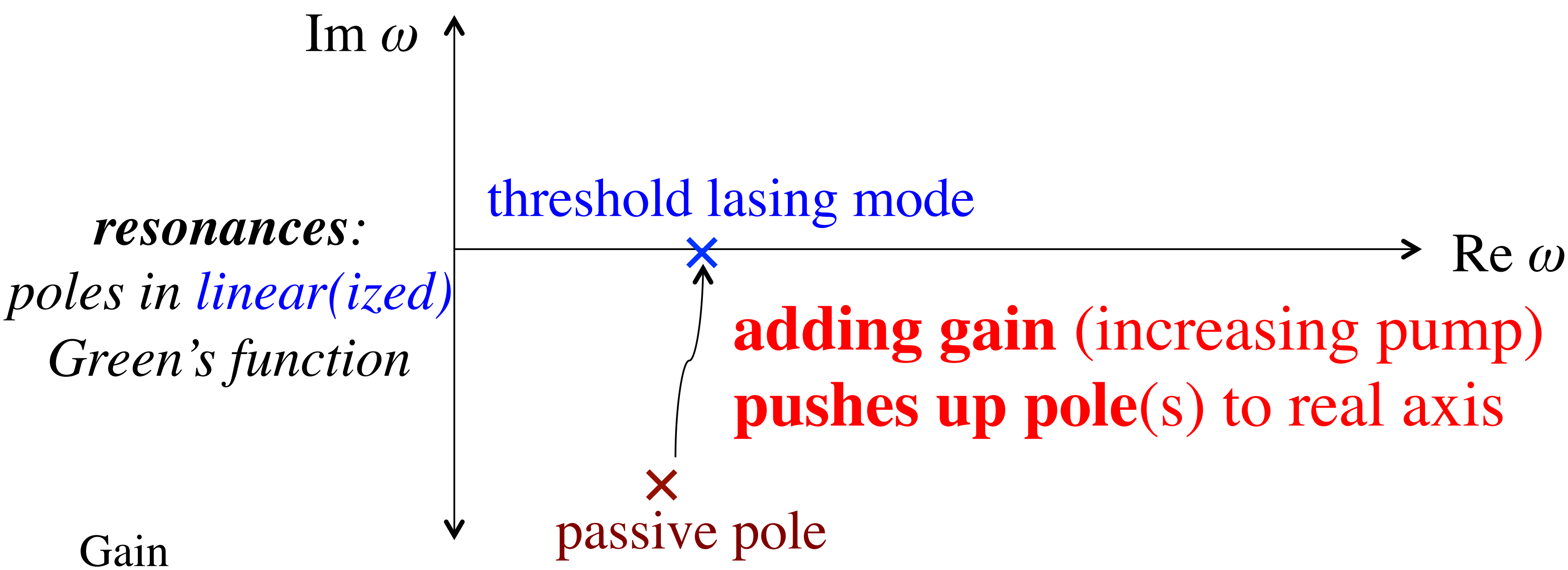
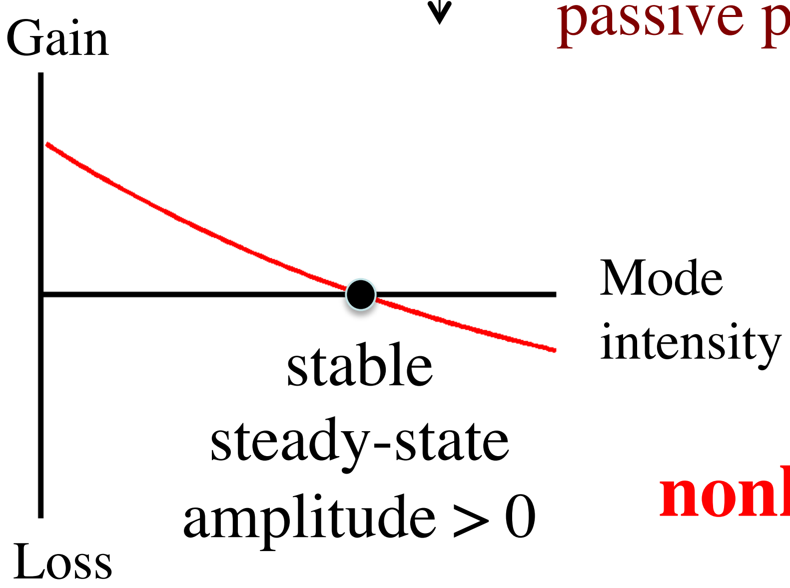
A qualitative picture of lasing





pump > threshold:

nonlinear gain stabilizes amplitude

Maxwell–Bloch equations:
simplest accurate spatio-temporal lasing model

- fully time-dependent, multiple unknown fields, nonlinear (Haken, Lamb, 1963): Maxwell + Lorentzian polarization resonance + 2-level atom population inversion

$$-\nabla \times \nabla \times (\mathbf{E}^{+}) - \varepsilon_c \ddot{\mathbf{E}}^{+} = \frac{1}{\varepsilon_0} \ddot{\mathbf{P}}^{+}$$

Polarization induces inversion

Inversion drives polarization

$$\dot{\mathbf{P}}^{+} = (-i\omega_a - \gamma_\perp)\mathbf{P}^{+} + \frac{1}{i\hbar}\mathbf{E}^{+} D$$

population inversion:

$$\dot{D} = \gamma_{||}(D_0 - D) - \frac{2}{i\hbar}[\mathbf{E}^{+} \cdot (\mathbf{P}^{+})^{*} - \mathbf{P}^{+} \cdot (\mathbf{E}^{+})^{*}]$$

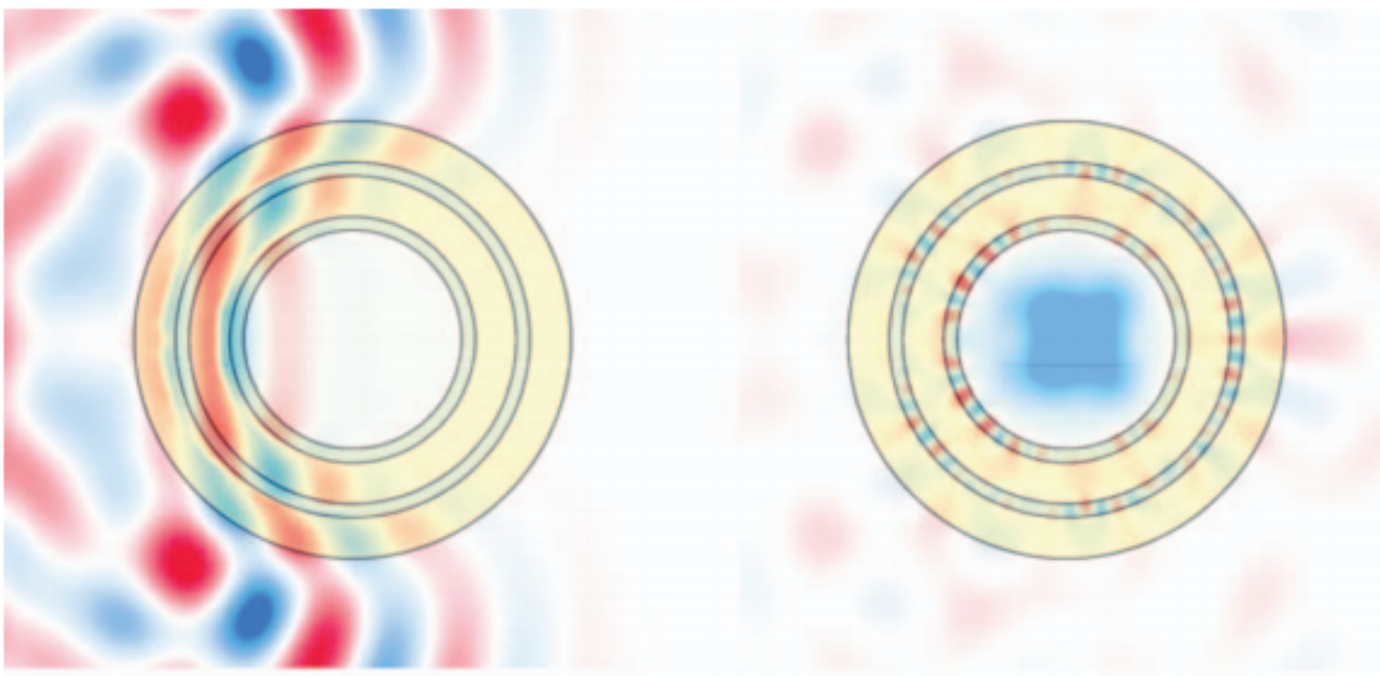
brute-force Maxwell–Bloch

FDTD (finite-difference time-domain)

simulations very expensive —

E and *D* change on very different timescales

— but do-able (barely)



[Bermel et. al. (PRB 2006)]

If a steady-state lasing solution exists, we'd rather solve for it directly *without* time-evolving

[Tureci, Stone, 2006]

$$\dot{D} = \gamma_{||}(D_0 - D) - \frac{2}{i\hbar}[\mathbf{E}^+ \cdot (\mathbf{P}^+)^* - \mathbf{P}^+ \cdot (\mathbf{E}^+)^*]$$

key assumption:

$\gamma_\perp$, $\Delta\omega >> \gamma_{||}$

valid for < 100μm microlasers

• "rotating-wave approximation"

fast oscillations average out to zero

… all oscillations are fast compared to $\gamma_{||}$

… leads to: $\dot{D} \approx 0$

stationary-inversion approximation (SIE)

before

$$-\nabla \times \nabla \times (\mathbf{E}^+) - \varepsilon_c \ddot{\mathbf{E}}^+ = \frac{1}{\varepsilon_0} \ddot{\mathbf{P}}^+$$

$$\dot{\mathbf{P}}^+ = (-i\omega_a - \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar}\mathbf{E}^+ D$$

$$\dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar}[\mathbf{E}^+ \cdot (\mathbf{P}^+)^* - \mathbf{P}^+ \cdot (\mathbf{E}^+)^*]$$

after:

Steady-State Ab-Initio Lasing Theory, "SALT"

[Tureci, Stone, 2006]

$$\nabla \times \nabla \times \mathbf{E}_m = \omega_m^2 \varepsilon_m \mathbf{E}_m$$

$$\varepsilon_m(\mathbf{x}) = \varepsilon_c(\mathbf{x}) + \frac{\gamma_\perp}{\omega_m - \omega_a + i\gamma_\perp} \left[\frac{D_0(\mathbf{x})}{1 + \sum \left| \frac{\gamma_\perp}{\omega_\nu - \omega_a + i\gamma_\perp} \mathbf{E}_\nu \right|^2} \right]$$

Still nontrivial to solve:
equation is nonlinear in both

eigenvalue ω_m ← easier

eigenvector $\mathbf{E}_m$ ← harder

New numerical solvers:

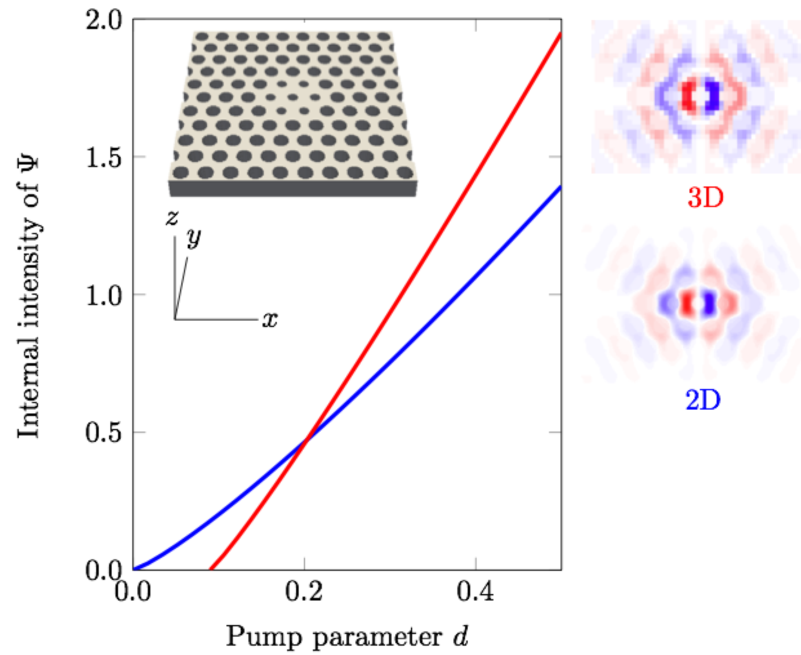
High-dimensional Newton from threshold modes

[Esterhazy et al., PRA (2014)]

SALT: lasing steady state
= "ordinary" EM eigenproblem

$$\nabla \times \nabla \times \mathbf{E}_m = \omega_m^2 \varepsilon_m \mathbf{E}_m$$

with nonlinear permittivity ε



$$\varepsilon_m = \varepsilon_c(\mathbf{x}) + \frac{\gamma_0}{\omega_m - \omega_0 + i\gamma_0} \frac{D_0(\mathbf{x}, d)}{1 + \sum_n \left| a_n \frac{\gamma_0}{\omega_n - \omega_0 + i\gamma_0} \mathbf{E}_n \right|^2}$$

full 3d nonlinear PDE solvers

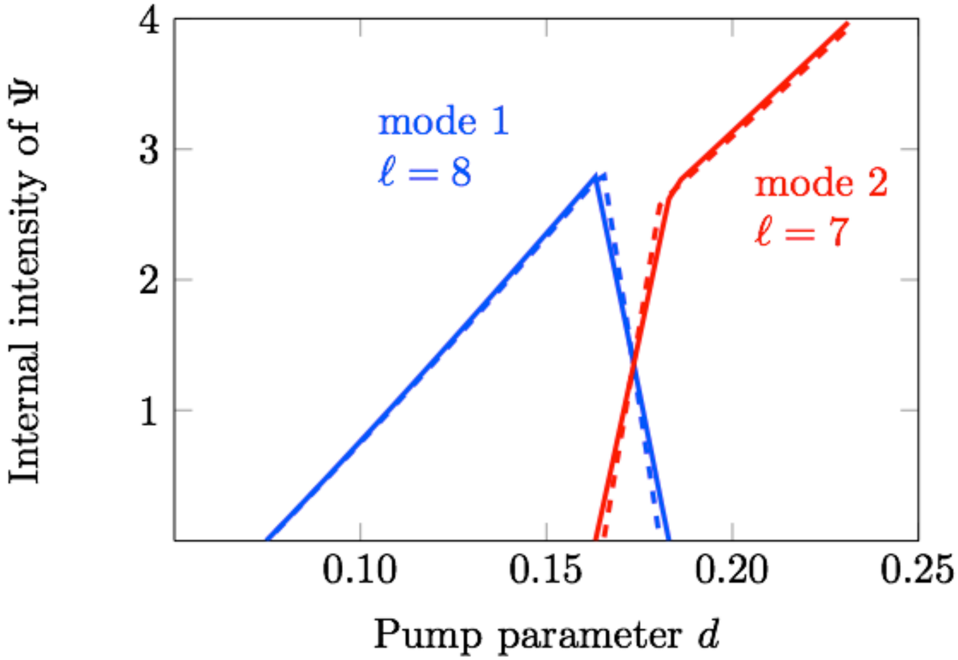
(Lorentzian gain spectrum, mode amplitudes a_n)

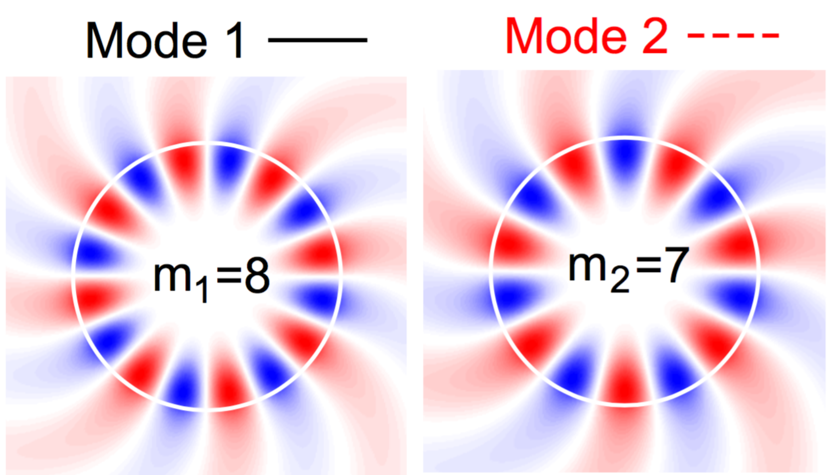
Fully nonlinear inter-modal interactions: “gain-switched lasing modes” in 2d

[Li Ge et al, Optics Express **24**, 41–54 (2016)]

Mode-switching in Microdisc Laser

(solid = Newton FDFD,
dashed = Bessel basis)





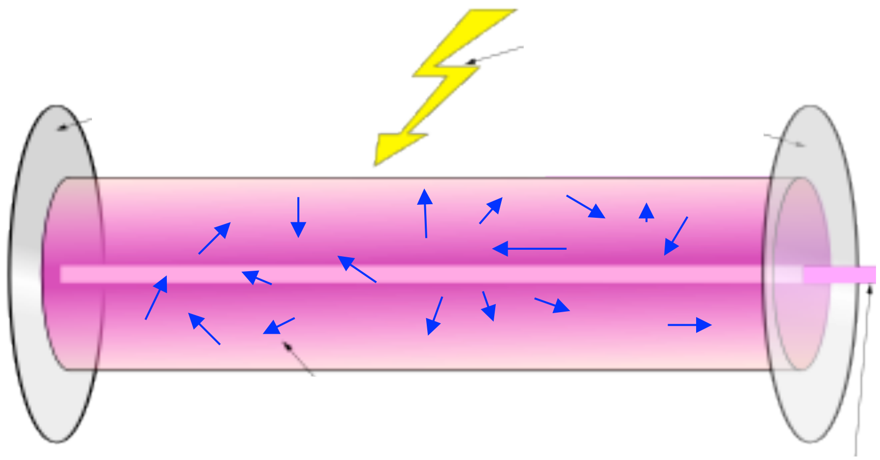
New analytical formulations (SALT)
+ new numerical solvers (lasing modes)

[many other variations:
including laser amplification “I-SALT”,
lasing in diffusive gases “C-SALT”, …]

…

New opportunities for *analytical* results, too.

Laser noise:



random (quantum/thermal) currents
"kick" the laser mode
⇒ Brownian phase drift = finite linewidth

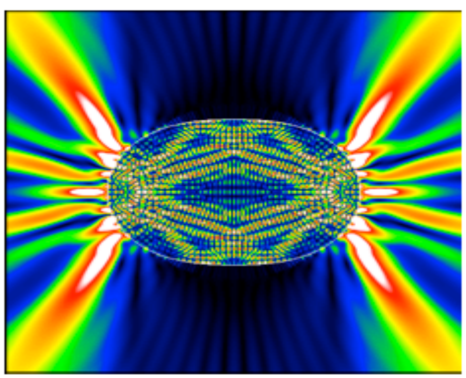
Linewidth formulas: a long history

$$\Gamma = \frac{\hbar\omega_0\gamma_c^2}{2P} \cdot \frac{N_2}{N_2 - N_1} \cdot \left| \frac{\int_C dx |\mathbf{E}_c|^2}{\int_C dx \mathbf{E}_c^2} \right|^2 \cdot \left(\frac{\gamma_\perp}{\gamma_\perp + \frac{\gamma_c}{2}} \right)^2 \cdot (1 + \alpha^2)$$

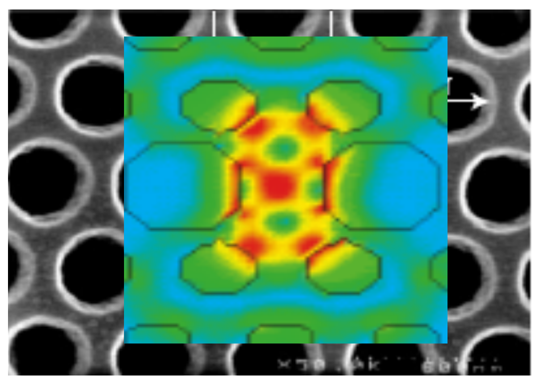
ST **I** **P** **B** **α**

- **S**chawlow-**T**ownes ('58) - inverse power 1/P scaling
- **I**ncomplete inversion ('67) - due to partial inversion
- **P**etermann ('79) - enhancement for lossy cavities
- **B**ad-cavity ('67) - reduction due to dispersion
- **α**-factor ('82) - coupling of intensity/phase fluctuations

… all make approximations invalid for *μ*-scale lasers…



chaotic cavity



photonic crystal

random laser

Starting point:

Maxwell–Bloch

electric field $$\nabla \times \nabla \times \mathbf{E} - \frac{\varepsilon_c}{c^2}\ddot{\mathbf{E}} = \frac{4\pi}{c^2}\left[\ddot{\mathbf{P}}^+ + (\ddot{\mathbf{P}}^+)^*\right]$$

gain polarization $$\dot{\mathbf{P}}^+ = -(i\omega_a + \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar}\mathbf{E}D$$

population inversion $$\dot{D} = \gamma_\|(D_0 - D) - \frac{2}{i\hbar}\mathbf{E}\cdot\left[(\mathbf{P}^*)^+ - \mathbf{P}^+\right]$$

[Arecchi & Bonifacio, 1965]

Starting point:

Langevin Maxwell–Bloch

electric field $$\nabla \times \nabla \times \mathbf{E} - \frac{\varepsilon_c}{c^2}\ddot{\mathbf{E}} = \frac{4\pi}{c^2}\left[\ddot{\mathbf{P}}^+ + (\ddot{\mathbf{P}}^+)^*\right] - \frac{4\pi}{c}\dot{\mathbf{J}}$$

noise

gain polarization $$\dot{\mathbf{P}}^+ = -(i\omega_a + \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar}\mathbf{E}D$$

population inversion $$\dot{D} = \gamma_\parallel(D_0 - D) - \frac{2}{i\hbar}\mathbf{E}\cdot\left[(\mathbf{P}^*)^+ - \mathbf{P}^+\right]$$

[Arecchi & Bonifacio, 1965]

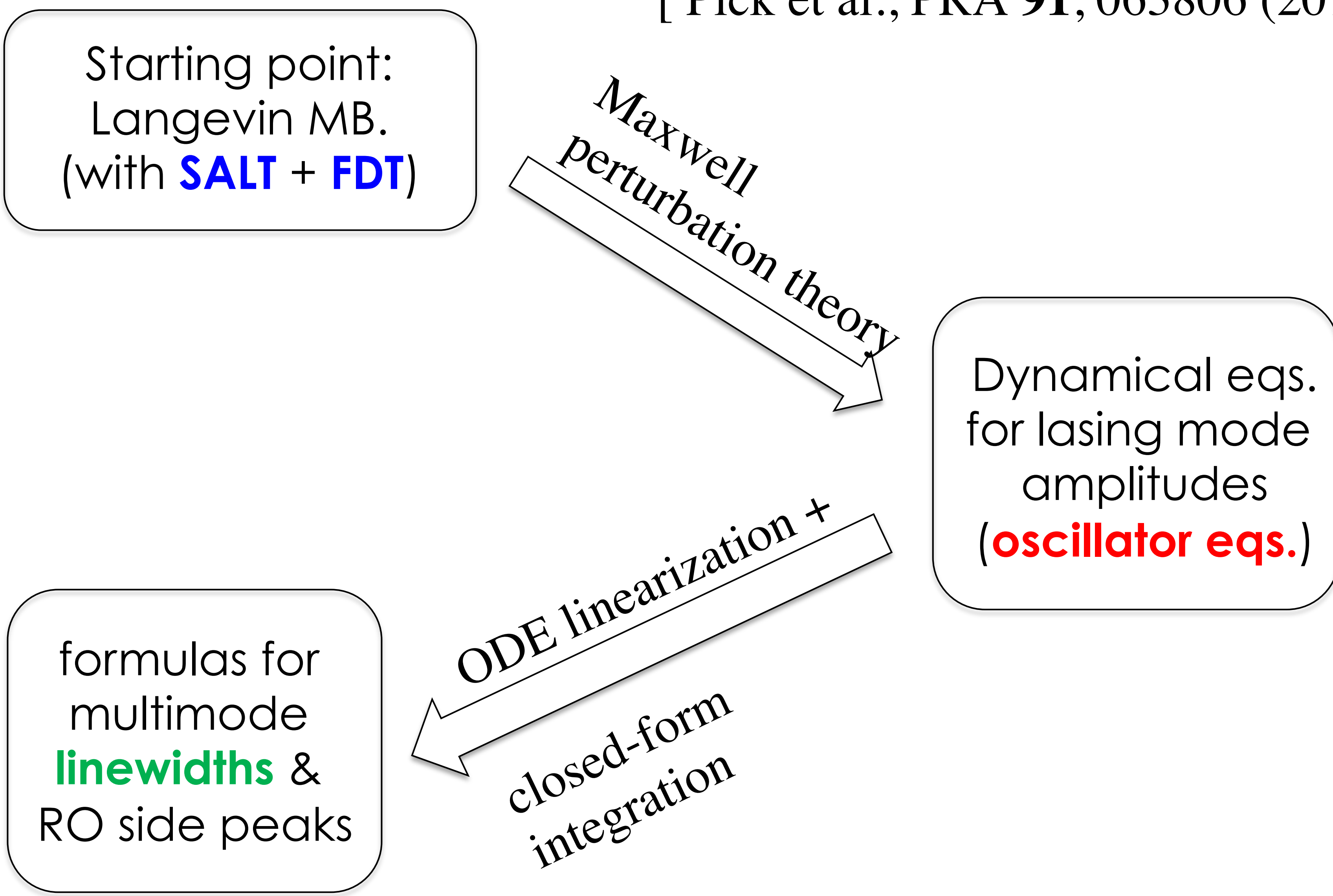
Noise correlations: fluctuation–dissipation theorem at $T < 0$

$$\langle J_i(\omega, x) J_j^*(\omega, x')\rangle = \frac{\omega}{\pi}\delta_{ij}\delta(x - x')\left[\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2kT}\right)\right]\mathrm{Im}\,\varepsilon(x)$$

[Callen & Welton, 1957]

The Noisy-SALT linewidth

[Pick et al., PRA **91**, 063806 (2015)]



Oscillator equations

Noise-free **SALT:** $\mathbf{E}(\mathbf{x},t) = \sum_\mu \mathbf{E}_\mu(\mathbf{x}) a_{\mu 0} e^{-i\omega_\mu t}$

SALT modes

Noisy **N-SALT:** $\mathbf{E}(\mathbf{x},t) = \sum_\mu \mathbf{E}_\mu(\mathbf{x}) a_\mu(t) e^{-i\omega_\mu t}$

Simple limit: Single-mode "class A" lasers

$$\frac{da_1}{dt} = \underbrace{C_{11}\left(a_{10}^2 - |a_1|^2\right) a_1}_{\text{instantaneous restoring force}} + f_1$$

often derived heuristically [Lax (1967)]

Most general dynamical equations (class A+B lasers)

$$\dot{a}_\mu = \underbrace{\sum_\nu \left[\int dx\, c_{\mu\nu}(x)\,\gamma(x) \int_{-\infty}^{t} dt' e^{-\gamma(x)(t-t')} \left(a_{\nu 0}^2 - |a_\nu(t')|^2\right)\right] a_\mu}_{\text{time-delayed, spatially inhomogeneous restoring force}} + f_\mu$$

Solving the oscillator equations

$$\dot{a}_\mu = \sum_\nu \left[\int dx\, c_{\mu\nu}(x)\, \gamma(x) \int_{-\infty}^{t} dt' e^{-\gamma(x)(t-t')} \left(a_{\nu 0}^2 - |a_\nu(t')|^2 \right) \right] a_\mu + f_\mu$$

Expand mode amplitudes around steady state:
$a_\mu = (a_{\mu 0} + \delta_\mu) \exp(i\varphi_\mu)$ [small noise = linearize in δ_μ]

- **Miracle #1:** can solve analytically for $< \varphi_\mu \varphi_\nu >$ correlation function, which gives linewidths.

- **Miracle #2:** $\gamma(x)$ exactly cancels and gives same answer as instantaneous model! The simple “class A” model is correct for “class B!”

Single-mode linewidth formula

[Pick et al., PRA **91**, 063806 (2015)]

cavity bandwith

$$\left|\frac{\int dx\,(\omega_0 \mathrm{Im}\,\varepsilon)\mathbf{E}_0{}^2}{\int dx\,\varepsilon\,\mathbf{E}_0{}^2}\right|$$

Petermann factor

$$\left|\frac{\int_P dx\,\mathrm{Im}\,\varepsilon\,|\mathbf{E}_0|^2}{\int dx\,\mathrm{Im}\,\varepsilon\,\mathbf{E}_0{}^2}\right|^2$$

Bad-cavity factor

$$\left|\frac{\int dx\,\varepsilon\,\mathbf{E}_0^2}{\int dx\,\mathbf{E}_0^2\left(\varepsilon+\frac{\omega_0}{2}\frac{\partial\varepsilon}{\partial\omega_0}\right)}\right|^2$$

$$\Gamma = \frac{\hbar\omega_0\tilde{\gamma}_0^2}{2P}\cdot\tilde{n}_{\mathrm{sp}}\cdot\tilde{K}\cdot\tilde{B}\cdot(1+\tilde{\alpha}^2)$$

ST **I** **P** **B** α

$$\frac{\int dx\left[\frac{1}{2}\coth\left(\frac{\hbar\omega\beta}{2}\right)-\frac{1}{2}\right]\mathrm{Im}\varepsilon|\mathbf{E}_0|^2}{\int_P dx\,\mathrm{Im}\,\varepsilon\,|\mathbf{E}_0|^2}$$

Incomplete inversion

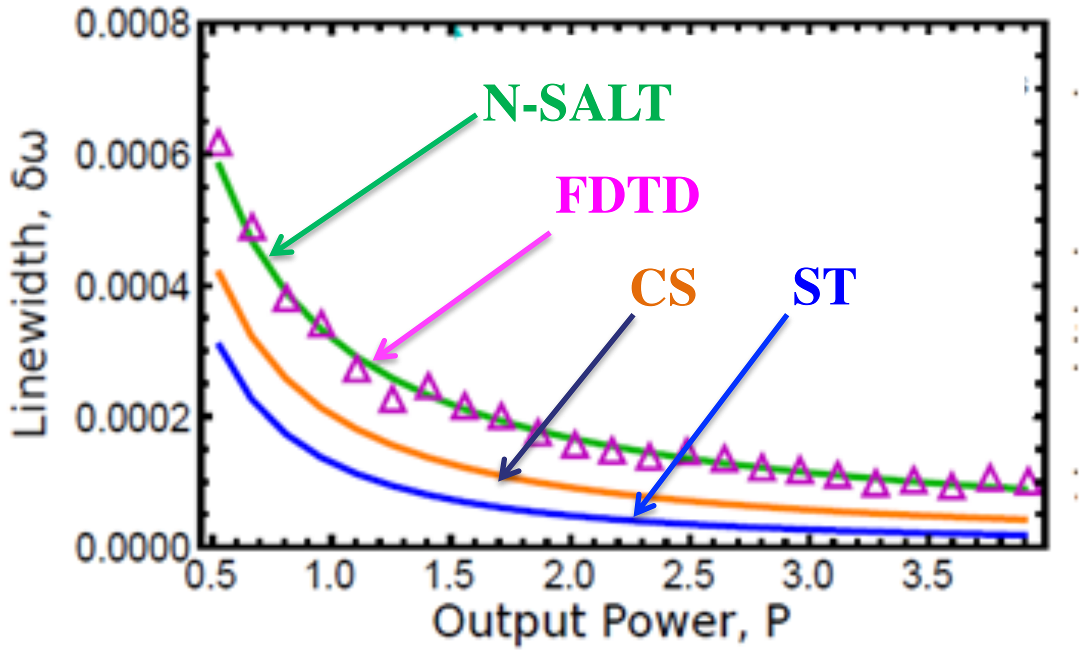
$$\mathrm{Im}\left[\frac{-i\omega_0^2\int\frac{\partial\varepsilon}{\partial|a|^2}\mathbf{E}_0^2}{\int\frac{\partial}{\partial\omega}(\omega^2\varepsilon)\mathbf{E}_0^2}\right]\Big/\mathrm{Re}\left[\frac{-i\omega_0^2\int\frac{\partial\varepsilon}{\partial|a|^2}\mathbf{E}_0^2}{\int\frac{\partial}{\partial\omega}(\omega^2\varepsilon)\mathbf{E}_0^2}\right]$$

α factor

Brute-force validation

A. Cerjan et al., *Opt. Exp.* 23, 28316 (2015)

Brute-force simulations of Langevin–Maxwell–Bloch show excellent agreement with N-SALT linewidth formula

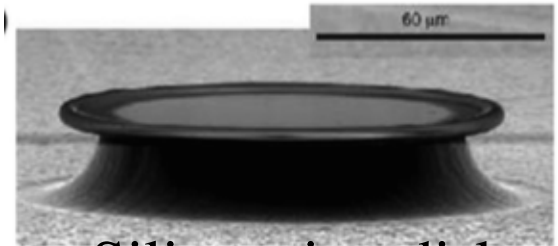


Only N-SALT captures **all relevant physics** in MB

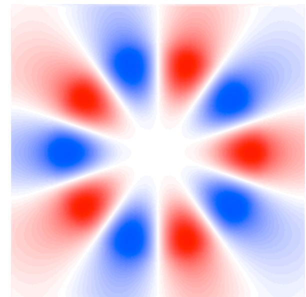
many other new analytical
& computational opportunities…

Lasing of Degenerate Modes

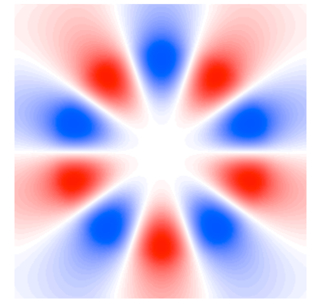
well-studied example: **whispering gallery modes**



Silica microdisk
[Armani et. al. 2003]



$\cos m\phi$

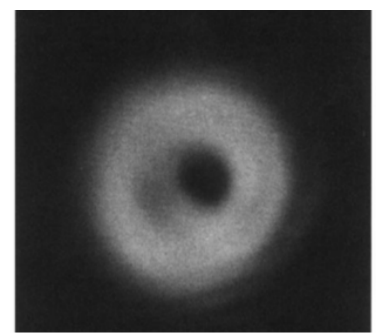


$\sin m\phi$

High-symmetry resonant cavities can have degenerate resonances, but almost the cases that have been studied above threshold are ring/disk resonators.

Lasing stable superposition:

$$e^{im\phi} = \cos m\phi + i \sin m\phi$$

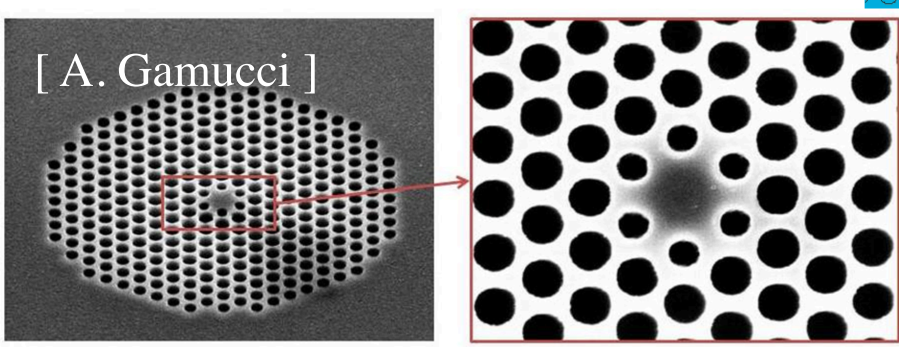


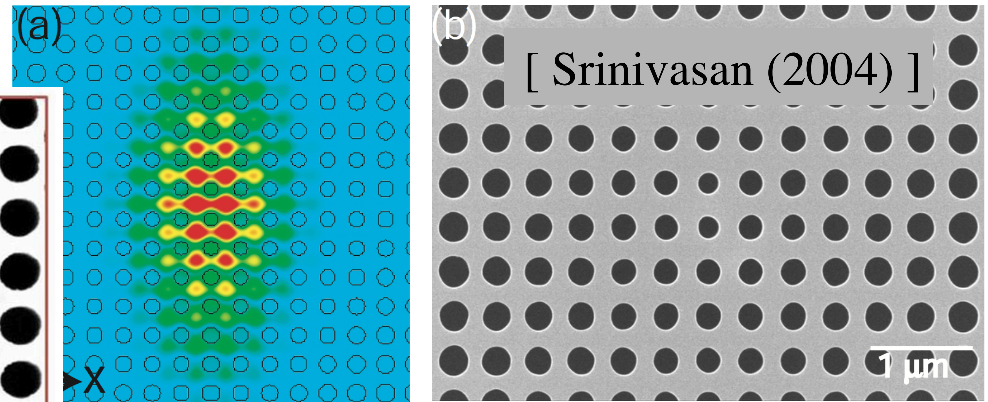
Intensity in He-Ne laser
[Tamm PRA 1998]

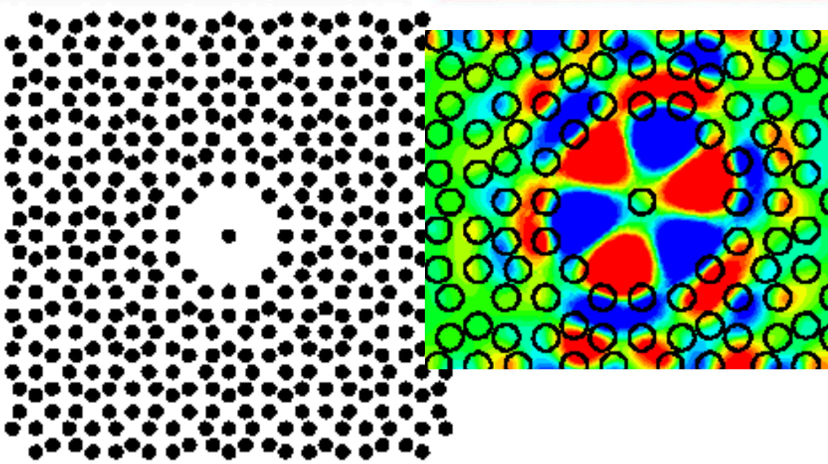
How do you find such modes?

Do SALT or SALT solvers need to be modified for degeneracies?

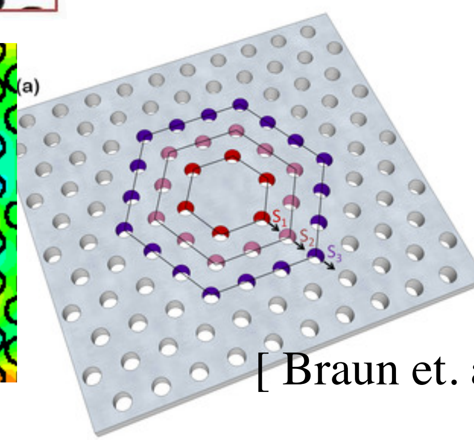
photonic-crystal (and quasicrystal) cavities have
discrete rotational symmetries



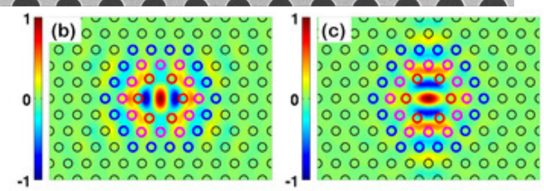




quasi-crystal
[Romero-Vivas (2005)]



[Braun et. al. 2003]



[Minkov (2014)]

… and so on???

Group theory: C_{nv} symmetry (= n-fold rotation + n mirror planes)
can have 2-fold degenerate modes, but how do they lase?

Why not just plug degenerate geometry into SALT?

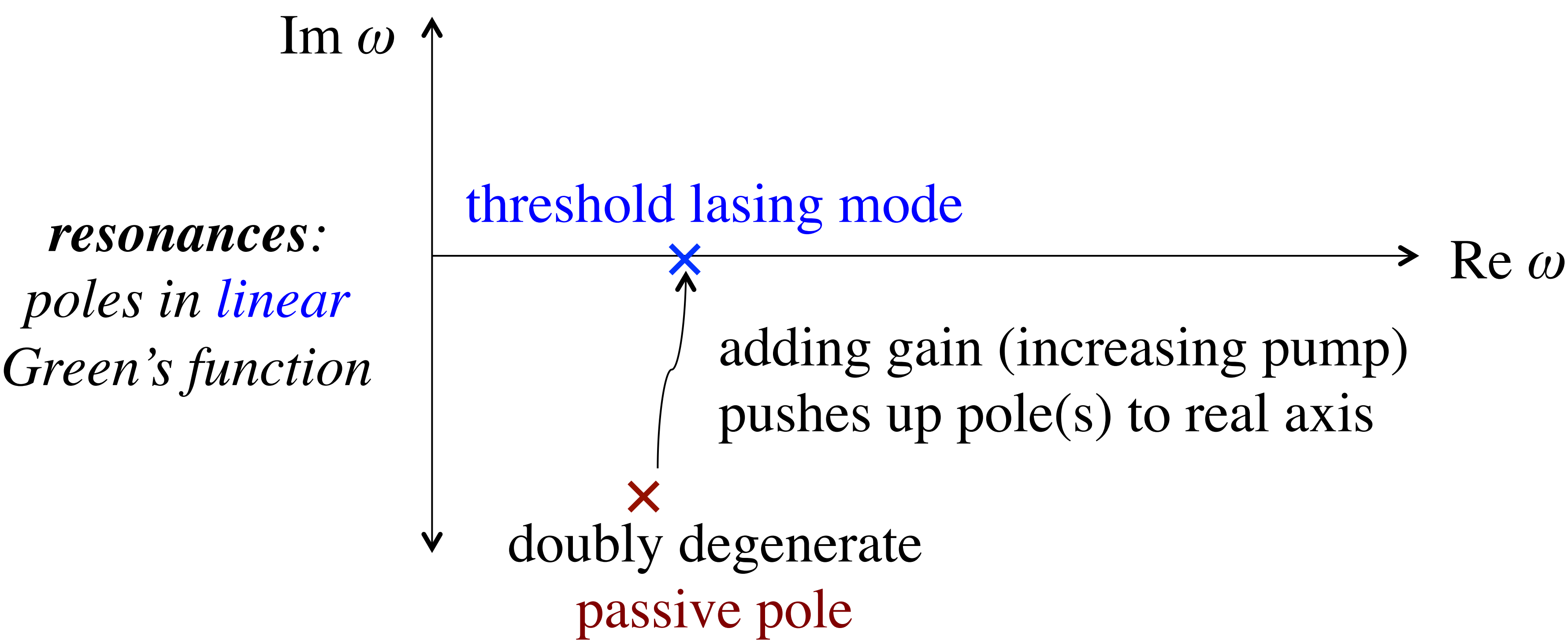
One of the challenges:

- A nonlinear solver (e.g. Newton) needs an initial guess via the threshold (linear) modes — but now there are two
  - — but what is the “right” (stable) superposition?
  - — are we sure a stable lasing mode exists?

Threshold perturbation theory

At a pump strength D_T, suppose we have two degenerate threshold modes $\psi_{1,2}$ (solving *linear* Maxwell eigenproblem)

… consider pump $D_0 = (1 + d)\, D_T$ for $0 \leq d << 1$,
& solve the nonlinear $d>0$ equations to lowest order in d



Perturbative lasing modes

First, find the steady-state d>0 modes (possibly unstable)

$$E = \sqrt{d}(a_1\psi_1 + a_2\psi_2) + \mathcal{O}(d^{3/2})$$
$$\omega = \omega_0 + \omega_1 d + \mathcal{O}(d^2)$$

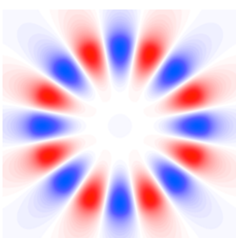
… plug into SALT, drop higher-order terms in d …

Straightforward to solve for all allowed $a_{1,2}$ superpositions.

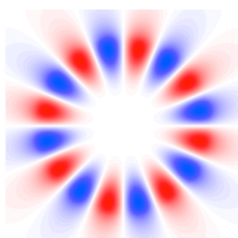
High-symmetry perturbative SALT

- Consider degenerate lasing modes coming from C_{nv} symmetry.

ring: $C_{\infty v}$

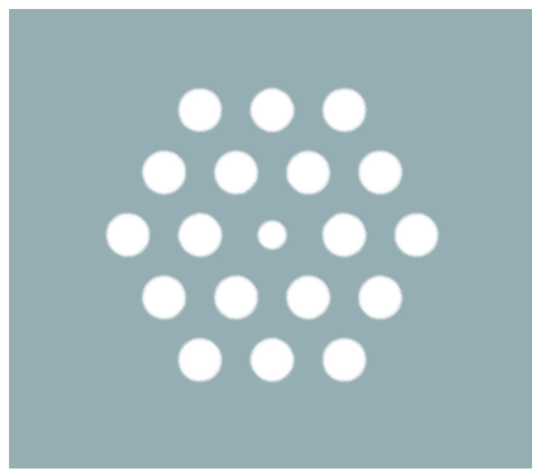


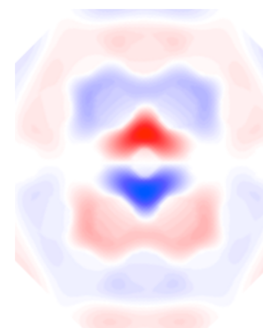
$\cos m\phi$



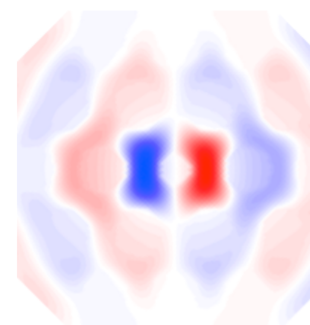
$\sin m\phi$

photonic crystal: C_{6v}





"cos-like"



"sin-like"

degenerate modes $\psi_{1,2}$ comein cos/sin-like even/odd pairs

Result: d>0 SALT solutions are always either
standing (ψ_1 or ψ_2) or circulating ($\psi_1 \pm i\, \psi_2$)!

Analytical Near-Threshold Stability

[following Burkhardt, Liertzer, Krimer, & Rotter (2015), who solve linear-stability numerically for any *d*]

MB solution = steady state + perturbation

$$E^{+}(x,t) = [E(x) + \delta E(x,t)]e^{-i\omega t}$$

$$P^{+}(x,t) = [P(x) + \delta P(x,t)]e^{-i\omega t}$$

$$\tilde{D}(x,t) = D(x) + \delta D(x,t)$$

$$u(t) = \begin{pmatrix} \delta E_R \\ \delta E_I \\ \delta P_R \\ \delta P_I \\ \delta D \end{pmatrix}$$

linearized MB equations, dropping $O(u^2)$

$$\left(C\frac{d^2}{dt^2} + B\frac{d}{dt} + A \right) u(t) = 0$$

stability:
all eigenvalues
σ have $\text{Re}\,\sigma < 0$

$ve^{\sigma t}$ eigensolutions: $(C\sigma^2 + B\sigma + A)v = 0$

Perturbative Stability Analysis

$ve^{\sigma t}$ eigensolutions: $(C\sigma^2 + B\sigma + A)v = 0$

expand perturbatively in *d*:

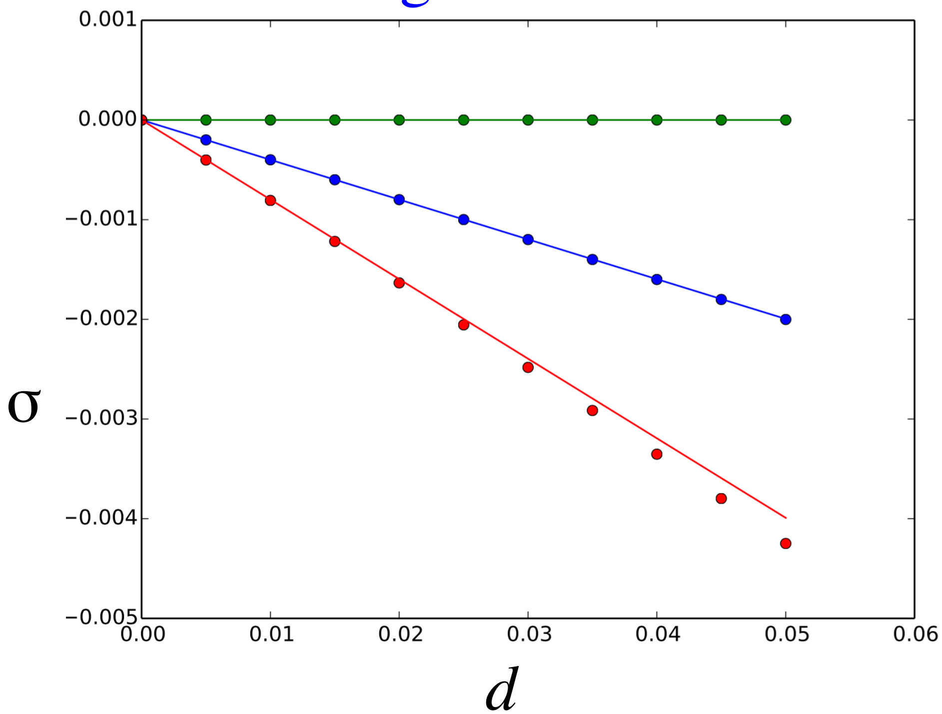
$$v = v_0 + v_{1/2}\sqrt{d} + v_1 d + \mathcal{O}(d^{3/2})$$
$$\sigma = \sigma_0 + \sigma_{1/2}\sqrt{d} + \sigma_1 d + \mathcal{O}(d^{3/2})$$

solve order-by-order … quite tedious, but analytical!

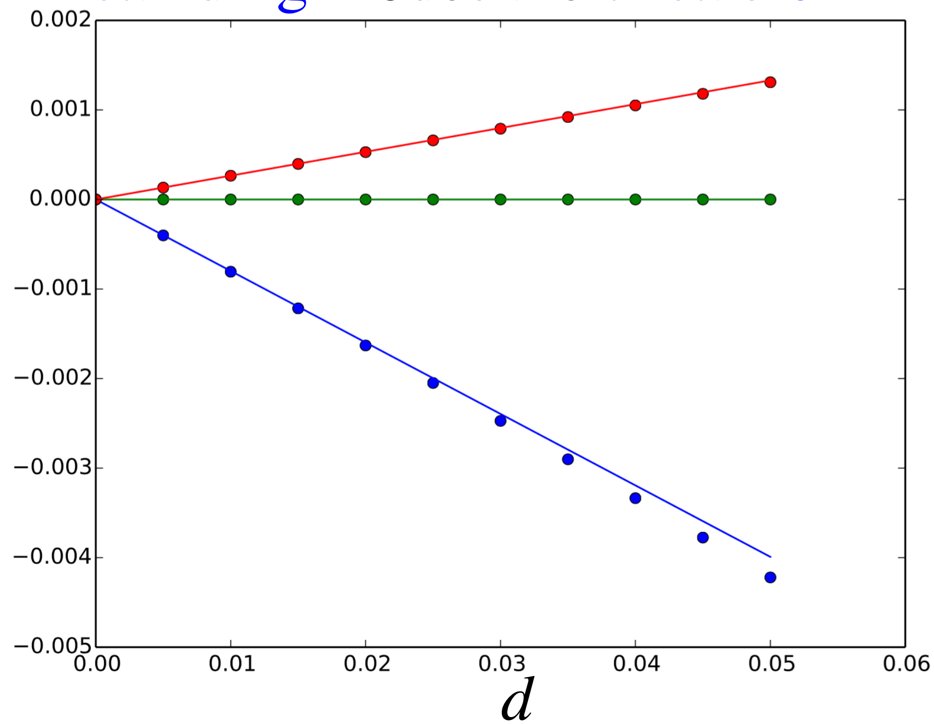
… many terms simplify depending on symmetry group.

Perturbative Stability Results (in 1d ring example)

circulating modes are stable



standing modes are unstable



Validated perturbation theory (lines)
against brute-force eigenvalues σ (dots) for 1d ring.

Result: symmetry + integrals of threshold modes = stability criteria for circulating/standing modes

stable degenerate solutions are almost always circulating (from "chiral" group representations)

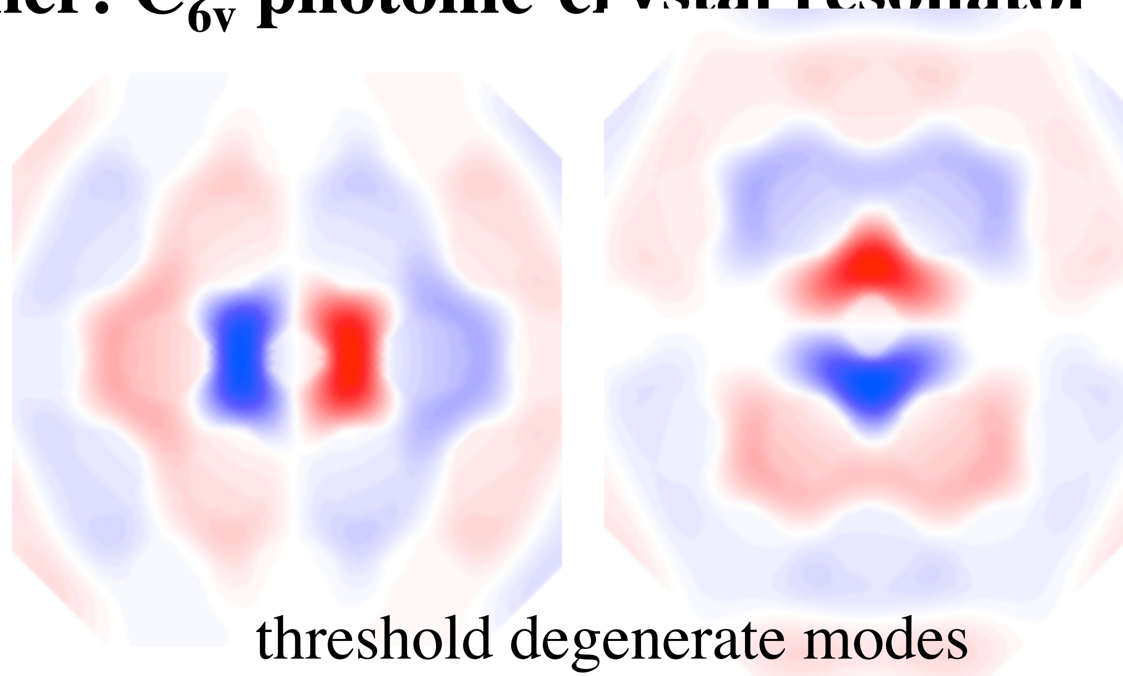
projection onto circulating mode:

$$\mathbf{E}_+ = \sum_{k=0}^{n-1} \exp\left(-\frac{2\pi i m k}{n}\right) R_n^k \mathbf{E},$$

Correct "initial guess" for above-threshold SALT solver.

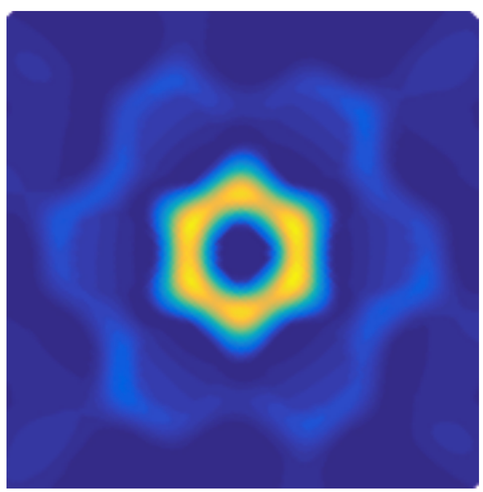
[Interesting point: C4v group (square) is very special, and can sometimes have stable standing-wave modes]

Putting it all together: C_{6v} photonic-crystal resonator



threshold degenerate modes

Stable lasing intensity:



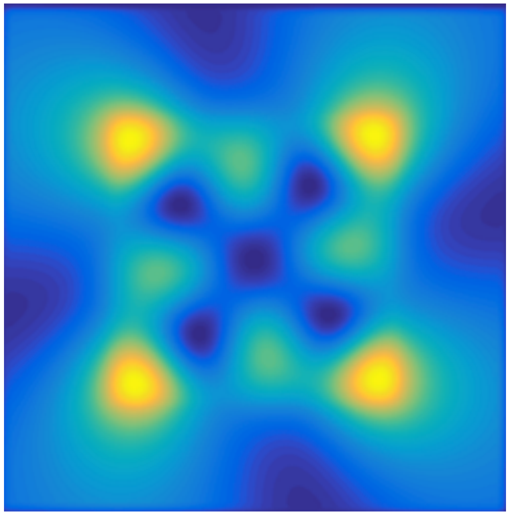
[Omitted details: techniques to correct for numerical symmetry breaking.]

Symmetry-breaking above threshold

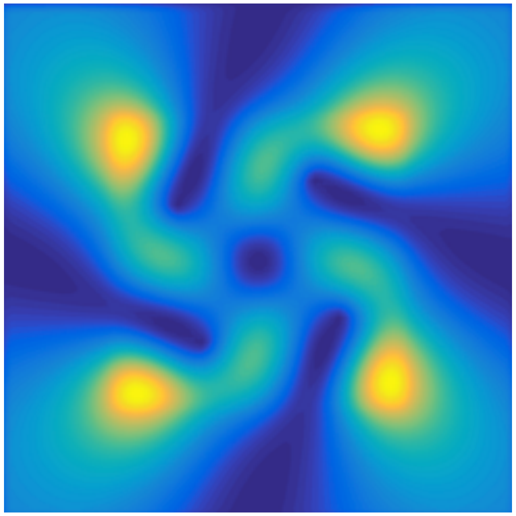
“spiral” intensity pattern of circulating mode generally breaks mirror symmetry above threshold — only C_n symmetry remains!

… what happens to degeneracy above threshold?

“chiral” lasing mode
in dielectric square



C_n does not have degeneracy
… except if we *also* have reciprocity
[Hopkins et al. arXiv:1412.1120v2 (2015)]



degenerate passive resonance
≠ mirror flip of lasing mode

New solvers, new formulations = many analytical (& computational) opportunities remaining

- Lyapunov stability of multi-mode SALT: well-established numerically & qualitatively plausible, but no rigorous analysis.
- Exceptional-point lasing
- Band-edge surface-emitting lasers (*continuum* of guided/leaky resonances)
- …

Thanks!

Adi Pick (Harvard)

David Liu (MIT)

Stefan Rotter
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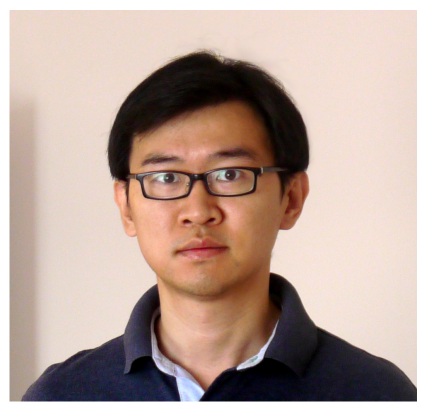

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