Inverse problems for time-harmonic Maxwell equations

Mikko Salo
University of Jyväskylä

Durham, 19 July 2016

Finnish Centre of Excellence in Inverse Problems Research

European Research Council
Established by the European Commission

Outline

1. Inverse problem for Maxwell equations
2. Matrix Schrödinger equation
3. Complex geometrical optics
4. Partial data

Calderón problem

Conductivity equation

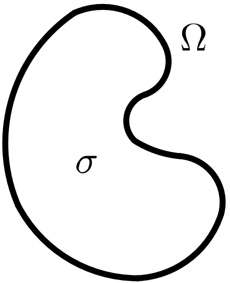
$$\begin{cases} \mathrm{div}(\sigma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $\sigma \in L^\infty(\Omega)$ positive scalar function (electrical conductivity).

Boundary measurements given by *Dirichlet-to-Neumann (DN) map*

$$\Lambda_\sigma : f \mapsto \sigma \nabla u \cdot \nu|_{\partial\Omega}.$$

Inverse problem: given Λ_σ, determine σ.



Maxwell equations

Consider (elliptic) Maxwell equations in $\Omega \subset \mathbb{R}^3$,

$$\begin{cases} \nabla \times E = i\omega\mu H, \\ \nabla \times H = -i\omega\varepsilon E. \end{cases}$$

Here Ω is a bounded C^∞ domain and

- $E, H : \Omega \to \mathbb{C}^3$ are electric and magnetic fields
- $\omega > 0$ is a fixed (non-resonant) frequency
- $\varepsilon, \mu \in C^\infty(\overline{\Omega}, \mathbb{C})$ and $\mathrm{Re}\,(\varepsilon), \mathrm{Re}\,(\mu) > 0$

Boundary measurements (*admittance map*)

$$\Lambda_{\varepsilon,\mu} : E_{\tan}|_{\partial\Omega} \mapsto H_{\tan}|_{\partial\Omega}.$$

Inverse problem: given $\Lambda_{\varepsilon,\mu}$, determine ε, μ.

Relation to Calderón problem

Maxwell equations with real μ_0, ε_0 and conductivity σ:

$$\begin{cases} \nabla \times E = i\omega\mu_0 H, \\ \nabla \times H = -i\omega(\varepsilon_0 + \frac{i}{\omega}\sigma)E. \end{cases}$$

Formal limit as $\omega \to 0$:

$$\begin{cases} \nabla \times E = 0, \\ \nabla \times H = \sigma E. \end{cases}$$

From first equation get $E = \nabla u$, then second equation implies the *conductivity equation*

$$\nabla \cdot \sigma \nabla u = 0.$$

Also $\Lambda_{\varepsilon,\mu} \rightsquigarrow \Lambda_\sigma$ as $\omega \to 0$ [Lassas 1997].

Maxwell inverse problem

Results (mostly for scalar ε, μ):

uniqueness	$\varepsilon, \mu \in C^3$	Ola-Päivärinta-Somersalo 1993
	$\varepsilon, \mu \in C^1$	Caro-Zhou 2014
log stability	$\varepsilon, \mu \in C^2$	Caro 2010
partial data	under	Caro-Ola-S 2009
	various	Brown-Marletta-Reyes 2016
	conditions	Chung-Ola-S-Tzou 2016
matrix ε, μ		Kenig-S-Uhlmann 2011

Outline

Elliptization

Maxwell equations in $\Omega \subset \mathbb{R}^3$,

$$\begin{cases} \nabla \times E = i\omega\mu H, \\ \nabla \times H = -i\omega\varepsilon E. \end{cases}$$

This is a 6×6 system, not elliptic as it is written! Since $\operatorname{div} \circ \operatorname{curl} = 0$, obtain constituent equations

$$\begin{cases} \nabla \cdot (\mu H) = 0, \\ \nabla \cdot (\varepsilon E) = 0. \end{cases}$$

Elliptization (Herz/Sommerfeld potentials, [Picard 1984], [Ola-Somersalo 1996]): adding two equations requires adding two extra unknowns, the scalar fields Φ and Ψ.

Elliptization

Maxwell equations become the 8×8 system

$$\left[\begin{bmatrix} * & \nabla\cdot & 0 & * \\ * & 0 & \nabla\times & * \\ * & \nabla\times & 0 & * \\ * & 0 & \nabla\cdot & * \end{bmatrix} + V(x) \right] \begin{bmatrix} \Phi \\ E \\ H \\ \Psi \end{bmatrix} = 0$$

where V is an 8×8 matrix function. Here $(E \quad H)^t$ will solve Maxwell iff $(0 \quad E \quad H \quad 0)^t$ solves the above system (that is, *need* $\Phi = \Psi = 0$ *in order to solve Maxwell*).

We are free to choose the $*$ entries so that the new system becomes elliptic. How to do this?

Geometric setup

Let (M, g) compact 3D Riemannian manifold with boundary.
Maxwell equations

$$\begin{cases} *dE = i\omega\mu H, \\ *dH = -i\omega\varepsilon E \end{cases}$$

Here

- E, H complex 1-forms on M
- ε, μ smooth functions in M with $\mathrm{Re}\,(\varepsilon), \mathrm{Re}\,(\mu) > 0$
- d *exterior derivative*
- $*$ *Hodge star* in (M, g), maps k-forms to $(3-k)$-forms

The adjoint of d is the *codifferential* $\delta = \pm * d *$. Recall that

d and δ act as $\pm$grad / curl / div.

Elliptization

The previous 8×8 system may be rewritten as

$$\left[\begin{bmatrix} * & \delta & 0 & * \\ * & 0 & d & * \\ * & d & 0 & * \\ * & 0 & \delta & * \end{bmatrix} + V(x)\right] \begin{bmatrix} \Phi \\ E \\ *H \\ *\Psi \end{bmatrix} = 0.$$

where Φ, Ψ are 0-forms and E, H are 1-forms. The vector $(\Phi \quad E \quad *H \quad *\Psi)^t$ identifies with the *graded differential form*

$$X = \Phi + E + *H + *\Psi.$$

There is a natural elliptic operator, the *Hodge Dirac operator*, acting on graded forms:

$$D = d + \delta.$$

Elliptization

Reduce Maxwell equations to a *Dirac equation* (8×8 system)

$$(D + V)X = 0$$

where $X = \Phi + E + *H + *\Psi$ is a graded differential form, and

$$D = d + \delta.$$

Here $D^2 = \Delta_g$ is the *Hodge Laplacian* acting on graded forms. For functions, Δ_g is the Laplace-Beltrami operator

$$\Delta_g u = \sum_{j,k=1}^{n} \frac{1}{\sqrt{\det\, g}} \frac{\partial}{\partial x_j} \left(\sqrt{\det\, g}\, g^{jk} \frac{\partial u}{\partial x_k} \right),$$

where $g = (g_{jk}), g^{-1} = (g^{jk})$.

Calderón problem

Recall the steps to solve the Calderón problem:

1. Substitute $u = \gamma^{-1/2} v$, conductivity equation $\mathrm{div}(\gamma \nabla u) = 0 \leftrightsquigarrow$ Schrödinger equation $(-\Delta + q)v = 0$.
2. Integral identity: for any u_j solving $(-\Delta + q_j)u_j = 0$,
$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0.$$
3. Insert *complex geometrical optics* solutions
$$u_j = e^{\rho_j \cdot x}(1 + r_j), \quad \rho \in \mathbb{C}^n, \quad \rho \cdot \rho = 0$$
to integral identity, recover Fourier transform of q.

Want to use a similar strategy for Maxwell equations.

Strategy [Ola-Somersalo (1996) in $\mathbb{R}^3$]

1. Reduce Maxwell equations to Dirac equation $(D+V)X=0$.
2. Rescale by $\varepsilon^{1/2}$ and $\mu^{1/2}$, obtain rescaled Dirac equation $(D+W)Y=0$.
3. Reduce to Schrödinger equation $(-\Delta_g+Q)Z=0$ by squaring.
4. Construct *complex geometrical optics* solutions Z.
5. Obtain solutions to Maxwell by showing that $\Phi=\Psi=0$ (need *uniqueness notion* for complex geometrical optics).
6. Insert solutions to an integral identity.
7. Recover ε and μ from nonlinear differential expressions by unique continuation.

Outline

1. Inverse problems for Maxwell equations
2. Matrix Schrödinger equation
3. Complex geometrical optics
4. Partial data

Complex geometrical optics

Recall exponential solutions for $\rho \in \mathbb{C}^n$ [Calderón 1980]

$$\Delta u = 0, \quad u = e^{\rho \cdot x}, \quad \rho \cdot \rho = 0.$$

If $q \in L^\infty(\Omega)$, CGO solutions [Sylvester-Uhlmann 1987]

$$(-\Delta + q)u = 0, \quad u = e^{\rho \cdot x}(1 + r),$$

where $\|r\|_{L^2} \to 0$ as $|\rho| \to \infty$.

If $\Omega \subset \mathbb{R}^3$ and ε, μ are scalar, matrix Schrödinger equation becomes

$$((-\Delta) I_{8\times 8} + Q)Z = 0.$$

Can use Sylvester-Uhlmann approach with *uniqueness notion* to produce CGO solutions with $\Phi = \Psi = 0$.

Complex geometrical optics

For *matrix ε, μ* or *partial data* need a new method, leading to:

Theorem (Kenig-S-Uhlmann 2011)

Let ε and μ be matrices conformal to

$$A(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x')^{-1} \end{pmatrix}$$

where g_0 is a simple metric[1]. Then $\Lambda_{\varepsilon,\mu}$ determines ε and μ.

Here, matrices ε and μ are *conformal* if

$$\varepsilon(x) = \alpha(x)\mu(x), \quad \alpha \text{ positive scalar function.}$$

[1] e.g. a small perturbation of the identity matrix

Dynamic Maxwell equations

Theorem (Kurylev-Lassas-Somersalo 2006)

Knowledge of $\Lambda_{\varepsilon,\mu}$ for *all* frequencies $\omega > 0$ determines any conformal real matrices ε, μ uniquely up to diffeomorphism.

(Reduces to an inverse problem for hyperbolic Maxwell equations.)

If ε, μ are not conformal, many open questions in both elliptic and hyperbolic cases:

- [Krupchyk-Kurylev-Lassas 2010] Recover Betti numbers of the domain Ω from $\Lambda_{\varepsilon,\mu}$ for all frequencies

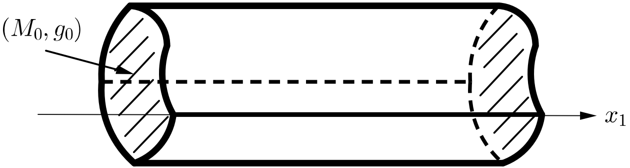
Complex geometrical optics

If $\Omega \subset \mathbb{R}^3$, Sylvester-Uhlmann obtain CGO solutions with *uniqueness notion* by *extending to $\mathbb{R}^3$* and *fixing decay at ∞*.

If (M, g) is compact and

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

get CGO solutions with *uniqueness notion* by *extending to a cylinder* and *requiring decay at ends + zero boundary values*.



Complex geometrical optics

Let $T = \mathbb{R} \times M_0$, $g = e \oplus g_0$, where (M_0, g_0) is a compact manifold with boundary. Write $x = (x_1, x')$, and define

$$\|f\|_{L^2_\delta(T)} = \|\langle x_1 \rangle^\delta f\|_{L^2(T)},$$
$$H^1_\delta(T) = \{f \in L^2_\delta(T)\,;\, df \in L^2_\delta(T)\},$$
$$H^1_{\delta,0}(T) = \{f \in H^1(T)\,;\, f|_{\partial T} = 0\}.$$

Theorem (Kenig-S-Uhlmann 2011)

Let $\delta > 1/2$. If $|\tau| \geq 1$ and $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$, then for any $f \in L^2_\delta(T)$ there is a unique solution $u \in H^1_{-\delta,0}(T)$ of

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = f \quad \text{in } T,$$
$$\|u\|_{L^2_{-\delta}(T)} \leq \frac{C}{|\tau|}\|f\|_{L^2_\delta(T)}.$$

Proof of norm estimates

Here $\mathrm{Spec}(-\Delta_{g_0}) = \{\lambda_l\}_{l=1}^{\infty}$ are Dirichlet eigenvalues of the Laplacian in (M_0, g_0), with eigenfunctions $\{\phi_l\}_{l=1}^{\infty}$ forming an orthonormal basis of $L^2(M_0)$:

$$-\Delta_{g_0}\phi_l = \lambda_l \phi_l \text{ in } M_0, \quad \phi_l|_{\partial M_0} = 0.$$

If $f \in L^2(T)$ write partial Fourier expansion

$$f(x_1, x') = \sum_{l=1}^{\infty} \tilde{f}(x_1, l)\phi_l(x'), \quad \tilde{f}(x_1, l) = (f(x_1, \,\cdot\,), \phi_l)_{L^2}.$$

Example: if $M_0 = \mathbb{T}^{n-1}$, eigenfunctions are $\{e^{im' \cdot x'}\}_{m' \in \mathbb{Z}^{n-1}}$.

Uniqueness

Assume $u \in H^1_{\delta,0}(T)$ and $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = 0$. Have

$$g = e \oplus g_0 \implies \Delta_g = \partial_1^2 + \Delta_{g_0}.$$

Taking partial Fourier coefficients in x' and Fourier transform in x_1, obtain

$$\begin{aligned} &e^{\tau x_1}\Delta_g e^{-\tau x_1} u = 0 \implies (-\partial_1^2 + 2\tau\partial_1 - \tau^2 - \Delta_{g_0})u = 0 \\ &\implies (-\partial_1^2 + 2\tau\partial_1 - \tau^2 + \lambda_l)\tilde{u}(\,\cdot\,, l) = 0 \\ &\implies (\xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_l)\hat{u}(\,\cdot\,, l) = 0. \end{aligned}$$

The symbol is nonvanishing since $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$ (look at the real and imaginary parts). Thus $\hat{u} = 0$.

Existence

Let $f \in L^2_\delta(T)$ with $\delta > 1/2$, $\tau \geq 1$. Have

$$e^{\tau x_1} \Delta_g e^{-\tau x_1} u = f \iff (-\partial_1^2 + 2\tau\partial_1 - \tau^2 - \Delta_{g_0})u = f$$
$$\iff (-\partial_1^2 + 2\tau\partial_1 - \tau^2 + \lambda_l)\tilde{u}(\,\cdot\,, l) = \tilde{f}(\,\cdot\,, l)$$

This is an ODE for the partial Fourier coefficients of u. Factorize:

$$(\partial_1 - [\tau + \sqrt{\lambda_l}])(\partial_1 - [\tau - \sqrt{\lambda_l}])\tilde{u}(\,\cdot\,, l) = -\tilde{f}(\,\cdot\,, l).$$

It is enough to solve these ODEs with suitable estimates.

Existence

Lemma

Let $a \in \mathbb{R} \setminus \{0\}$. The equation

$$u' - au = f \quad \text{in } \mathbb{R}$$

has a unique solution $u \in \mathscr{S}'(\mathbb{R})$ for any $f \in \mathscr{S}'(\mathbb{R})$. The solution operator S_a satisfies

$$\|S_a f\|_{L^2_\delta} \leq \frac{C_\delta}{|a|} \|f\|_{L^2_\delta} \quad \text{if } |a| \geq 1 \text{ and } \delta \in \mathbb{R},$$
$$\|S_a f\|_{L^2_{-\delta}} \leq C_\delta \|f\|_{L^2_\delta} \quad \text{if } a \neq 0 \text{ and } \delta > 1/2.$$

Proof of Lemma

Since $a \neq 0$,

$$u' - au = f \iff (i\xi - a)\hat{u} = \hat{f} \iff \hat{u} = \frac{1}{i\xi - a}\hat{f}.$$

Have unique solution $u \in \mathscr{S}'$ for any $f \in \mathscr{S}'$. If $|a| \geq 1$,

$$\|u\|_{L^2_\delta} = \|\hat{u}\|_{H^\delta} \leq \|(i\xi - a)^{-1}\|_{C^k}\|\hat{f}\|_{H^\delta} \leq \frac{C_\delta}{|a|}\|f\|_{L^2_\delta}.$$

If $a > 0$, have $u(x) = -\int_x^\infty f(t)e^{-a(t-x)}\,dt$ so $\|u\|_{L^\infty} \leq \|f\|_{L^1}$.
Since $\delta > 1/2$,

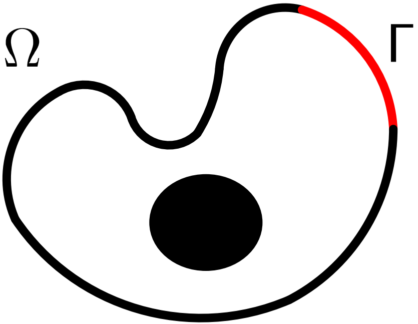
$$\|u\|_{L^2_{-\delta}} \leq \|u\|_{L^\infty}\|\langle x\rangle^{-\delta}\|_{L^2} \leq C_\delta\|f\|_{L^1} = C_\delta \int \langle t\rangle^{-\delta}\langle t\rangle^{\delta}|f|\,dt$$
$$\leq C_\delta\|f\|_{L^2_\delta}. \quad \square$$

Outline

1. Inverse problems for Maxwell equations
2. Matrix Schrödinger equation
3. Complex geometrical optics
4. Partial data

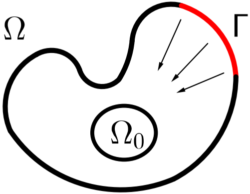
Local data problem

Prescribe $E_{\mathrm{tan}}|_\Gamma$, measure $H_{\mathrm{tan}}|_\Gamma$:



Local data problem

Theorem (Brown-Marletta-Reyes 2016)
Let $\varepsilon, \mu \in C^2(\overline{\Omega})$ be *a priori known near $\partial\Omega$*.
If $\Gamma \subset \partial\Omega$ is open, boundary measurements on Γ determine ε, μ.



Extends scalar result of [Ammari-Uhlmann 2004]. Ideas:

- boundary map on Γ + known coefficients near $\partial\Omega$ $\rightsquigarrow$ full boundary map on a subdomain $\Omega_0 \subset\subset \Omega$
- uses the *Runge approximation property* (solutions in Ω_0 approximated by solutions in Ω vanishing on $\partial\Omega \setminus \Gamma$), follows from unique continuation principle

Partial data problem

Theorem (Chung-Ola-S-Tzou 2016)
Let $\Omega \subset \mathbb{R}^3$ be strictly convex and $\varepsilon, \mu \in C^3(\overline{\Omega})$. If $\Gamma \subset \partial\Omega$ is open, measuring $H_{\tan}|_{\Gamma}$ for any $E_{\tan}|_{\partial\Omega}$ determines ε and μ.

Extends scalar result of [Kenig-Sjöstrand-Uhlmann 2007]. Ideas:

- CGO solutions for matrix Schrödinger equation

$$(-\Delta_g + Q)Z = 0$$

- control $Z|_{\Gamma^c}$ via *Carleman estimates with boundary terms* [Chung-S-Tzou 2016]
- *relative/absolute boundary conditions* for Hodge Laplace $\rightsquigarrow$ good boundary conditions for Maxwell

Partial data problem

Matrix Schrödinger equation

$$(-\Delta_g + Q)u = 0$$

Relative boundary conditions $(tu, t\delta u)$, where $t = i^*$ is the tangential part of a differential form, lead to a well-posed BVP.

If $u = (\Phi \quad E \quad *H \quad *\Psi)^t$ with Φ, Ψ 0-forms and E, H 1-forms, *relative BC* correspond to fixing

$$\Phi|_{\partial\Omega},\ E_{\tan}|_{\partial\Omega},\ \nabla \cdot E|_{\partial\Omega},\ \nu \cdot H|_{\partial\Omega},\ (\nabla \times H)_{\tan}|_{\partial\Omega},\ \partial_\nu \Psi|_{\partial\Omega}.$$

If $\Phi = \Psi = 0$, this leads to CGO solutions for Maxwell with $E_{\tan}$ and $H_{\tan}$ vanishing on a (large) part of $\partial\Omega$.

Open questions

1. Solve the Maxwell inverse problem without reducing to a second order equation or extending to a larger set.

2. Can one determine $\varepsilon, \mu \in W^{1,3}$ in $\Omega \subset \mathbb{R}^3$ from $\Lambda_{\varepsilon,\mu}$?

3. Is it possible in some cases to recover matrix ε, μ that are not conformal?