Twistors and Integrability

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Outline

Introduction

- Integrability arose in classical applied maths.
- Twistor theory (50 years old) has roots in relativity etc.
- Similar structures, for example geometric.
- Twistor-integrable side: impact in geometry & math phys.

Integrability via Lax Pairs

- Integrability from commuting linear operators $[L_1, L_2] = 0$.
- The L_a depend on a parameter ζ (spectral parameter).
- Require $[L_1(\zeta), L_2(\zeta)] = 0$ for all $\zeta \in \mathbb{C}$.
- $\longrightarrow$ conserved quantities, construction of solutions etc.

Example: sine-Gordon Equation

With $f = f(u, v)$, $g = g(u, v)$, $\phi = \phi(u, v)$, take

$$L_1 = 2\partial_u + \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} + \zeta \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\phi/2} \\ \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \end{pmatrix},$$

$$L_2 = -2\zeta\partial_v + \zeta \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \\ \mathrm{e}^{\mathrm{i}\phi/2} & 0 \end{pmatrix},$$

where $\partial_u = \partial/\partial u$. Then $[L_1, L_2] = 0$ is equivalent to

$$\phi_{uv} + \sin\phi = 0.$$

The Lax pair has the form
$L_1 = (\partial_1 + A_1) + \zeta(\partial_3 + A_3)$, $L_2 = (\partial_2 + A_2) + \zeta(\partial_4 + A_4)$:
four dimensions, quaternionic structure.

Geometry and Gauge Theory

- Coordinates z^μ, with $\mu = 1, \ldots, 4$.
- $n \times n$ matrices A_μ, operators $D_\mu = \partial_\mu + A_\mu$.
- Geometry: *vector bundle* with fibre $\mathbb{C}^n$ over each z.
- *Connection* with *covariant derivative* $\Psi \mapsto D_\mu \Psi$.
- *Curvature* $F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.
- Physics: *gauge potential* A_μ and *gauge field* $F_{\mu\nu}$.
- *Gauge transformation*
  $\Psi \mapsto \Lambda^{-1}\Psi$, $D_\mu \Psi \mapsto \Lambda^{-1} D_\mu \Psi$, $F_{\mu\nu} \mapsto \Lambda^{-1} F_{\mu\nu} \Lambda$.

General Lax Pairs and SDYM

- So generally $L_1 = D_1 + \zeta D_3$ and $L_2 = D_2 + \zeta D_4$.
- Then $[L_1, L_2] = 0$ is equivalent to

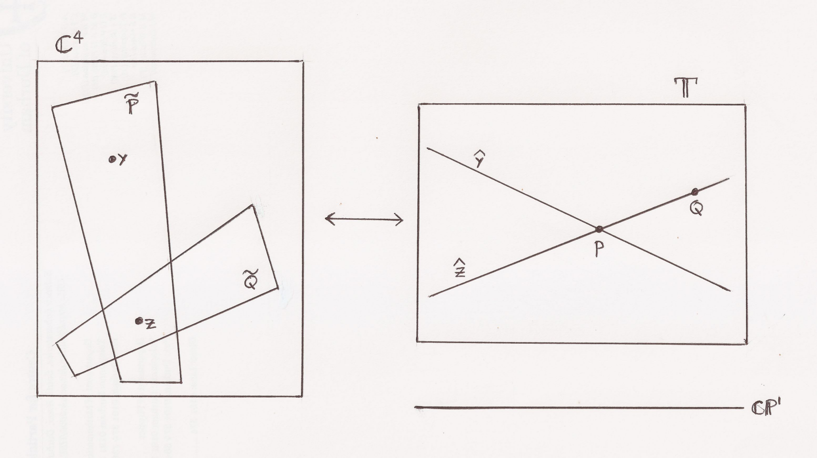
$$F_{12} = F_{34} = F_{14} + F_{32} = 0. \qquad \text{(SDYM)}$$

- These are the *self-dual Yang-Mills* equations.
- Nonlinear coupled PDEs for $A_\mu(z^\nu)$, integrable.
- Sine-Gordon, KdV, nonlinear Schrödinger, Toda etc are *reductions* of SDYM.

Twistor Space as a Quotient

- Take $z^\mu \in \mathbb{C}^4$ and $\zeta \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.
- So $(z^\mu, \zeta) \in \mathbb{F} = \mathbb{C}^4 \times \mathbb{CP}^1$.
- The vector fields $\partial_1 + \zeta\partial_3$ and $\partial_2 + \zeta\partial_4$ live in $\mathbb{F}$.
- Quotient is 3-dim complex manifold $\mathbb{T}$: *twistor space*.
- Correspondence $\mathbb{C}^4 \leftrightarrow \mathbb{T}$ is classical algebraic geometry.
- Solutions of SDYM on $\mathbb{C}^4$ correspond to *holomorphic vector bundles* on $\mathbb{T}$: a nonlinear integral transform.
- No equations on $\mathbb{T}$-side, except holomorphic structure.
- Analogous to Inverse Scattering Transform.

Twistor Correspondence



Reductions and Generalizations.

- Impose boundary and global conditions.
- Eg dimensional reduction and algebraic constraints.
- BPS monopoles: take (t, x^1, x^2, x^3) real, and put

$$z^1 = t + ix^3,\ z^4 = t - ix^3,\ z^2 = i(x^1 + ix^2),\ z^3 = i(x^1 - ix^2).$$

- Assume fields independent of t, write $\Phi = A_t$, get

$$D_1\Phi = F_{23},\ D_2\Phi = F_{31},\ D_3\Phi = F_{12}. \qquad \text{(Bog)}$$

- BC $|\Phi| \to 1$ & $|F_{jk}| \to 0$ as $r \to \infty$ in $\mathbb{R}^3$.
- Topological classification $\to$ monopole number p.
- Higher-dim generalization: Lax $2m$-plet with $m \geq 2$.
- Reductions give hierarchies such as KdV and NLS.

Moduli Spaces

- In many cases, solution space is $\cup_p \mathcal{M}_p$.
- $\mathcal{M}_p$ is the *moduli space* of p static solitons.
- For SU(2) monopoles, $\mathcal{M}_p$ is a $4p$-dim manifold.
- Comes equipped with a natural *hyperkähler* metric.
- Dynamics not integrable, but approximated by geodesics.

Reciprocity

- Kind of duality transform (nonlinear integral transform).
- ADHM transform, Nahm transform, and generalizations.
- Related to Fourier-Mukai transform in algebraic geometry.
- SDYM in $\mathbb{R}^4$: let $\mathcal{S}_{d,k}$ be the reduced system where
  - the fields depend on only d coordinates;
  - they are periodic in k coordinates;
  - they satisfy appropriate BCs in $d-k$ dimensions.
- Then $\mathcal{S}_{d,k} \cong \mathcal{S}_{4-d+k,k}$.
- The soliton number p and the rank n get interchanged.
- $k=0$, $d=3$: monopoles (PDE) from Nahm eqns (ODE).

Self-Dual Einstein Equations

- Historically, this came before the gauge-theory version.
- Use vector fields $V = V^\mu(x^\alpha)\partial_\mu$ on a 4-dim manifold.
- Lax pair $L_1 = V_1 + \zeta V_2$, $L_2 = V_3 + \zeta V_4$.
- Surfaces $\widetilde{P}$ etc become 'curved'.
- Encodes a curved metric on the 4-dimensional space:
- Self-dual solution of Einstein's vacuum equations.
- Generalizes to $4k$ dimensions: *hyperkähler* structure.
- Of great interest in geometry, GR, string theory etc.

Integrable Lattice Gauge System

Integrable Lattice Gauge System

Discrete Systems from ADHM Data

- SDYM instanton fields on $\mathbb{R}^4$ correspond to algebraic data (ADHM): matrices satisfying quadratic algebraic relations.
- Imposing symmetry in $\mathbb{R}^4$ makes these into lattice eqns.
- Circle symmetry $\to$ discrete version of Nahm equations

$$\frac{d}{ds}T_j = \frac{1}{2}\varepsilon_{jkl}[T_k, T_l].$$

- T^2 symmetry $\to$ discrete version of Hitchin equations.
- On $\mathbb{R}^2$, two Higgs fields (Φ_1, Φ_2), gauge field $F = F_{xy}$,

$$F = [\Phi_1, \Phi_2], \quad D_x\Phi_1 = -D_y\Phi_2, \quad D_x\Phi_2 = D_y\Phi_1.$$

Lattice Gauge Theory & Discrete Hitchin Eqns

- 2-dim lattice with $x, y \in \mathbb{Z}^2$, gauge group $\mathrm{U}(p)$.
- Local $\mathrm{U}(p)$ gauge invariance on lattice: $\psi \mapsto \Lambda^{-1}\psi$.
- Standard lattice gauge assigns $A \in \mathrm{U}(p)$ to each link.
- In our case, assign $A \in \mathrm{GL}(p, \mathbb{C})$ to each link.
- Write B for the x-links, C for the y-links.
- Lattice curvature is $\Omega = C^{-1} B_{+y}^{-1} C_{+x} B$.
- One of our lattice eqns is $\Omega = 1$, and the other is

$$(BB^*)_{-x} + (CC^*)_{-y} = B^* B + C^* C.$$

Some Features

- Corresponding lattice linear system is

$$B^*\psi_{+x} + \zeta(C\psi)_{-y} = 0, \quad C^*\psi_{+y} - \zeta(B\psi)_{-x} = 0.$$

- Continuum limit: lattice spacing h, let $h \to 0$ with

$$B = 1 - h(A_x - i\Phi_1), \quad C = 1 - h(A_y - i\Phi_2).$$

- U(1) case: solving $\Omega = 1$ gives $B = \exp(\Delta_x^+\phi)$, $C = \exp(\Delta_Y^+\phi)$, leaving nonlinear discrete Laplace eqn

$$\Delta_x^- \exp(2\Delta_x^+\phi) + \Delta_y^- \exp(2\Delta_y^+\phi) = 0.$$

- T^2-sym instantons correspond to solns of this (with BCs).