Argument shift method and Manakov operators: applications to differential geometry

Alexey Bolsinov
Loughborough University, UK

LMS EPSRC DURHAM SYMPOSIUM ON
Geometric and Algebraic Aspects of Integrability
25 July – 4 August, 2016
Grey College, Durham University, UK

What is it about?

Review on joint papers with V.Matveev, V.Kiosak, S.Rosemann, D.Tsonev and A.Konyaev

What is it about?

Review on joint papers with V.Matveev, V.Kiosak, S.Rosemann, D.Tsonev and A.Konyaev

Around the following observation:

The curvature tensors of some interesting Riemannian metrics
coincide with
the Hamiltonians of multi-dimensional rigid bodies

What is it about?

Review on joint papers with V.Matveev, V.Kiosak, S.Rosemann, D.Tsonev and A.Konyaev

Around the following observation:

The curvature tensors of some interesting Riemannian metrics
coincide with
the Hamiltonians of multi-dimensional rigid bodies

Applications (for indefinite metrics):

- Obstructions to the existence of a projectively equivalent partner
- Pseudo-Riemannian analog of the Fubini theorem
- New class of holonomy groups
- New class of symmetric spaces
- Yano-Obata conjecture
- Local description of Bochner-flat Kähler metrics

Pre-history

Let $\mathfrak{g}$ be a semisimple Lie algebra, $R : \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{g}$ a symmetric linear operator.
Euler equations on $\mathfrak{g}^*$

$$\frac{dx}{dt} = [x, R(x)] \tag{1}$$

are Hamiltonian with $H = \frac{1}{2}\langle R(x), x\rangle$.
For which R, are the equations (1) integrable?

Pre-history

Let $\mathfrak{g}$ be a semisimple Lie algebra, $R : \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{g}$ a symmetric linear operator.
Euler equations on $\mathfrak{g}^*$

$$\frac{dx}{dt} = [x, R(x)] \tag{1}$$

are Hamiltonian with $H = \frac{1}{2}\langle R(x), x\rangle$.
For which R, are the equations (1) integrable?

Definition

$R : \mathrm{so}(n) \to \mathrm{so}(n)$ is called a Manakov operator (with parameters A and B), if

$$[R(X), A] = [X, B] \quad \text{for all } X \in \mathrm{so}(g) \tag{2}$$

where A and B are some fixed symmetric matrices.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $R : \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{g}$ a symmetric linear operator.
Euler equations on $\mathfrak{g}^*$

$$\frac{dx}{dt} = [x, R(x)] \tag{1}$$

are Hamiltonian with $H = \frac{1}{2}\langle R(x), x\rangle$.
For which R, are the equations (1) integrable?

Definition

$R : \mathrm{so}(n) \to \mathrm{so}(n)$ is called a Manakov operator (with parameters A and B), if

$$[R(X), A] = [X, B] \quad \text{for all } X \in \mathrm{so}(g) \tag{2}$$

where A and B are some fixed symmetric matrices.

Theorem (Manakov, Mischenko, Fomenko)

Let R satisfy (2). Then

- (1) *can be rewritten as* $\frac{d}{dt}(X + \lambda A) = [X + \lambda A, R(X) + \lambda B]$;
- $\mathrm{Tr}(X + \lambda A)^k$ *are commuting first integrals of* (1);
- *if A is regular, then* (1) *are completely integrable.*

Properties of Manakov operators

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\Big|_{t=0} p(A + tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\Big|_{t=0} p(A + tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.
3. if $B = 0 = p_{\min}(A)$, then $R_0 = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX)$ still defines a non-trivial Manakov operator whose image is contained in $\mathfrak{g}_A$. Moreover, if for each eigenvalues of A there are at most 2 Jordan blocks, then the image R_0 coincides with $\mathfrak{g}_A$.

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\Big|_{t=0} p(A + tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.
3. if $B = 0 = p_{\min}(A)$, then $R_0 = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX)$ still defines a non-trivial Manakov operator whose image is contained in $\mathfrak{g}_A$. Moreover, if for each eigenvalues of A there are at most 2 Jordan blocks, then the image R_0 coincides with $\mathfrak{g}_A$.
4. R_0 satisfies the Bianchi identity: $R_0(u \wedge v)w + \text{ (cyclic) } = 0$.

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\big|_{t=0} p(A+tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.
3. if $B = 0 = p_{\min}(A)$, then $R_0 = \frac{d}{dt}\big|_{t=0} p_{\min}(A+tX)$ still defines a non-trivial Manakov operator whose image is contained in $\mathfrak{g}_A$. Moreover, if for each eigenvalues of A there are at most 2 Jordan blocks, then the image R_0 coincides with $\mathfrak{g}_A$.
4. R_0 satisfies the Bianchi identity: $R_0(u \wedge v)w + \text{ (cyclic) } = 0$.
5. If in addition $p(A) = 0$, then R_0 satisfies the second Bianchi identity $[R_0(X), R_0(Y)] = R_0[R_0(X), Y]$.

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\Big|_{t=0} p(A + tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.
3. if $B = 0 = p_{\min}(A)$, then $R_0 = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX)$ still defines a non-trivial Manakov operator whose image is contained in $\mathfrak{g}_A$. Moreover, if for each eigenvalues of A there are at most 2 Jordan blocks, then the image R_0 coincides with $\mathfrak{g}_A$.
4. R_0 satisfies the Bianchi identity: $R_0(u \wedge v)w + \text{ (cyclic) } = 0$.
5. If in addition $p(A) = 0$, then R_0 satisfies the second Bianchi identity $[R_0(X), R_0(Y)] = R_0[R_0(X), Y]$.
6. Let R satisfy two identities $[R(X), A] = [X, B]$ and $[R(X), A'] = [X, B']$, where $A' \neq a\,A + b \cdot \mathrm{id}$. Then $R(X) = k \cdot X \bmod \mathfrak{g}_A$. In particular, if A is regular, then $R = k \cdot \mathrm{id}$.

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of the centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\Big|_{t=0} p(A + tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.
3. if $B = 0 = p_{\min}(A)$, then $R_0 = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX)$ still defines a non-trivial Manakov operator whose image is contained in $\mathfrak{g}_A$. Moreover, if for each eigenvalues of A there are at most 2 Jordan blocks, then the image R_0 coincides with $\mathfrak{g}_A$.
4. R_0 satisfies the Bianchi identity: $R_0(u \wedge v)w +$ (cyclic) $= 0$.
5. If in addition $p(A) = 0$, then R_0 satisfies the second Bianchi identity $[R_0(X), R_0(Y)] = R_0[R_0(X), Y]$.
6. Let R satisfy two identities $[R(X), A] = [X, B]$ and $[R(X), A'] = [X, B']$, where $A' \neq a\,A + b \cdot \mathrm{id}$. Then $R(X) = k \cdot X \mod \mathfrak{g}_A$. In particular, if A is regular, then $R = k \cdot \mathrm{id}$.
7. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A. Then $\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}$ are eigenvalues of R. Moreover, if A has a nontrivial Jordan λ_i-block, then $p'(\lambda_i)$ is an eigenvalue of R.

Riemann curvature tensor (quick reminder and "new" point of view)

Let ∇ be the Levi-Civita connection of a pseudo-Riemannian metric g.

Definition

The Riemann curvature tensor $R = (R^l_{ij\ k})$ is defined by (formula from a text-book):

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In other words, R can be understood as a map

$$R : (X,Y) \mapsto R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(TM).$$

Algebraic symmetries:

- $R(X,Y) = -R(X,Y)$, i.e., $R : \Lambda^2 V \to \mathrm{gl}(V)$, $V = T_x M$;
- $g(R(X,Y)Z, W) = -g(R(X,Y)W, Z)$, i.e. $R(X,Y) \in \mathrm{so}(g)$;
- $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (Bianchi identity);
- $g(R(X,Y)Z, W) = -g(R(Z,W)X, Y)$.

Conclusion: $R : \mathrm{so}(g) \to \mathrm{so}(g)$ which is symmetric and satisfying Bianchi.

Riemann curvature tensor (quick reminder and "new" point of view)

Let ∇ be the Levi-Civita connection of a pseudo-Riemannian metric g.

Definition

The Riemann curvature tensor $R = (R^l_{ij\ k})$ is defined by (formula from a text-book):

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In other words, R can be understood as a map

$$R : (X,Y) \mapsto R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(TM).$$

Algebraic symmetries:

- $R(X,Y) = -R(X,Y)$, i.e., $R : \Lambda^2 V \to \mathrm{gl}(V)$, $V = T_x M$;
- $g(R(X,Y)Z, W) = -g(R(X,Y)W, Z)$, i.e. $R(X,Y) \in \mathrm{so}(g)$;
- $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (Bianchi identity);
- $g(R(X,Y)Z, W) = -g(R(Z,W)X, Y)$.

Conclusion: $R : \mathrm{so}(g) \to \mathrm{so}(g)$ which is symmetric and satisfying Bianchi.

Easy observations:

- constant curvature $\Leftrightarrow$ $R = \mathrm{const} \cdot \mathrm{id}$
- Weyl tensor vanishes $\Leftrightarrow$ $R(X) = AX + XA$
  (cf., in rigid body dynamics: $M(\Omega) = J\Omega + \Omega J$)

Projectively equivalent metrics

Definition

g and $\bar{g}$ are projectively equivalent if they have the same (unparametrised) geodesics. Notation: $g \underset{\text{proj}}{\simeq} \bar{g}$.

Main equation: Let $A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$. Then $g \underset{\text{proj}}{\simeq} \bar{g}$ if and only if

$$\nabla_u A = \tfrac{1}{2}\big(u \otimes d\,\mathrm{tr}\,A + (u \otimes d\,\mathrm{tr}\,A)^*\big).$$

Definition

g and $\bar{g}$ are projectively equivalent if they have the same (unparametrised) geodesics. Notation: $g \underset{\text{proj}}{\simeq} \bar{g}$.

Main equation: Let $A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$. Then $g \underset{\text{proj}}{\simeq} \bar{g}$ if and only if

$$\nabla_u A = \tfrac{1}{2}\big(u \otimes d\,\mathrm{tr}\,A + (u \otimes d\,\mathrm{tr}\,A)^*\big).$$

Theorem (B., Matveev)

Let $g \underset{\text{proj}}{\simeq} \bar{g}$. Then the Riemann curvature tensor of g is a Manakov operator:

$$[R(X), A] = [B, X] \quad \textit{for all } X \in \mathrm{so}(g), \textit{ where } B = \tfrac{1}{2}\nabla\big(\mathrm{grad}\ \mathrm{tr}\,A\big).$$

Proof.

Consider the compatibility condition for the main equation. $\square$

Definition

g and $\bar{g}$ are projectively equivalent if they have the same (unparametrised) geodesics. Notation: $g \underset{\text{proj}}{\simeq} \bar{g}$.

Main equation: Let $A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$. Then $g \underset{\text{proj}}{\simeq} \bar{g}$ if and only if

$$\nabla_u A = \tfrac{1}{2}\big(u \otimes d\operatorname{tr} A + (u \otimes d\operatorname{tr} A)^*\big).$$

Theorem (B., Matveev)

Let $g \underset{\text{proj}}{\simeq} \bar{g}$. Then the Riemann curvature tensor of g is a Manakov operator:

$$[R(X), A] = [B, X] \quad \textit{for all } X \in \mathrm{so}(g), \textit{ where } B = \tfrac{1}{2}\nabla\big(\mathrm{grad}\operatorname{tr} A\big).$$

Proof.

Consider the compatibility condition for the main equation. □

Theorem (B., Matveev, Kiosak)

Let g, $\bar{g}$ and $\hat{g}$ be projectively equivalent. Assume that these metrics are linearly independent and g and $\hat{g}$ are strictly non-proportional, then g, $\bar{g}$ and $\hat{g}$ are metrics of constant sectional curvature.

Proof.

Apply Property 6. □

Definition

Let M be a smooth manifold endowed with an affine symmetric connection ∇. The holonomy group of ∇ is a subgroup $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_x M)$ that consists of the linear operators $A : T_x M \to T_x M$ being 'parallel transport transformations' along closed loops $\gamma(t)$ with $\gamma(0) = \gamma(1) = x$.

Problem. Given a subgroup $H \subset GL(n, \mathbb{R})$, can it be realised as the holonomy group for an appropriate symmetric connection on M^n?

Riemannian case and irreducible case: the problem is completely solved (Marcel Berger, D. V. Alekseevskii, R. Bryant, D. Joyce, L. Schwahhöfer, S. Merkulov).

Pseudo-Riemannian case: many fundamental results but still open (L. Bérard Bergery, A. Ikemakhen, C. Boubel, D. V. Alekseevskii, T. Leistner, A. Galaev).

Definition

Let M be a smooth manifold endowed with an affine symmetric connection ∇. The holonomy group of ∇ is a subgroup $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_x M)$ that consists of the linear operators $A : T_x M \to T_x M$ being 'parallel transport transformations' along closed loops $\gamma(t)$ with $\gamma(0) = \gamma(1) = x$.

Problem. Given a subgroup $H \subset GL(n, \mathbb{R})$, can it be realised as the holonomy group for an appropriate symmetric connection on M^n?

Riemannian case and irreducible case: the problem is completely solved (Marcel Berger, D. V. Alekseevskii, R. Bryant, D. Joyce, L. Schwahhöfer, S. Merkulov).

Pseudo-Riemannian case: many fundamental results but still open (L. Bérard Bergery, A. Ikemakhen, C. Boubel, D. V. Alekseevskii, T. Leistner, A. Galaev).

Theorem (B., Tsonev)

For every g-symmetric operator $A : V \to V$, its centraliser in $\mathrm{SO}(g)$ (the identity connected component of)

$$G_A = \{Y \in \mathrm{SO}(g) \mid YA = AY\}$$

is a holonomy group for a certain (pseudo)-Riemannian metric.

Definition

A map $R : \Lambda^2 V \to \mathrm{gl}(V)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V.$$

Definition

Let $\mathfrak{h} \subset \mathrm{gl}(V)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R : \Lambda^2 V \to \mathrm{gl}(V)$ such that $\mathrm{Im}\, R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0,\ u, v, w \in V\}.$$

We say that $\mathfrak{h}$ is a Berger algebra if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \mathrm{span}\{R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}),\ u, v \in V\}.$$

Berger test:

Let ∇ be a symmetric affine connection on TM. Then the Lie algebra $\mathfrak{hol}(\nabla)$ of its holonomy group $\mathrm{Hol}(\nabla)$ is Berger.

Definition

A map $R : \mathrm{so}(g) \to \mathrm{so}(g)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V,$$

where $u \wedge v = u \otimes g(v) - v \otimes g(u) \in \mathrm{so}(g)$.

Definition

Let $\mathfrak{h} \subset \mathrm{so}(g)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R : \mathrm{so}(g) \to \mathrm{so}(g)$ such that $\mathrm{Im}\, R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0,\ u, v, w \in V\}.$$

We say that $\mathfrak{h}$ is a Berger algebra if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \mathrm{span}\{R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}),\ u, v \in V\}.$$

Berger test:

Let ∇ be a Levi-Civita connection on (M, g). Then the Lie algebra $\mathfrak{hol}(\nabla) \subset \mathrm{so}(g)$ of its holonomy group $\mathrm{Hol}(\nabla)$ is Berger.

We have

$$\mathfrak{g}_A = \{X \in \mathrm{so}(g) \mid XA = AX\}$$

and we need to construct formal curvature tensors $R : \mathrm{so}(g) \to \mathrm{so}(g)$ whose images generate $\mathfrak{g}_A$.

Ideally, we want one single formal curvature tensor R such that $\mathrm{Im}\, R = \mathfrak{g}_A$.

Question: How to find R?

Step one: Berger test for $\mathfrak{g}_A$ and Magic Formula 1

We have

$$\mathfrak{g}_A = \{X \in \mathrm{so}(g) \mid XA = AX\}$$

and we need to construct formal curvature tensors $R : \mathrm{so}(g) \to \mathrm{so}(g)$ whose images generate $\mathfrak{g}_A$.
Ideally, we want one single formal curvature tensor R such that $\mathrm{Im}\, R = \mathfrak{g}_A$.

Question: How to find R?

Answer: Apply Properties 3 and 4, i.e. define a linear mapping $R : \mathrm{so}(g) \to \mathrm{so}(g)$ by:

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX), \tag{3}$$

where $p_{\min}(\lambda)$ is the minimal polynomial of A.

Conclusion: $\mathfrak{g}_A$ is a Berger algebra.

Step two: Realisation and Magic Formula 2

We need to find an example of g such that $\mathfrak{hol}(\nabla) = \mathfrak{g}_A$. The idea is natural:

- set $A(x) = \text{const}$
- try to find the desired metric $g(x)$ in the form constant + quadratic:

$$g_{ij}(x) = g^0_{ij} + \sum \mathcal{B}_{ij,pq} x^p x^q. \tag{4}$$

Question: How to find $\mathcal{B}$?
It is more convenient to work with "operators" rather than "forms":

$$\mathcal{B} = \sum \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha \quad \longrightarrow \quad B = \sum C_\alpha \otimes D_\alpha,$$

where C_α and D_α are the g_0-symmetric operators corresponding to $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$.
In terms of B, the answer is amasingly simple $B = \frac{1}{2} R(\otimes)$, i.e.

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX) \quad \mapsto \quad B = \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot \otimes),$$

Conclusion: The metric g defined by (4) satisfies two properties:
1) A is covariantly constant, i.e. $\mathfrak{hol}(\nabla) \subset \mathfrak{g}_A$ and
2) the curvature tensor at the origin is $R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX)$, and
therefore $\mathrm{Im}\, R = \mathfrak{g}_A \subset \mathfrak{hol}(\nabla)$ (hence solving the realisation problem)

A new (?) class of pseudo-Riemannian symmetric spaces

Construction via $\mathbb{Z}_2$-graded Lie algebras

A homogeneous space G/H is (pseudo-)Riemannian symmetric if the corresponding Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ satisfy the following conditions:

- $\mathfrak{g} = \mathfrak{h} + V$ is a $\mathbb{Z}_2$-grading, i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, V] \subset V$ and $[V, V] \subset \mathfrak{h}$,
- V admits an $\mathfrak{h}$-invariant inner product.

A new (?) class of pseudo-Riemannian symmetric spaces

Construction via $\mathbb{Z}_2$-graded Lie algebras

A homogeneous space G/H is (pseudo-)Riemannian symmetric if the corresponding Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ satisfy the following conditions:

- $\mathfrak{g} = \mathfrak{h} + V$ is a $\mathbb{Z}_2$-grading, i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, V] \subset V$ and $[V, V] \subset \mathfrak{h}$,
- V admits an $\mathfrak{h}$-invariant inner product.

In our situation, we take $R_0 : \mathrm{so}(g, V) \to \mathrm{so}(g, V)$ defined by $R_0(X) = \frac{d}{dt}|_{t=0} p(A + tX)$ with $p(A) = 0$ and $X \in \mathrm{so}(g)$.
Then we simply set $\mathfrak{h} = \mathrm{Im}\, R_0$ and consider $\mathfrak{g} = \mathfrak{h} + V$. To complete the construction and get a $\mathbb{Z}_2$-grading on $\mathfrak{g}$, we need to define $[u, v] \in \mathfrak{h}$ for $u, v \in V$. The answer is given by the formal curvature tensor R_0:

$$[u, v] = R_0(u \wedge v).$$

The Jacobi identity for $\mathfrak{g}$ follows from the first and second Bianchi identities (Properties 4 and 5).

Conclusion: The decomposition $\mathfrak{g} = \mathfrak{h} + V$ defines a $\mathbb{Z}_2$-grading and therefore G/H is a symmetric (pseudo)-Riemannian space.

Observation 1. For Kähler manifolds, the curvature tensor can be understood as a linear map on the unitary Lie algebra

$$R : \mathrm{u}(g) \to \mathrm{u}(g)$$

Observation 2. The definition of Manakov operators still makes sense:

$$[R(X), A] = [X, B], \quad \text{for } X \in \mathrm{u}(g) \text{ and } A, B \text{ being } g\text{-Hermitian} \tag{5}$$

and Properties 1–7 have natural generalisations.

Definition

A curve $\gamma(t)$ on a Kähler manifold (M, g, J) is called *J-planar*, if

$$\nabla_\gamma \dot\gamma =$$

where $\alpha, \beta \in \mathbb{R}$, and J is the complex structure on M. Two Kähler metrics g and $\bar g$ on a complex manifold (M, J) are called *c-projectively equivalent*, if they have the same J-planar curves.

Observation 3. Let g and $\bar g$ be c-projectively equivalent Kähler metrics. Then the Riemann curvature tensor of g is a Manakov operator in the sense of (5), where $A = \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{2(n+1)}} \bar g^{-1} g$ and $B = \frac{1}{2}\nabla(\mathrm{grad}\,\mathrm{tr}\, A)$.

Observation 1. For Kähler manifolds, the curvature tensor can be understood as a linear map on the unitary Lie algebra

$$R : \mathrm{u}(g) \to \mathrm{u}(g)$$

Observation 2. The definition of Manakov operators still makes sense:

$$[R(X), A] = [X, B], \quad \text{for } X \in \mathrm{u}(g) \text{ and } A, B \text{ being } g\text{-Hermitian} \tag{5}$$

and Properties 1–7 have natural generalisations.

Definition

A curve $\gamma(t)$ on a Kähler manifold (M, g, J) is called *J-planar*, if

$$\nabla_{\gamma}\dot{\gamma} = \alpha\dot{\gamma} + \beta J\dot{\gamma}$$

where $\alpha, \beta \in \mathbb{R}$, and J is the complex structure on M. Two Kähler metrics g and $\bar{g}$ on a complex manifold (M, J) are called *c-projectively equivalent*, if they have the same J-planar curves.

Observation 3. Let g and $\bar{g}$ be c-projectively equivalent Kähler metrics. Then the Riemann curvature tensor of g is a Manakov operator in the sense of (5), where $A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{2(n+1)}} \bar{g}^{-1} g$ and $B = \frac{1}{2}\nabla(\mathrm{grad}\,\mathrm{tr}\, A)$.

Definition

A vector field ξ on a Kähler manifold is called *c-projective*, if the flow of ξ preserves J-planar curves. A c-projective vector field is called *essential* if its flow changes the Levi-Civita connection.

Theorem (B., Matveev, Rosemann)

Let (M, g, J) be a closed connected Kähler manifold of arbitrary signature which admits an essential c-projective vector field. Then the manifold is isometric to $\mathbb{C}P^n$ with the Fubini-Study metric.

One of the ingredients of the proof is Property 7 for Jordan blocks.

Definition

A vector field ξ on a Kähler manifold is called *c-projective*, if the flow of ξ preserves J-planar curves. A c-projective vector field is called *essential* if its flow changes the Levi-Civita connection.

Theorem (B., Matveev, Rosemann)

Let (M, g, J) be a closed connected Kähler manifold of arbitrary signature which admits an essential c-projective vector field. Then the manifold is isometric to $\mathbb{C}P^n$ with the Fubini-Study metric.

One of the ingredients of the proof is Property 7 for Jordan blocks.

Theorem (B., Matveev, Rosemann (in progress))

A local description of Bochner-flat Kähler metrics of arbitrary signature.

The proof uses a Kähler modification of the Magic formula and Kähler analogs of the pseudo-Riemannian symmetric spaces discussed above.

Thanks for your attention